UNIVERSAL POINCARÉ DUALITY FOR THE INTERSECTION HOMOLOGY OF BRANCHED AND PARTIAL COVERINGS OF A PSEUDOMANIFOLD

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Submitted to the Graduate Faculty of the
College of Science and Engineering
Texas Christian University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

May 2016

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ACKNOWLEDGEMENTS

I am deeply indebted to my advisor, Greg Friedman. Without his support, patience, and incredible attention to detail this dissertation would not be possible. I am honored to have been his first doctoral student. I'm also grateful to Ken Richardson whose dedication to the new program and support for each individual graduate student has been truly above and beyond. My special thanks are also extended to Bob Doran. I'm grateful for his effort in reestablishing the math doctoral program and his encouraging words to complete my doctoral degree at TCU. I would like to thank everyone in the TCU math department for all of their help. I've been very fortunate to work on my doctorate in a nurturing environment thanks to open door policies.

I am especially grateful for the love and support of my parents who have always instilled in me the confidence to achieve anything I've wanted. I would also like to acknowledge all of the close friends I've made over the past few years spent at TCU who have always been supportive in all facets of my life.

Contents

1 Introduction

Intersection homology, originally developed by Goresky and MacPherson (12) as a tool to study singular spaces, is a homology theory that controls how singular simplices are allowed to intersect singularities. An important feature of intersection homology is that it satisfies a version of Poincaré duality for *stratified spaces* —spaces that are not quite manifolds, but are comprised of manifold layers. Goresky and Macpherson first proved Poincaré duality for *piecewise-linear pseudomanifolds* (12) (these include algebraic and analytic varieties) by defining an intersection pairing and later extended their results to *topological pseudomanifolds* (13) using sheaf-theoretic methods.

More recently, Friedman and McClure have given a new proof of Poincaré duality for intersection (co)homology by defining cup and cap products on intersection (co)homology and establishing the existence of fundamental classes for oriented topological pseudomanifolds (11). In (10), the authors go further and prove that regular covers of orientable pseudomanifolds satisfy universal Poincaré duality (in the sense of Ranicki (17)) for intersection (co)homology. Furthermore, they show there is a symmetric signature for Witt spaces. Here, universal Poincaré duality is a version for regular covering spaces (possibly non-compact), which is equivariant over the action by the group of deck transformations, and its importance stems from surgery theory. We should remark that while there may be interest in establishing Poincaré duality for intersection (co)homology via cap products for its own right, in order to employ Ranicki's algebraic techniques to define symmetric signatures, it is necessary to establish duality through a concrete isomorphism as opposed to the abstract isomorphisms achieved with sheaf-theoretic methods (i.e. Verdier duality).

A primary purpose of the present dissertation is to extend the work of Friedman and McClure (10) by proving a universal Poincaré duality theorem using covers over the regular strata. In particular, our approach will allow us to also consider possibly non-orientable topological pseudomanifolds. For example, the cone on projective space, $c\mathbb{R}\mathrm{P}^2$, is a non-orientable pseudomanifold that has no non-trivial covers. Consider, though, the cone of the orientation cover $S^2 \to \mathbb{R}\mathrm{P}^2$. While the map, $cS^2 \to c\mathbb{R}\mathrm{P}^2$, is not an even cover, it is a branched cover in the sense of Fox (4). Thus, we are led to consider topological branched covers. Beyond issues as in the previous example, there is also historical precedence to study the intersection homology of covers of the regular stratum. For instance, intersection homology with local coefficients defined solely over the regular stratum were studied by Goresky and MacPherson in (13).

We now give an outline of our results by section.

Section 2: Definition and basic properties. In this section we introduce intersection chain complexes for covering spaces of the regular stratum. These are defined through what we call *extended simplices*, and their definition is motivated by intersection homology with local coefficients defined over the top stratum of a pseudomanifold. Since extended simplices are a vital concept to the thesis, we present their definition below.

Definition 1.0.1 (Definition 2.1.2)**.** Let X be a stratified pseudomanifold, and let X_{reg} denote the set of regular points. We also let Σ denote the set of singular points. Let ν be any covering of X_{reg} and let R be a commutative ring with unity. We will call an ordered pair $(\widetilde{\sigma}, \sigma)$ an *extended* j-simplex where $\sigma : \Delta^j \to X$ and $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu)$[1] is a lift of σ. Moreover, we will call $\mathrm{im}(\sigma)$ the *base image* of $(\widetilde{\sigma}, \sigma)$ and $\mathrm{im}(\widetilde{\sigma})$ the *lifted image* of $(\widetilde{\sigma}, \sigma)$.

[1] We use ν to denote the data of a covering space. Here denotes $E(\nu)$ is the total space of the cover.

We define $S_*^\nu(X;R)$ by

$$S_*^\nu(X;R) := \text{The free } R\text{-module generated by extended simplices modded out}$$
$$\text{by extended simplices whose base image lie in } \Sigma.$$

For a perversity $\overline{p}$ defined on the strata of X, we can then determine $\overline{p}$-allowability of an extended simplex in an analogous fashion to intersection homology with local coefficients by defining an extended simplex $(\widetilde{\sigma}, \sigma)$ to be $\overline{p}$-allowable if σ is $\overline{p}$-allowable. What's more, we may consider the chain complex consisting of $\overline{p}$-intersection chains of $S_*^\nu(X;R)$, which we denote by $I^{\overline{p}}S_*^\nu(X;R)$, and whose homology we denote by $I^{\overline{p}}H_*^\nu(X;R)$.

In terms of generalizing the universal duality results of Friedman and McClure (10), the cover ν of X_{reg} takes on the role of what in their cases are covers of the entire space X.

Beyond their utility to define intersection homology of covering spaces of the regular stratum, extended simplices may also be applied to form another approach to intersection homology with local coefficients. This is the content of Section 2.2. Recall for ordinary homology there are two approaches to local coefficients (14, Section 3.H). One is to consider the homology of chain complexes of the form $A \otimes_{R[\pi]} S_*(\widetilde{X};R)$, where $\widetilde{X} \to X$ is a regular cover of X and A is a right $R[\pi]$-module. For the second, we consider local coefficient systems $\mathcal{E} \to X$, whose fibers are R-modules, and lifts of singular simplices to $\mathcal{E}$ with operations taking place in $\mathcal{E}$. It's well known these two approaches are equivalent (14, Proposition 3H.4), and because of this, the terminology of local coefficients is used interchangeably between the two. In order to reduce confusion, we will refer to the former as homology with *twisted coefficients* and to the latter as homology with *local coefficients*. Because intersection homology with local coefficients may be defined with local coefficients defined solely over the regular stratum, the approach of homology with twisted coefficients motivates our definition of intersection homology with twisted coefficients, reproduced below.

Definition 1.0.2 (Definition 2.2.7)**.** Let ν be a connected regular cover of X_{reg} with deck transformation group π, and let A be a right $R[\pi]$-module. The $\overline{p}$-*intersection chain complex with twisted coefficients* is defined to be the $\overline{p}$-intersection chains of $A \otimes_{R[\pi]} S^{\nu}_{*}(X; R)$ and is denoted by $I^{\overline{p}}\widetilde{S}^{\nu}_{*}(X; A)$. The homology of this chain complex is denoted $I^{\overline{p}}\widetilde{H}^{\nu}_{*}(X; A)$.

As expected, the approach of intersection homology with twisted coefficients is equivalent to intersection homology with local coefficients, which we prove in the theorem restated below.

Theorem 1.0.3 (Theorem 2.2.11)**.** *Let X be a stratified pseudomanifold with X_{reg} connected and with perversity $\overline{p} \leq \overline{t}$. Let ν be a regular cover of X_{reg} with deck transformation group π, and let A be a right $R[\pi]$-module (R a commutative ring with unity). Let $p : \mathcal{E} \to X_{reg}$ denote the system of local coefficient R-modules over X_{reg} associated to the $R[\pi]$-module A and the cover ν. Then there is an isomorphism of chain complexes between the intersection chain complex with twisted coefficients $I^{\overline{p}}\widetilde{S}^{\nu}_{*}(X; A)$ and the intersection chain complex of local coefficient system R-modules $I^{\overline{p}}S_{*}(X; \mathcal{E})$.*

This theorem allows us to carry over to intersection homology with twisted coefficients all the standard results of intersection homology such as the cone formula, excision, and Mayer-Vietoris long exact sequences. Moreover, by proving a generalization of Shapiro's lemma (Lemma 2.2.13), we can apply the above theorem to also carry over these standard results to $I^{\overline{p}}H^{\nu}_{*}(X;R)$.

Section 3: Cross product and Künneth theorem. Following the direction of Friedman, we prove a version of the Künneth theorem for the intersection homology of regular stratum covers. As pointed out by Friedman (see for instance (5), (6)), the approach of acyclic models is unavailable to intersection homology since the intersection homology of a contractible space can be nontrivial. As an alternative approach, Friedman uses shuffle-products to define a cross product map and proves a bi-perversity version of the Künneth theorem.

More precisely, let X and Y be stratified pseudomanifolds with perversities $\overline{p}$ and $\overline{q}$, respectively. The idea presented by Friedman [2] is to consider a perversity $Q_{\overline{p},\overline{q}}$defined on the strata of $X \times Y$ by

$$Q_{\overline{p},\overline{q}}(S \times S') = \begin{cases} \overline{p}(S) + \overline{q}(S') + 2 & \text{if } S, S' \text{ are both singular} \\ \overline{p}(S) & \text{if } S \text{ is singular and } S' \text{is regular} \\ \overline{q}(S') & \text{if } S \text{ is regular and } S' \text{ is singular} \\ 0 & \text{if } S, S' \text{ are both regular.} \end{cases}$$

He then shows that the cross product induces a quasi-isomorphism $I^{\overline{p}}S_{*}(X;F) \otimes_F I^{\overline{q}}S_{*}(Y;F) \to I^{Q_{\overline{p},\overline{q}}}S_{*}(X \times Y;F)$, where F a field [3]. Applying the techniques used by Friedman, we are also able to prove a Künneth theorem. The non-relative version is presented below.

[2]Friedman proves the Künneth theorem for other choices of $Q_{\overline{p},\overline{q}}$ as well, but the choice we use will be sufficient for our purposes.

[3]Friedman actually proves a Künneth theorem over any Dedekind domain (5, Theorem 6.56)

Theorem 1.0.4 (Theorem 3.4.1). *Let F be a field and let X and Y be stratified pseudomanifolds with perversities $\overline{p} \leq \overline{t}$ and $\overline{q} \leq \overline{t}$; respectively. Let ν be a cover for X_{reg} and ϑ a cover for Y_{reg}. Then the cross product induces an isomorphism*

$$\times : H_*(I^{\overline{p}}S_*^{\nu}(X;F) \otimes I^{\overline{q}}S_*^{\vartheta}(Y;F)) \to I^Q H_*^{\nu\times\vartheta}(X \times Y;F).$$

Besides being a computational tool, the Künneth theorem is also a vital step in defining cup and cap products for intersection (co)homology (11). Just as acyclic methods are unavailable to intersection homology, the "front face/back face" method of defining cup products also fails for intersection cohomology. The issue here is that while a simplex may satisfy allowability conditions, there is no guarantee that the front face and back face of the simplex will also be allowable. We refer the reader to Friedman's book in progress (5, Subsection 7.2.1) for a detailed discussion on the topic of products in intersection (co)homology.

Section 4: Finitely branched coverings of pseudomanifolds. Branched covers may be described topologically in the language of *spreads* developed by R.H. Fox (4) [4]. A spread is a map $g : Y \to Z$ between T_1 spaces such that the connected components of pre-images of open sets form a basis for the topology of Y. We call a point of Z an ordinary point if it may be evenly covered by the spread. The set of ordinary points is denoted by Z_o. A *complete spread* (Definition 4.1.4) is, heuristically speaking, a spread with no "missing pieces". For example, the map $\mathbb{C} - \{0\} \xrightarrow{z^2} \mathbb{C}$ is a non-complete spread; however, this spread may be completed by including 0 in the domain. In fact, every spread has a unique completion (Definition 4.1.6) according to the following theorem due to Fox.

[4]Besides Fox's original paper, a recent treatment of spreads and branched covers is given in (1) and includes interesting examples of more exotic spreads. For example, it is possible for a spread to have a fiber homeomorphic to the Cantor set.

Proposition 1.0.5 (Proposition 4.1.8)**.** *Let $f : X \to Z$ be a spread. Then f has a completion $g : Y \to Z$; that is, g is a complete spread that is an extension of f. Moreover, the completion is unique in the following sense. If $g' : Y' \to Z$ is any other completion of $f : X \to Z$, then there exists a homeomorphism $\phi : Y \to Y'$ such that $g'\phi = g$ and $\phi|_X = id_X$.*

It's important to realize that spreads and their completions are sensitive to target spaces. For example, the double cover $\mathbb{C} - \{0\} \xrightarrow{z^2} \mathbb{C} - \{0\}$ is a complete spread, but as we noted earlier the map $\mathbb{C} - \{0\} \xrightarrow{z^2} \mathbb{C}$ is not complete.

Another consequence in our study of spreads is we reprove Padilla's functoriality of normalization (16) by taking the completion of the inclusion of the regular strata, $X_{reg} \hookrightarrow X$. The completion is the normalization (16) of X (Example 4.1.7).

Next, we give a loose definition of topological branched covers. A *branched covering* is, a complete spread $g : Y \to Z$ such that "most" of the points of Z are ordinary. More technically, Z_o (the set of ordinary points) is a dense subset. What's more, the definition also requires $g^{-1}(Z_o)$ to be dense. There are other connectivity assumptions, but we will not discuss those now. See Definition 4.2.3 for the full definition.

Fox also defines a branching index to give a measurement to the "amount of branching". As an illustrating example, take the map $\mathbb{C} \xrightarrow{z^2} \mathbb{C}$. This is a branched cover and we have that the branching index at the origin is 2 and the branching index of any other point is 1.

With Fox's machinery laid out, we then go on to prove the main result of this section. Let Z be a normal[5] connected pseudomanifold, and consider a cover ν of Z_{reg}. We prove that the completion of $E(\nu) \to Z$, where $E(\nu)$ denotes the total space of the cover ν, is a

[5]We need Z to be normal here from the definition of branched coverings which will require Z_{reg} to be locally connected in Z.

branched cover (this follows by Proposition 4.2.4); moreover, if it is a finitely branched cover we have the following.

Theorem 1.0.6 (Theorem 4.3.3)**.** *Let Z be a connected normal stratified n-dimensional pseudomanifold, and let ν be the data associated to an unbranched covering of Z_{reg}. Let $g : Y \to Z$ be the branched covering associated to the pre-branched covering $E(\nu) \to Z$. If $g : Y \to Z$ is a finitely branched covering, then Y is a connected normal stratified n-dimensional pseudomanifold with stratification induced by the filtration $Y^i = g^{-1}(Z^i)$ where Z^i is the filtration inducing the stratification of Z.*

The proof is by induction on depth with the inductive step provided by Lemma 4.3.1. The idea is that in order to preserve compact links in the completion, we must require finite branching indices. We also show in Proposition 4.3.4 that branching indices are controlled by the pseudomanifold's strata.

In Section 4.4 we show that the intersection homology of a finitely branched cover is isomorphic to the intersection homology of the underlying regular stratum covering. More precisely, if ν is a cover of X_{reg} such that the completed branched cover $\widetilde{X} \to X$ of ν is finitely branched, then in Theorem 4.4.3 we show there is an isomorphism

$$I^{\overline{p}}H_*^{\nu}(X;R) \cong I^{\overline{p}}H_*(\widetilde{X};R).$$

The proof is similar to the proof that normalizations of pseudomanifolds preserve intersection homology. Because of this isomorphism we use the terms intersection homology of branched cover and the intersection homology of a regular stratum cover interchangeably.

Section 5: Fundamental classes with twisted coefficients Friedman and McClure show the existence and uniqueness of fundamental classes for oriented pseudomanifolds in (11). This section generalizes these results to possibly non-orientable pseudomanifolds with coefficients twisted by the orientation character.

We begin by recalling one construction of twisted fundamental classes from manifold theory. Our direction follows the approach in Hatcher (14, Example 3H.3). If M^n is a connected closed n-manifold, we may consider the orientation cover, $\widehat{M} \to M$, which possesses a non-trivial orientation-reversing deck transformation involution $\tau : \widehat{M} \to \widehat{M}$. By an abuse of notation, we also use τ to denote the isomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$. For a commutative ring with unity R, we let R^τ denote the right $R[\mathbb{Z}_2]$-module induced by τ. Let $S_*(M; R^\tau) = R^\tau \otimes_{R[\mathbb{Z}_2]} S_*(\widehat{M}; R)$. From (14, Example 3H.2), there is a long exact sequence

$$\to H_j(M; R^\tau) \to H_j(\widehat{M}; R) \to H_j(M; R) \to H_{j-1}(M; R^\tau) \to .$$

Using that $\widehat{M}$ is an orientable manifold one can show the existence and uniqueness of a twisted fundamental class $\Gamma \in H_n(M; R^\tau)$ that generates $H_n(M; R^\tau) \cong R$. Furthermore, the twisted fundamental class has the defining property of being the unique element mapping to the fundamental class of $\widehat{M}$ in the long exact sequence above, with $\widehat{M}$ given the canonical choice of orientation from the construction of orientation covers.

One difficulty with attempting to generalize the above construction of twisted fundamental classes to pseudomanifolds is that pseudomanifolds do not have orientation covers as we saw earlier in the case of $c\mathbb{R}\mathrm{P}^2$. However, let X be a normal stratified pseudomanifold, and consider the orientation cover $\widehat{X_{reg}} \to X_{reg}$. Applying the results of Section 4, the orientation cover of X_{reg} extends to a *branched orientation cover* $\widehat{X} \to X$ (Proposition 5.2.1); moreover, there is a nontrivial orientation-reversing deck transformation involution $\tau : \widehat{X} \to \widehat{X}$. Continuing to follow the approach from manifold theory, we show in this section that there is a long exact sequence in the proposition below.

Proposition 1.0.7 (Proposition 5.2.4)**.** *Let R be a commutative ring with unity and also assume $\frac{1}{2} \in R$. Let X be a stratified normal pseudomanifold with branched orientation cover $\widehat{X} \to X$, and let $U \subset X$ be open. Let $\bar{p} \leq \bar{t}$ be a perversity on X.Then there exists a long exact sequence*

$$\longrightarrow I^{\bar{p}}H_j(X, U; R^\tau) \longrightarrow I^{\bar{p}}H_j(\widehat{X}, \widehat{U}; R) \longrightarrow I^{\bar{p}}H_j(X, U; R) \longrightarrow I^{\bar{p}}H_{j-1}(X, U; R^\tau) \longrightarrow$$

The reader may have noticed the extra assumption that $\frac{1}{2} \in R$. This assumption is not too unreasonable since in the case R is a field (which duality requires for intersection homology without further restrictions on the base ring) and R is characteristic 2, then it is orientable and duality results hold by the work of Friedman and McClure. The reason this is needed to prove the proposition above is to show the map $I^{\bar{p}}S_*(\widehat{X}; R) \to I^{\bar{p}}S_*(X; R)$, which is induced by the branched cover $\widehat{X} \to X$, is surjective. In the case of manifolds there is no issue because one may simply lift singular simplices. However, if $x \in I^{\bar{p}}S_*(X; R)$ and $\widehat{x}$ is a choice of "lift", then while $\widehat{x}$ is $\bar{p}$-allowable, there is no guarantee that $\partial\widehat{x}$ will also be $\bar{p}$-allowable, since cancellations which occur downstairs in X for ∂x may not occur upstairs in $\widehat{X}$ for $\partial\widehat{x}$. If we instead consider $\frac{1}{2}\widehat{x} + \frac{1}{2}\tau\widehat{x}$, then this will be a $\bar{p}$-intersection chain that maps to x.

Once we have the long exact sequence above, we then apply results of Friedman and McClure (11) on orientable pseudomanifolds to prove existence and uniqueness of twisted fundamental classes over compact subsets. The key theorem is restated below.

Theorem 1.0.8 (Theorem 5.3.1)**.** *Let X be a normal stratified n-dimensional pseudomanifold with orientation branched cover $p : \widehat{X} \to X$. Let $\bar{p} \leq \bar{t}$ be a perversity on X. Let R be a commutative ring with unity and assume $\frac{1}{2} \in R$. For a subspace $A \subset X$, let $\widehat{A} = p^{-1}(A)$ and let $K \subset X$ be compact.*

1. *$I^{\bar{p}}H_i(X, X-K; R^\tau) = 0$ for $i > n$.*
2. *There exists a unique $\Gamma_K \in I^{\bar{p}}H_n(X, X-K; R^\tau)$ such that $\Gamma_K \mapsto \Gamma_{\widehat{K}}$ in the exact sequence of Proposition 5.2.4 where $\Gamma_{\widehat{K}}$ is the fundamental class over $\widehat{K}$ (see (11, Definition 5.9) for the definition of fundamental classes over a compact set in the orientable case) with $\widehat{X}$ given the tautological orientation (Remark 5.2.2).*
3. *If $L \subset K$ is compact, then Γ_K maps to Γ_L under the map $I^{\bar{p}}H_n(X, X-K; R^\tau) \to I^{\bar{p}}H_n(X, X-L; R^\tau)$.*

For pseudomanifolds not necessarily normal we use normalizations to prove a generalization of the proposition above to all pseudomanifolds (Theorem 5.4.4).

Section 6: Technical preliminaries. This is a technical section which may be seen as our analogue of (10, Section 6). It's here we see the algebraic problems that arise when attempting to prove a universal duality theorem for coverings of the regular stratum. In particular, a key step in Friedman and McClure's proof of universal duality for oriented pseudomanifolds is the quasi-isomorphism below (10, Proposition 6.1), which appears in the definition of their algebraic diagonal map (which is used to define the cap product).

$$F \otimes_{F[\pi]} I^{\overline{p}}S_*(\widetilde{X};F) \to I^{\overline{p}}S_*(X;F)$$

Here $\widetilde{X} \to X$ is a regular cover. For ordinary homology, the above map is an isomorphism and is an easy exercise. One just uses that $S_*(\widetilde{X};F)$ is a free $F[\pi]$-module with basis given by choosing lifts of singular simplices which generate $S_*(X;F)$. For intersection homology, though, there are allowability issues. For example, if $x \in I^{\overline{p}}S_*(X;F)$ then we can find an element $y \in F \otimes_{F[\pi]} S_*(\widetilde{X};F)$ which maps to x, but we also need to have that we can find a y of the form $y = 1 \otimes z$ where $z \in I^{\overline{p}}S_*(\widetilde{X};F)$. While we can find y such that z is $\overline{p}$-allowable, there is no guarantee that ∂z will be allowable since the cancellations which occur for ∂x may not occur upstairs in the lift ∂z.

Nevertheless, Friedman and McClure show the map above is a quasi-isomorphism. The idea of their proof is to make a local to global argument since in the case W is evenly covered we have that $F \otimes_{F[\pi]} I^{\overline{p}}S_*(\widetilde{W};F) \cong I^{\overline{p}}S_*(W;F)$ as chain complexes.

We would like to consider branched covers of X. In the case $\widetilde{X}$ is branched cover, though, we lose the ability to cover X by evenly covered open sets; we are not able to a priori make the same local to global argument of Friedman and McClure. For if W is any open subset of X, all we know is that $I^{\overline{p}}S_*(\widetilde{W};F)$ is an $F[\pi]$-submodule of the free $F[\pi]$-module $S_*(\widetilde{W};F)$. In general, though, we have no knowledge of the group ring $F[\pi]$ (for example, if it's a PID). Even if we did know that $F[\pi]$ were a PID, which would imply $I^{\overline{p}}S_*(\widetilde{W};F)$ is a free $F[\pi]$-module; in order to mimic the local to global argument of Friedman and McClure, we need to know that $H_*(F \otimes_{F[\pi]} I^{\overline{p}}S_*(\widetilde{W};F))$ satisfies the local computations of intersection homology such as a cone formula. There are issues though in proving a cone formula. For example, if $W = cL$ with L a $(k-1)$-dimensional compact pseudomanifold, then a cone formula would require $H_{k-1-\overline{p}(\{v\})}\left(F \otimes_{F[\pi]} I^{\overline{p}}S_*(\widetilde{cL};F)\right) = 0$. However, from the universal coefficients theorem we have

$$H_{k-1-\overline{p}(\{v\})}\left(F \otimes_{F[\pi]} I^{\overline{p}}S_*(\widetilde{cL};F)\right)\cong F \otimes_{F[\pi]} I^{\overline{p}}H_{k-1-\overline{p}(\{v\})}(\widetilde{cL};F) \oplus \operatorname{Tor}_1^{F[\pi]}\left(I^{\overline{p}}H_{k-2-\overline{p}(\{v\})}(\widetilde{cL};F), F\right).$$

By the cone formula for ordinary intersection homology, we know that the left factor in the direct sum above vanishes, but there is no guarantee that the torsion term vanishes. Thus, we do not have a cone formula in general even in the case $F[\pi]$ is a PID.

However, as we saw in Section 4, if we want the branched cover $\widetilde{X}$ to be a pseudomanifold we need to assume the branching indices are all finite. For the sake of simplicity, assume that $\widetilde{X} \to X$ is a finitely fibered regular branched cover so that it is also finitely branched. This implies that π is a finite group, so we can apply Maschke's theorem which states that if $\operatorname{char}(F)$ does not divide $|\pi|$ (always the case for $F = \mathbb{Q}$), then $F[\pi]$ is semi-simple. In particular, every module over $F[\pi]$ is projective (18, Theorem 4.2.2), in particular, flat. With this knowledge, we can achieve a cone formula (Lemma 6.2.9) to extend Friedman and McClure's local to global argument to our case whenever we have finitely branched covers, at the cost of restrictions on the characteristic of the base field.

Actually, by Lemma 6.2.7 all we need to make a local to global argument is to assume that $\widetilde{X}$ is a finitely branched covering and that $\operatorname{char}(F)$ does not divide branching indices, which includes the possibility of infinite fibers. The quasi-isomorphism in Proposition 6.2.11 is then one main ingredient that allows us to extend cap products later in Section 7 to include cap products for finitely branched covers. The other main ingredient is Proposition 6.2.12, which is a generalization of (10, Proposition 6.5). We go into more detail later, but essentially Proposition 6.2.12 says the Künneth theorem for $X \times X$ is preserved upon tensoring over the diagonal action of $F[\pi]$.

Section 6.3 is dedicated to universal cohomology (Definition 6.3.1). This is the dual to intersection homology of finitely branched covers in our universal duality theorem. We also prove that universal cohomology satisfies all the expected local computations such as a version of the cone formula. Our proof of these local results again relies on finitely branched covers and using that the group ring of deck transformations is "locally" semi-simple with appropriate assumptions on the characteristic of the underlying field.

Section 7: Poincaré duality theorems. The last section is dedicated to the main results of the thesis. In Section 7.1 we define the relevant algebraic diagonal map needed to define a cap product for finitely branched covers by using our results from Section 6. We also show that our cap product has all the expected naturality properties in Proposition 7.1.3 and Proposition 7.1.4. In Section 7.2 we prove universal duality for finitely branched covers. We first define universal cohomology with compact supports in the obvious way as well as the duality map defined via cap products. In the theorem below, $I^c_{\overline{p}}\overline{H}^*_\nu(X;F)$ denotes universal intersection cohomology with compact supports. The condition of F being ν-good in the theorem below is our terminology which states that the characteristic of F cannot divide the branching indices of the branched cover induced by ν. We note that $\mathbb{Q}$ will always be ν-good, and that if X is compact there are only finitely many integers char(F) cannot divide for the theorem to hold (Corollary 4.3.6).

Theorem 1.0.9 (Theorem 7.2.1)**.** *Let X be a stratified pseudomanifold with X_{reg} connected and with perversity $\overline{0} \leq \overline{p} \leq \overline{t}$ and dual perversity $\overline{q}$. Let ν be locally finite unbranched oriented regular connected cover of X_{reg} with deck transformation group π. Let F be a ν-good field. Then,*

$$\mathscr{D} : I^c_{\overline{p}}\overline{H}^*_{\nu}(X;F) \to I^{\overline{q}}H^{\nu}_*(X;F)$$

is an isomorphism of $F[\pi]$-modules.

In the case X is compact, the map in the theorem is given by $\mathscr{D}(\alpha) = (-1)^{\dim(\alpha)\cdot\dim(X)}\alpha \cap \Gamma$, where $\alpha \in I^{\overline{p}}\overline{H}^*_{\nu}(X;F)$ and Γ is the twisted fundamental class of X. For the sign in our definition of the duality map, see (9, Section 4.1), where the sign appears to make the duality map a chain map of appropriate degree. Our argument to prove universal duality for finitely branched covers follows that of (11, Theorem 6.3), which in turn, follows Hatcher's proof of Poincaré duality in (14). The argument is through induction on depth(X) by making a local to global to argument and a Zorn's lemma argument using a diagram between Mayer-Vietoris long exact sequences. Commutativity of this diagram is shown in Section 7.3.

Finally, in Section 7.4 we show there is a non-universal Poincaré duality via cap products for a special class of coefficient systems (with pseudomanifolds possibly non-orientable). In the theorem below, the tildes indicate that the coefficients are twisted. We also note that A^t denotes the left $F[\pi]$-module on A induce by the twisted involution.

Theorem 1.0.10 (Theorem 7.4.3)**.** *Let X be a stratified n-dimensional pseudomanifold with X_{reg} connected, and let ν be a connected locally finite unbranched oriented regular cover of X_{reg} with deck transformation group π. Let F be a ν-good field and A be a right $F[\pi]$-module. Then*

$$\mathscr{D} : I^{c}_{\overline{p}}\widetilde{H}^{*}_{\nu}(X; A^{t}) \to I^{\overline{q}}\widetilde{H}^{\nu}_{*}(X; A)$$

is an isomorphism of F-vector spaces where $\overline{p}$ and $\overline{q}$ are dual perversities.

As a last corollary, we extend the duality results of (11) to include cases of possibly non-orientable pseudomanifolds. We recall from earlier that F^{τ} denotes the $F[\mathbb{Z}_2]$-module with underlying field F and $\mathbb{Z}_2$-action induced by the isomorphism $\tau : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$. We also use $\mathfrak{o}$ in the corollary to denote the orientation cover of X_{reg}.

Corollary 1.0.11. *Let X be a stratified pseudomanifold, and let F be a field with char$F \neq 2$. Let $\overline{0} \leq \overline{p} \leq \overline{t}$ be a perversity on X with dual perversity $\overline{q}$. Then we have F-vector space isomorphisms*

- $\mathscr{D} : I^{c}_{\overline{p}}\widetilde{H}^{*}_{\mathfrak{o}}(X; F^{\tau}) \to I^{\overline{q}}H_{*}(X; F)$
- $\mathscr{D} : I^{c}_{\overline{p}}H^{*}(X; F) \to I^{\overline{q}}H_{*}(X; F^{\tau}) = I^{\overline{q}}\widetilde{H}^{\mathfrak{o}}_{*}(X; F^{\tau})$.

2 Definition and basic properties

2.1 Intersection homology for coverings of the regular stratum

Set up and definition

Let X be a stratified pseudomanifold with singular set Σ. We also let $X_{reg} = X - \Sigma$ be the set of regular points of X. Let $\nu = (E(\nu), X_{reg}, p_\nu)$ denote the data for a covering of X_{reg}. That is, $E(\nu)$ is the total space, X_{reg} is the base space, and $p_\nu : E(\nu) \to X_{reg}$ is the covering map.

Remark 2.1.1. We note that although this is not standard notation for covering spaces, this will be a notational aid for our purposes. Our notation is inspired by notation used by some authors for vector bundles.

Definition 2.1.2. Let R be a commutative ring with unity. We will call an ordered pair $(\widetilde{\sigma}, \sigma)$ an *extended* j-simplex where $\sigma : \Delta^j \to X$ and $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu)$ is a lift of σ. Moreover, we will call $\mathrm{im}(\sigma)$ the *base image* of $(\widetilde{\sigma}, \sigma)$ and $\mathrm{im}(\widetilde{\sigma})$ the *lifted image* of $(\widetilde{\sigma}, \sigma)$. We define $S_j^\nu(X; R)$ by

$$\begin{aligned} S_j^\nu(X;R) :=\ &\text{The free } R\text{-module generated by extended } j\text{-simplices modded out} \\ &\text{by extended } j\text{-simplices whose base image lie in } \Sigma. \end{aligned}$$

We also use the standard notation $S_*^\nu(X;R) := \bigoplus_j S_j^\nu(X;R)$. More generally, for a right R-module M we define $S_*^\nu(X;M) := M \otimes_R S_*^\nu(X;R)$.

Before verifying that this forms a chain complex under the obvious boundary map, we take a moment to go over how one should think of extended simplices. The cover ν only covers X_{reg}, so a generic k-simplex $\sigma : \Delta^k \to X$ will not be guaranteed to a have a lift to $E(\nu)$. Thus, we consider the best possible lifts, namely, we lift $\sigma|_{\sigma^{-1}(X_{reg})} : \sigma^{-1}(X_{reg}) \to X_{reg}$. So an extended k-simplex $(\widetilde{\sigma}, \sigma)$ simultaneously lifts "most" of σ while keeping track of the k-simplex which was lifted. The utility here is that we can consider how σ intersects the strata of X to determine allowability. This is sort of a covering space version of intersection homology with local coefficients, where the local coefficients are over the regular stratum.

We may turn $S^{\nu}_*(X; M)$ into a chain complex as follows. If Δ^j has ordered vertices $v_0, \dots, v_j$, then we will use the convention that for $k = 0, \dots, j$ the k-th face of Δ^j is the face spanned by $v_0, \dots, v_{k-1}, v_{k+1}, \dots, v_n$. So for $k = 0, \dots, j$ we let ∂^j_k be the map which takes Δ^{j-1} to the k-th face of Δ^j. Then we have a boundary map given by

$$\partial\left(\sum_i m_i(\widetilde{\sigma}_i, \sigma_i)\right) = \sum_{i,k} (-1)^k m_i(\widetilde{\sigma_i\partial^j_k}, \sigma_i\partial^j_k).$$

Here $\widetilde{\sigma\partial^j_k} = \widetilde{\sigma}|_{(\sigma\partial^j_k)^{-1}(X_{reg})}$. We note that if $\sigma^{-1}(X_{reg}) = \emptyset$ then we also have that $(\sigma\partial^j_k)^{-1}(X_{reg}) = \emptyset$. Hence, $(\widetilde{\sigma\partial^j_k}, \sigma\partial^j_k) = 0$ by definition which means that $\partial(\widetilde{\sigma}, \sigma) = 0$ whenever $\sigma^{-1}(X_{reg}) = \emptyset$. Thus, we have a well defined map $\partial : S^{\nu}_j(X; M) \to S^{\nu}_{j-1}(X; M)$ and we have $\partial\partial = 0$ (which we verify below) so that $S^{\nu}_*(X; M)$ is a chain complex.

Proof that $\partial\partial = 0$. First, notice for $k \leq l$ we have $\partial^j_k\partial^{j-1}_l = \partial^j_{l+1}\partial^{j-1}_k$ and for $k > l$ we have $\partial^j_k\partial^{j-1}_l = \partial^j_l\partial^{j-1}_{k-1}$. The former follows since the l-th face of the k-th face of Δ^j will be the l-th face of the $j-1$-simplex spanned by the ordered vertices $v_0, \dots, v_{k-1}, v_{k+1}, \dots, v_j$ which whenever $k \leq l$ is by definition the $j-2$ simplex spanned by the vertices $v_0, \dots, v_{k-1}, v_{k+1}, \dots, v_l, v_{l+2}, \dots, v_j$.

However, this is the same as the $(l+1)$-st face of the k-th face of Δ^j since this is the $(l+1)$-st face of the $j-1$ simplex spanned by the ordered vertices $v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_j$ which whenever $k \leq l$ is by definition the $j-2$ simplex spanned by the vertices $v_0, \ldots, v_{k-1}, v_k, \ldots, v_l, v_{l+2}, \ldots, v_j$. So we see that the k-th face of the l-th face of Δ^j is the same as the $(l+1)$-st face of the k-th face of Δ^j which means that $\partial_k^j \partial_l^j = \partial_{l+1}^j \partial_k^j$. For the case $k > l$ we claim this is actually the previous case after a change of variables. To see this let $k' = l$ and let $l' = k - 1$. Then notice that if $0 \leq l \leq j-1$, then $0 \leq k' \leq j$ so that $\partial_{k'}^j$ is defined. Also, if $k > l$, then $k > 0$ and so $1 \leq k \leq j$ which means that $0 \leq l' \leq j-1$ so that $\partial_{l'}^{j-1}$ is also defined. But now if $k > l$ then that means that $k - 1 \geq l$ so that $l' \geq k'$. Thus, $\partial_l^j \partial_{k-1}^{j-1} = \partial_{k'}^j \partial_{l'}^{j-1}$ and $k' \leq l'$. Thus, we can use the above argument to see that $\partial_{k'}^j \partial_{l'}^{j-1} = \partial_{l'+1}^j \partial_{k'}^{j-1}$. Plugging back in k and l we see that $\partial_{l'+1}^j \partial_{k'}^{j-1} = \partial_k^j \partial_l^{j-1}$. Hence, we have shown that for $k > l$, $\partial_k^j \partial_l^{j-1} = \partial_l^j \partial_{k-1}^{j-1}$.

To ease the computation in what follows we will set $\beta_{k,l} = (\sigma \widetilde{\partial_k^j \partial_l^{j-1}, \sigma} \partial_k^j \partial_l^{j-1})$. Thus, for $k = 0, \ldots, j$ and for $l = 0, \ldots, j-1$ we have that

$$\beta_{k,l} = \beta_{l,k-1} \text{ whenever } k > l. \tag{1}$$

First note by a change of variable $t = l+1$ we have $\sum_{k \leq l} (-1)^k (-1)^l \beta_{k,l} = \sum_{k \leq t-1} (-1)^k (-1)^{t-1} \beta_{k,t-1}$. Of course t is a dummy variable so we can just write this as $\sum_{k \leq l-1} (-1)^k (-1)^{l-1} \beta_{k,l-1} = \sum_{k<l} (-1)^k (-1)^{l-1} \beta_{k,l-1} = \sum_{l>k} (-1)^k (-1)^{l-1} \beta_{k,l-1}$. We make a further variable change by relabeling k to be l and l to be k. That is, we have $\sum_{l>k} (-1)^k (-1)^{l-1} \beta_{k,l-1} = \sum_{k>l} (-1)^l (-1)^{k-1} \beta_{l,k-1}$. However, whenever $k > l$ we have by 1.1 that $\beta_{l,k-1} = \beta_{k,l}$ Hence, we have shown that

$$\sum_{k \leq l} (-1)^k (-1)^l \beta_{k,l} = \sum_{k>l} (-1)^{k-1} (-1)^l \beta_{k,l}. \tag{2}$$

So we have

$$
\begin{aligned}
\partial\partial(\widetilde{\sigma},\sigma) =& \partial\left(\sum_k (-1)^k(\widetilde{\sigma\partial_k^j}, \sigma\partial_k^j)\right) \\
=& \sum_k \sum_l (-1)^k(-1)^l \beta_{k,l} \\
=& \sum_{k\le l} (-1)^k(-1)^l \beta_{k,l} + \sum_{k>l} (-1)^k(-1)^l \beta_{k,l} \\
=& \sum_{k>l} (-1)^{k-1}(-1)^l \beta_{k,l} + \sum_{k>l} (-1)^k(-1)^l \beta_{k,l} \qquad \text{by (1.2)} \\
=& -\sum_{k>l} (-1)^k(-1)^l \beta_{k,l} + \sum_{k>l} (-1)^k(-1)^l \beta_{k,l} \\
=& 0.
\end{aligned}
$$

Hence, $\partial\partial = 0$. □

Before defining intersection chain complexes, we recall the definition of perversities. A *perversity* on a stratified pseudomanifold X is a function $\overline{p} : \{\text{strata of } X\} \to \mathbb{Z}$. We will call an extended simplex $(\widetilde{\sigma},\sigma)$ $\overline{p}$-allowable if σ is $\overline{p}$-allowable. Recall that a singular k-simplex $\sigma : \Delta^k \to X$ is $\overline{p}$-allowable if for each stratum S of X,

$$\sigma^{-1}(S) \subset (k - \operatorname{codim}(S) + \overline{p}(S)) - \text{skeleton of } \Delta^k$$

We will call a chain $\xi \in S_*^{\nu}(X;M)$ $\overline{p}$-allowable if ξ may be written as a sum $\xi = \sum m_i(\widetilde{\sigma}_i,\sigma_i)$, where $m_i \in M$ and $m_i \neq 0$ and each σ_i is $\overline{p}$-allowable.

Next, we define a few submodules of $S^{\nu}_{*}(X;M)$. While we do not study these submodules for their own right, they come up often enough in arguments we make in proving various results about intersection chain complexes that we introduce new notation. Before we make the definition, though, we prove a consequence of the condition $\overline{p} \leq \overline{t}$ that will be used in the definition below. We recall for the reader that $\overline{t}$ is the top perversity and is defined by $\overline{t}(S) = \text{codim}(S) - 2$ for each singular stratum. We prove that if $\sigma : \Delta^k \to X$ is $\overline{p}$-allowable, then $\text{int}(\Delta^k) \subset \sigma^{-1}(X_{reg})$ and $\text{int}(\partial\Delta^k) \subset (\sigma \circ \partial_j)^{-1}(X_{reg})$, where ∂_j is the j-th face map. If σ is a $\overline{p}$-allowable k-simplex and $\overline{p} \leq \overline{t}$, then for each singular stratum S we have

$$\begin{aligned}
\sigma^{-1}(S) &\subset (k - \text{codim}(S) + \overline{p}(S)) - \text{skeleton of } \Delta^k \\
&\subset (k - \text{codim}(S) + \overline{t}(S)) - \text{skeleton of } \Delta^k \\
&= (k - \text{codim}(S) + \text{codim}(S) - 2) - \text{skeleton of } \Delta^k \\
&= (k-2) - \text{skeleton of } \Delta^k
\end{aligned}$$

So if $x \in \text{int}(\Delta^k) \cup \text{int}(\partial\Delta^k)$, it cannot be the case that $x \in \sigma^{-1}(S)$ for any singular stratum S (we are assuming X has no singular strata of codimension 1); and therefore, we must have $x \in \sigma^{-1}(X_{reg})$ since $\sigma(x)$ must belong to some stratum of X.

Definition 2.1.3. We define $\widehat{S}^{\nu}_{k}(X;M)$ to be the submodule of $S^{\nu}_{k}(X;M)$ generated by extended k-simplices $(\widetilde{\sigma}, \sigma)$ with $\text{int}(\Delta^k) \subset \sigma^{-1}(X_{reg})$.

Next, we define ${}^{\overline{p}}S^{\nu}_{k}(X;M)$ to be the submodule of $S^{\nu}_{k}(X;M)$ generated by $\overline{p}$-allowable extended k-simplices.

Notice that if $\overline{p} \leq \overline{t}$ (a condition we always require in our paper), then by our work preceding the definition we have the inclusions ${}^{\overline{p}}S^{\nu}_{*}(X;M) \subset \widehat{S}^{\nu}_{*}(X;M) \subset S^{\nu}_{*}(X;M)$ as our above work shows the generators of ${}^{\overline{p}}S^{\nu}_{*}(X;M)$ satisfy the definition of $\widehat{S}^{\nu}_{*}(X;M)$. Our above work also shows $\partial\left({}^{\overline{p}}S^{\nu}_{*}(X;M)\right) \subset \widehat{S}^{\nu}_{*}(X;M)$ so that the boundary map satisfies

$$\partial : {}^{\overline{p}}S^{\nu}_{*}(X;M) \to \widehat{S}^{\nu}_{*}(X;M).$$

An important point is that ${}^{\overline{p}}S^{\nu}_{*}(X;M)$ and $\widehat{S}^{\nu}_{*}(X;M)$ will not in general be chain complexes. This is because while $x \in {}^{\overline{p}}S^{\nu}_{k}(X;M)$ may be a sum of $\overline{p}$-allowable k-simplices, there is no guarantee that ∂x will be a sum of allowable $(k-1)$-simplices.

We also note that if ν is a regular cover, then $\widehat{S}^{\nu}_{*}(X;R)$, and ${}^{\overline{p}}S^{\nu}_{*}(X;R)$ are free $R[\pi]$-modules, where π denotes the group of deck transformations of the cover ν. To see this, for each k-simplex σ with $\mathrm{int}(\Delta^k) \subset \sigma^{-1}(X_{reg})$ choose a single lift $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu)$. Then, $\{(\widetilde{\sigma}, \sigma)\}$ provides an $R[\pi]$-basis for $\widehat{S}^{\nu}_{*}(X;R)$ (here we are using that the deck transformation action of a regular cover is transitive). The same proof shows that ${}^{\overline{p}}S^{\nu}_{*}(X;R)$ is also free $R[\pi]$-modules.

In the proceeding paper we will want to show numerous intersection chain complexes are isomorphic. The property of being a free $R[\pi]$-module will allow us to make arguments which mimic standard arguments from ordinary homological algebra and we will then apply Lemma 2.2.10 below to show intersection chain complexes are also isomorphic.

We may now define the intersection homology for a covering of the regular stratum.

Definition 2.1.4. We define the *ν-extended intersection chain complex* of X, denoted $I^{\overline{p}}S^{\nu}_{*}(X;M)$, to be all chains $\xi \in S^{\nu}_{*}(X;M)$ such that ξ and $\partial\xi$ are both $\overline{p}$-allowable.

First, we can see this forms an R-module from the equality $I^{\overline{p}}S_*^{\nu}(X;M) = {}^{\overline{p}}S_*^{\nu}(X;M) \cap \partial^{-1}\left({}^{\overline{p}}S_*^{\nu}(X;M)\right)$. Secondly, $I^{\overline{p}}S_*^{\nu}(X;M)$ is indeed a chain complex because if $\xi \in I^{\overline{p}}S_*^{\nu}(X;M)$, then $\partial\xi$ is by definition $\overline{p}$-allowable and $\partial\partial\xi = 0$ is trivially $\overline{p}$-allowable so we have $\partial\xi \in I^{\overline{p}}S_*^{\nu}(X;M)$. The *$\nu$-extended intersection homology* of X, denoted $I^{\overline{p}}H_*^{\nu}(X;M)$, is the homology of this chain complex. Also, observe that $I^{\overline{p}}S_*^{\nu}(X;M) \subset {}^{\overline{p}}S_*^{\nu}(X;M) \subset \widehat{S}_*^{\nu}(X;M)$.

Before turning to the next section, we take a moment to compare our ν-extended intersection chain complexes with ordinary intersection chain complexes. In (10) the authors study regular covers of stratified pseudomanifolds $\widetilde{X} \to X$ and they prove a version of universal Poincaré duality for intersection homology. However, for intersection homology, a well-known fact is that one may define intersection homology with local coefficients defined only on the regular stratum. This motivates our definition of $I^{\overline{p}}H_*^{\nu}(X;R)$, where the cover ν is a cover of the regular stratum of X. So one may think of ν as playing a similar role to a regular cover $\widetilde{X} \to X$.

One of our main results is that under relatively mild conditions, there is also a version of universal Poincaré duality for intersection homology of a regular cover defined solely over the regular stratum. Thus, one should think of $I^{\overline{p}}H_*^{\nu}(X;R)$ as the intersection homology of a *branched cover* of X. In fact, in Section 4 we make the connection precise by identifying our definition of ν-extended intersection homology to the intersection homology of branched covers (in the topological sense of Fox (4)) of pseudomanifolds.

Restriction to Open Subsets and Intersection Homology of Pairs

Let X be a stratified pseudomanifold and let $\nu = (E(\nu), X_{reg}, p)$ denote the data associated to a covering of X_{reg}. Let $U \subset X$ be an open subset with inclusion map $i : U \hookrightarrow X$. We denote the restriction of ν to U by $i^*\nu = (p^{-1}(U), U, p|_U)$. Thus, we may form $I^{\overline{p}}S_*^{i^*\nu}(U; R)$ (R a commutative ring with unity). For a chain $\xi \in S_*^{\nu}(X; R)$ let $|\xi|$ denote the base support. Observe that $\xi \in I^{\overline{p}}S_*^{i^*\nu}(U; R) \iff \xi \in I^{\overline{p}}S_*^{\nu}(X; R)$ and $|\xi| \subset U$. So elements of $I^{\overline{p}}S_*^{i^*\nu}(U; R)$ correspond to elements of $I^{\overline{p}}S_*^{\nu}(X; R)$ which have base support in U. Hence, we have an injection

$$I^{\overline{p}}S_*^{i^*\nu}(U; R) \hookrightarrow I^{\overline{p}}S_*^{\nu}(X; R).$$

When the context is clear we will usually just write $I^{\overline{p}}S_*^{\nu}(U; R)$ for $I^{\overline{p}}S_*^{i^*\nu}(U; R)$. We also define the ν-extended $\overline{p}$-intersection chain complex of the pair (X, U) to be

$$I^{\overline{p}}S_*^{\nu}(X, U; R) := I^{\overline{p}}S_*^{\nu}(X; R)/I^{\overline{p}}S_*^{\nu}(U; R).$$

We denote the homology of this chain complex by $I^{\overline{p}}H_*^{\nu}(X, U; R)$. We also have the short exact sequence

$$0 \longrightarrow I^{\overline{p}}S_*^{\nu}(U; R) \to I^{\overline{p}}S_*^{\nu}(X; R) \to I^{\overline{p}}S_*^{\nu}(X, U; R) \to 0$$

which by standard commutative algebra induces a long exact sequence on homology. We state this as a proposition below.

Proposition 2.1.5. *Let X be a stratified pseudomanifold with $U \subset X$ open and let ν denote the data associated to a covering of X_{reg}. There is a long exact sequence*

$$\longrightarrow I^{\overline{p}}H_i^{\nu}(U;R) \longrightarrow I^{\overline{p}}H_i^{\nu}(X;R) \longrightarrow I^{\overline{p}}H_i^{\nu}(X,U;R) \longrightarrow I^{\overline{p}}H_{i-1}^{\nu}(U;R) \longrightarrow$$

Connected Components

Many arguments we make in proofs will involve breaking up spaces and coverings into their connected components. We first relate the ν-extended intersection homology in terms of the connected components of ν (assuming ν covers a connected space). Afterwards, we relate the ν-extended intersection homology of a space in terms of the connected components of the space. The proofs of these results are elementary and follow exactly as in ordinary homology.

Proposition 2.1.6. *Let R be a commutative ring with unity and let X be a stratified pseudomanifold with perversity $\overline{p} \leq \overline{t}$. Suppose X_{reg} is path connected. Let ν denote the data associated to a cover of X_{reg} and let ν_i denote the connected components of ν. That is, $E(\nu) = \coprod_i E(\nu_i)$ with each $E(\nu_i)$ connected. Then we have an isomorphism of chain complexes $I^{\overline{p}}S_*^{\nu}(X;R) \cong \bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X;R)$.*

Proof. Let $\iota : \bigoplus_i \widehat{S}_*^{\nu_i}(X;R) \to \widehat{S}_*^{\nu}(X;R)$ be the inclusion map on summands of $\bigoplus_i \widetilde{S}_*^{\nu_i}(X;R)$ and extended linearly. That is, $\iota = \oplus \iota_i$ where $\iota_i : \widehat{S}_*^{\nu_i}(X;R) \hookrightarrow \widehat{S}_*^{\nu}(X;R)$ is the inclusion map. Clearly ι maps extended singular simplices to extended singular simplices uniquely. Moreover, if $(\widetilde{\sigma}, \sigma)$ is a generator of $\widehat{S}_k^{\nu}(X;R)$ then $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu)$.

However, $\mathrm{int}(\Delta^k) \subset \sigma^{-1}(X_{reg})$ by definition of $\widehat{S}_*^{\nu}(X;R)$ so that $\sigma^{-1}(X_{reg})$ is simply-connected, thus, path-connected which means $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu_i)$ for some i. Thus, ι is bijective on extended singular simplices which means ι is an isomorphism.

Now notice $\iota(I^{\overline{p}}S_*^{\nu_i}(X;R)) \subset I^{\overline{p}}S_*^{\nu}(X;R)$ from definitions. Also recall from our previous work that $\bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X;R) \subset \bigoplus_i \widehat{S}_*^{\nu_i}(X;R)$. So we have an injective map $\iota' : \bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X;R) \to I^{\overline{p}}S_*^{\nu}(X;R)$. We need only show it is also surjective. Suppose $\xi \in I^{\overline{p}}S_*^{\nu}(X;R)$. Write $\xi = \sum_k r_k(\widetilde{\sigma_k}, \sigma_k)$ where $r_k \in R$. Then each σ_k is $\overline{p}$-allowable. Now for each k we have that the lifted image of $(\widetilde{\sigma_k}, \sigma_k)$ is in $E(\nu_{i(k)})$ for some $i(k)$. Let

$$\xi_j = \sum_{\substack{k: \\ i(k)=j}} r_k(\widetilde{\sigma_k}, \sigma_k)$$

Then we have that $\xi = \sum_j \xi_j$, each ξ_j is $\overline{p}$-allowable. Notice that only finitely many of the ξ_j will be non-zero since $|\xi|$ is compact. Thus, if we set $\xi' = \oplus_j \xi_j$, then $\iota(\xi') = \xi$.

Similarly, because $\partial\xi$ is $\overline{p}$-allowable we can find η with each extended simplex with non-zero coefficient in the sum $\overline{p}$-allowable and such that $\iota(\eta) = \partial\xi$. But we have $\iota(\eta) = \partial\xi = \partial\iota(\xi') = \iota(\partial\xi')$. As ι is injective this means $\eta = \partial\xi'$. Hence, $\partial\xi'$ is $\overline{p}$-allowable. Thus, $\xi' \in \bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X;R)$ so that ι' is surjective and therefore an isomorphism.

The last statement of the theorem follows because if we set $\overline{\partial} = \oplus_i \partial_{\nu_i}$, where ∂_{ν_i} is the restriction of ∂ to the sub-chain complex $I^{\overline{p}}S_*^{\nu_i}(X;R)$, then we have $\overline{\partial}\iota = \iota\partial$. This is because if $(\widetilde{\sigma}, \sigma)$ is an extended simplex and has lifted image in ν_i then the boundary must also have lifted support in ν_i. Thus, $\partial(\widetilde{\sigma}, \sigma) = \partial_{\nu_i}(\widetilde{\sigma}_i, \sigma)$. Thus, ι is a chain map and therefore is an isomorphism of chain complexes. The proposition now follows. $\square$

Proposition 2.1.7. *Let R be a commutative ring with unity. Let X be a stratified pseudomanifold with perversity $\overline{p} \leq \overline{t}$. Let $X = \coprod X_i$ be the path components of X and let ν be the data associated to a covering of X_{reg}. Notice that $X_{reg} = \coprod_i (X_i)_{reg}$. Define ν_i to be the restriction of ν to $(X_i)_{reg}$ so that $\nu = (\coprod_i E(\nu_i), X_{reg}, p_\nu)$. We have an isomorphism of chain complexes*

$$I^{\overline{p}}S_*^{\nu}(X;R) \cong \bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X_i;R).$$

Proof. Let ι_k be the inclusion map $I^{\overline{p}}S_*^{\nu_k}(X_k;R) \hookrightarrow I^{\overline{p}}S_*^{\nu}(X;R)$ and let ι be ι_k on the k-th summand and extended linearly to all of $\bigoplus_i I^{\overline{p}}S_*^{\nu_i}(X_i;G)$. The argument now proceeds in a analogous fashion as the previous proposition. $\square$

Invariance Under Normalization

Recall that the normalization ((16)) of a pseudomanifold X is a finite-to-one proper surjection $\mathbf{n} : X^N \to X$ such that X^N is a normal pseudomanifold, $\mathbf{n}|_{\mathbf{n}^{-1}(X_{reg})} : \mathbf{n}^{-1}(X_{reg}) \to X_{reg}$ is a homeomorphism, $\mathbf{n}$ sends i dimensional strata of X^N to i-dimensional strata of X, and for any point $x \in \Sigma$, $\mathbf{n}^{-1}(x)$ is a disjoint union of points and the number of such points equals the number of regular components of any link of x. An important property of normalization is that intersection homology is preserved. We prove this also holds for ν-extended intersection homology. Given a perversity $\overline{p}$ on X we define $\overline{p}^N$ by $\overline{p}^N(S) = \overline{p}(S')$ where S' is the unique stratum such that $\mathbf{n}(S) \subset S'$.

Because $\mathbf{n}$ restricts to a homeomorphism $(X^N)_{reg} \to X_{reg}$ (so ν is a cover for both $(X^N)_{reg}$ and X_{reg}) and sends i dimensional strata to i dimensional strata we have a well defined map $\mathbf{n} : I^{\overline{p}^N} S^{\nu}_{*}(X^N; R) \to I^{\overline{p}} S^{\nu}_{*}(X; R)$. The next proposition says this map is actually an isomorphism.

Proposition 2.1.8. *Let X be a stratified pseudomanifold with ν the associated data for a cover of X_{reg}. Let $\mathbf{n} : X^N \to X$ be the normalization of X. Then $\mathbf{n} : I^{\overline{p}^N} S^{\nu}_{*}(X^N; R) \to I^{\overline{p}} S^{\nu}_{*}(X; R)$ is an isomorphism.*

Proof. We first show the map is injective. Suppose $(\widetilde{\sigma}, \sigma)$ and $(\widetilde{\tau}, \tau)$ are extended simplices in $\widetilde{S}^{\nu}_{k}(X^N; R)$ with $\mathbf{n}((\widetilde{\sigma}, \sigma)) = \mathbf{n}((\widetilde{\tau}, \tau))$. Then $\widetilde{\sigma} = \widetilde{\tau}$ and $\mathbf{n}\sigma = \mathbf{n}\tau$. Hence, $(\mathbf{n}\sigma)|_{\sigma^{-1}((X^N)_{reg})} = (\mathbf{n}\tau)|_{\tau^{-1}((X^N)_{reg})}$. However, $\mathbf{n}$ restricts to a homeomorphism on $(X^N)_{reg}$ which means $\sigma|_{\sigma^{-1}((X^N)_{reg})} = \tau|_{\tau^{-1}((X^N)_{reg})}$ so by density we have $\sigma = \tau$. Thus, because $\mathbf{n}$ is injective on extended simplices it must be injective on $\overline{p}$-intersection chains.

Next, we show the map is surjective. Let $\xi \in I^{\overline{p}} S^{\nu}_{*}(X; R)$ and write $\xi = \sum_i r_i(\widetilde{\sigma}_i, \sigma_i)$ where $r_i \in R$. By (16, Proposition 2.6) each σ_i has a lift σ_i^N such that $\mathbf{n}\sigma_i^N = \sigma_i$. By definition of $\overline{p}^N$ each σ_i^N is $\overline{p}^N$-allowable because each σ_i is $\overline{p}$-allowable. Let $\xi^N = \sum_i r_i(\widetilde{\sigma}_i, \sigma_i)$. If we can show $\partial\xi^N$ is $\overline{p}^N$-allowable we will be done. However, by the same argument we can find η^N mapping to $\partial\xi$ with η^N $\overline{p}^N$-allowable. But as we saw above $\mathbf{n}$ is injective on extended simplices which means we must have $\eta^N = \partial\xi^N$ so that ξ^N is a $\overline{p}^N$-intersection chain as desired.

$\square$

2.2 Intersection homology with twisted coefficients

In this subsection we show how we can use coverings of the regular stratum to define an intersection homology with twisted coefficients. We prove this approach is equivalent to the more frequently used approach of intersection homology with local coefficient systems. Before turning to intersection homology, we recall these concepts for ordinary homology.

Ordinary Homology with Twisted Coefficients and Local Systems of Coefficients

Recall for a path connected space X with universal cover $\widetilde{X}$ and fundamental group $\pi = \pi_1(X)$, the homology with twisted coefficients in a right $R[\pi]$-module A (R a commutative ring with unity) is defined to be the homology of the chain complex

$$S_*(X; A) = A \otimes_{R[\pi]} S_*(\widetilde{X}; R).$$

where π acts on the left of $S_*(\widetilde{X}; R)$ via the identification of π with the group of deck transformations of $\widetilde{X}$ and where the boundary map of this chain complex is $\partial \otimes 1$. If the $R[\pi]$-module is given by a representation $\rho : \pi \to \mathrm{Aut}_R(A)$, then some authors also use the notation $S_*(X; A_\rho)$ (2, Definition 5.3).

Remark 2.2.1. There may be cause for confusion in notation as $S_*(X; A)$ may now have two different meanings. On the one hand, this may mean the chain complex $A \otimes_R S_*(X; R)$, while on the other hand, as defined above it may mean $A \otimes_{R[\pi]} S_*(\widetilde{X}; R)$. There does not seem to be an accepted way around this in the literature, except perhaps using the representation ρ in the notation as we noted above. However, this is often cumbersome and usually only clutters notation. We must therefore take care to make the context clear of which definition we mean.

Notice, though, if A is given the trivial right $R[\pi]$-module structure, then $A \otimes_R S_*(X;R)$ and $A \otimes_{R[\pi]} S_*(\widetilde{X};R)$ are isomorphic chain complexes.

Remark 2.2.2. There is a more efficient description of homology with twisted coefficients using regular covers. To see this, let $\widetilde{X} \to X$ be the universal cover of X, let $\pi = \pi_1(X)$, and let A be a right $R[\pi]$-module with representation $\rho : \pi \to \mathrm{Aut}_R(A)$. Let $H = \ker(\rho)$ and let X' be the regular cover associated to H. Let $\pi' = \pi/H$ which is isomorphic to the deck transformation group of X' from standard covering space theory and let $\rho' : \pi' \to \mathrm{Aut}_R(A)$ be the map induced by ρ. Then we have the isomorphism of chain complexes

$$A_\rho \otimes_{R[\pi]} S_*(\widetilde{X};R) \cong A_{\rho'} \otimes_{R[\pi']} S_*(X';R).$$

For more, see comments preceding (14, Example 3.H3).

Remark 2.2.3. There is a pullback construction for twisted coefficients defined in the following way. Let $f : (X, x_0) \to (Y, y_0)$ be a map of path connected spaces and suppose A_ρ is a right $R[\pi_1(Y)]$-module given by a representation $\rho : \pi_1(Y) \to \mathrm{Aut}_R(A)$. Then f^*A_ρ is a right $R[\pi_1(X)]$-module given by the representation that is the composition $\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \xrightarrow{\rho} \mathrm{Aut}_R(A)$. Moreover, if $p_X : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and $p_Y : (\widetilde{Y}, \widetilde{y_0}) \to (Y, y_0)$ are the respective universal covers of X and Y, there is a map $\widetilde{f} : (\widetilde{X}, \widetilde{x_0}) \to (\widetilde{Y}, \widetilde{y_0})$

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{x_0}) \arrow[r, "\widetilde{f}"] \arrow[d, "p_X"'] & (\widetilde{Y}, \widetilde{y_0}) \arrow[d, "p_Y"'] \\
(X, x_0) \arrow[r, "f"] & (Y, y_0)
\end{tikzcd}$$

commutes and we have an induced chain map $f_\# : S_*(X; f^*A_\rho) \to S_*(Y; A_\rho)$ defined by

$f_\#(a \otimes \sigma) = a \otimes \widetilde{f} \circ \sigma$. Let us verify this is well-defined. Consider $a\alpha^{-1} \otimes \alpha \cdot \sigma$ where $\sigma : \Delta^k \to \widetilde{X}$, $a \in A$, and $\alpha \in \pi_1(Y)$. Let τ_α denote the deck transformation associated to α from the identification of the fundamental group with the group of deck transformations. In order to show this construction is well-defined we first show that $\widetilde{f}\tau_\alpha = \tau_{f_*\alpha}\widetilde{f}$. First note that both $\widetilde{f}\tau_\alpha$ and $\tau_{f_*\alpha}\widetilde{f}$ are lifts of f. Therefore, in order to show equality it suffices to show they agree on a single point. On the one hand, $\widetilde{f}\tau_\alpha(\widetilde{x_0}) = \widetilde{f}(\widetilde{\alpha}(1))$ by definition of τ_α. On the other hand, $\tau_{f_*\alpha}\widetilde{f}(\widetilde{x_0}) = \tau_{f_*\alpha}(\widetilde{y_0}) = \widetilde{f\alpha}(1) = \widetilde{f}\widetilde{\alpha}(1)$. The last equality follows because $\widetilde{f}\widetilde{\alpha}$ is a lift of $f\alpha$ and $\widetilde{f}\widetilde{\alpha}(0) = \widetilde{f}(\widetilde{x_0}) = \widetilde{y_0}$ so by uniqueness of lifts $\widetilde{f\alpha} = \widetilde{f}\widetilde{\alpha}$. Hence, we see that $\widetilde{f}\tau_\alpha = \tau_{f_*\alpha}\widetilde{f}$ since they both agree at $\widetilde{x_0}$. In particular, this shows that $f(\alpha \cdot \sigma) = f_*(\alpha) \cdot \sigma$.

We also have that by definition of f^*A_ρ that for $a \in f^*A_\rho$, $a \cdot \alpha^{-1} = a \cdot f_*(\alpha^{-1})$, which of course is $a \cdot f_*(\alpha)^{-1}$. Altogether then, we have shown that $f_\#(a \otimes \sigma) = a \otimes \widetilde{f}\sigma$ and $f_\#(a \cdot \alpha^{-1} \otimes \alpha \cdot \sigma) = a \cdot \alpha^{-1} \otimes f(\alpha \cdot \sigma)$ which is $a \cdot f_*(\alpha)^{-1} \otimes f_*(\alpha) \cdot \widetilde{f}\sigma$ by our above work and this is the same as $a \otimes \widetilde{f}\sigma$. Hence, our map is in fact well-defined.

An equivalent approach to homology with twisted coefficients is homology on a system of local coefficients which is defined as follows. A local coefficient system of R modules over X is a covering space $p : \mathcal{E} \to X$ whose fibers are R-modules and is such that the operations of multiplication by elements of R and addition fiberwise are continuous. One then considers the chain complex $S_k(X; \mathcal{E})$ of formal sums

$$\sum_j a_j \sigma_j$$

where $\sigma_j : \Delta^k \to X$ and $a_j : \Delta^k \to \mathcal{E}$ is a lift of σ_j. Addition is given by $a\sigma + b\sigma = (a + b)\sigma$ where $a + b$ is defined by $(a + b)(x) = a(x) + b(x)$ which is continuous. Similarly R acts by

$r(a\sigma) = ra\sigma$ where $(ra)(x) = r \cdot a(x)$ which is also continuous. The boundary map of this chain complex is given by

$$\partial \left(\sum_i a_i \sigma_i \right) := \sum_{i,j} (-1)^j (a_i \circ \partial_j) \sigma_i \circ \partial_j$$

and the homology is denoted $H_*(X; \mathcal{E})$.

Remark 2.2.4. There is also a pullback construction for local coefficient systems. Let $f : X \to Y$ be a map and let $p : \mathcal{E} \to Y$ be a system of local coefficients over Y. Then f pulls back $\mathcal{E}$ to a system of local coefficients over X:

$$f^* \mathcal{E} \to X$$

where $f^* \mathcal{E} \subset X \times \mathcal{E}$ consists of all ordered pairs $(x, e) \in X \times \mathcal{E}$ such that $f(x) = p(e)$.

For more on the constructions of homology with twisted coefficients and homology with local coefficients we refer the reader to (2, Chapter 5) or (14, Chapter 3.H). As mentioned above, these two approaches are actually equivalent. The equivalence goes as follows. For every right $R[\pi]$-module A, there is an associated system of local coefficients $p : \mathcal{E}(X, \pi, A) \to X$ where the total space $\mathcal{E}(X, \pi, A)$ is defined to be the quotient of $A \times \widetilde{X}$ by the relations $(a, \widetilde{x}) \sim (a \cdot \alpha, \alpha^{-1} \widetilde{x})$ for all $\widetilde{x} \in \widetilde{X}$, $a \in A$, and $\alpha \in \pi$. More on this construction may be found in the comments preceding (14, Theorem 1.38). Writing $\mathcal{E}$ for $\mathcal{E}(X, \pi, A)$, one may then prove there is an isomorphism of chain complexes (14, Proposition 3H.4) (and therefore an induced isomorphism on homology)

$$S_*(X;A) = S_*(\widetilde{X};R) \otimes_{R[\pi]} A \cong S_*(X;\mathcal{E}).$$

Moreover, the construction is functorial in the sense that if $f : Y \to X$, then $S_*(Y; f^*A) \cong S_*(Y; f^*\mathcal{E})$ and the diagram below commutes.

$$\begin{tikzcd}
S_*(Y;f^*A) \arrow[d] \arrow[r, phantom, "\cong"] & S_*(Y;f^*\mathcal{E}) \arrow[d] \\
S_*(X;A) \arrow[r, phantom, "\cong"] & S_*(X;\mathcal{E})
\end{tikzcd}$$

Remark 2.2.5. Many authors use the terms twisted coefficients and local coefficients interchangeably, and for good reason since from the comments above they form isomorphic categories. The aim of the next section is to prove the same holds for intersection homology (in fact, for coefficients defined only on the regular stratum). Therefore, we must use care with our terminology. We will always use *twisted coefficients* to mean the algebraic approach which uses the universal cover and tensor products over the group ring of deck transformations (or equivalently a regular cover and its deck transformations as we saw above). On the other hand, *local coefficients* will always mean the more geometric approach which uses covering spaces whose fibers are R-modules, where R is a commutative ring with unity.

Intersection Homology with Twisted Coefficients and Local Systems of Coefficients

We first recall for the reader intersection homology with a system of local coefficients. Let X be a stratified pseudomanifold and let $p : \mathcal{E} \to X_{reg}$ be a system of local coefficients of

R-modules (R a commutative ring with unity) over X_{reg}. Then $S_k(X;\mathcal{E})$ is defined to be formal sums of the form

$$\sum_j a_j \sigma_j$$

where $\sigma_j : \Delta^k \to X$ and $a_j : \sigma_j^{-1}(X_{reg}) \to \mathcal{E}$. If $\sigma_j^{-1}(X_{reg}) = \emptyset$, then a_j is taken to be 0. As in ordinary homology with local coefficients, addition is taken point-wise in $\mathcal{E}$ as is multiplication by elements of R. We may then define the boundary map by

$$\partial \left(\sum_j a_j \sigma_j \right) = \sum_{i,j} (-1)^i (a_j \circ \partial_i) \sigma_j \circ \partial_i$$

where $a_j \circ \partial_i$ is really $(a_j \circ \partial_i)|_{(a_j \circ \partial_i)^{-1}(X_{reg})}$. The same computation we made in Section 2.1 shows that $\partial\partial = 0$ and we may consider $\bar{p}$-intersection chains defined in the usual way. With the above laid out, we make the following definition.

Definition 2.2.6 (Intersection homology with local coefficients)**.** The *$\bar{p}$-intersection chain complex with local coefficients* is defined to be the $\bar{p}$-intersection chains of $S_*(X;\mathcal{E})$ and is denoted $I^{\bar{p}}S_*(X;\mathcal{E})$. The *$\bar{p}$-intersection homology with local coefficients* is denoted $I^{\bar{p}}H_*(X;\mathcal{E})$.

So we see that intersection homology may be defined on a coefficient system living over only X_{reg}. Of course this means these will not be an invariants of X; rather, they will depend heavily on X_{reg} and the stratification chosen for X.

Next, we define intersection homology with twisted coefficients. Recall from Remark 2.2.2 that we may use regular covers to define twisted coefficients. Although incorporating this into notation is not standard, this will be convenient for our purposes since our main results will rely on assumptions on the regular cover used to define twisted coefficients. Therefore, instead of making statements of our theorems awkward, we opt to incorporate the regular cover into our notation for intersection homology with twisted coefficients.

Let X be a stratified pseudomanifold with X_{reg} connected and let ν denote the data of a regular cover of X_{reg} with deck transformation group π. Notice that π acts on the left of $S_*^{\nu}(X;R)$ (R a commutative ring with unity). Explicitly, for $\xi \in S_*^{\nu}(X;R)$ and $\alpha \in \pi$, if we write $\xi = \sum_j r_j(\widetilde{\sigma}_j, \sigma_j)$ with $r_j \in R$, we define

$$\alpha \cdot \xi := \sum_i r_i(\alpha \cdot \widetilde{\sigma}_i, \sigma_i)$$

where $\alpha \cdot \widetilde{\sigma}$ is the result of the left deck transformation action of α. From this equation we see that the action of π preserves the base image of extended chains. In particular, π also acts on $\widehat{S}_*^{\nu}(X;R)$ and ${}^{\overline{p}}S_*^{\nu}(X;R)$.

Next, let A be any right $R[\pi]$-module. Consider the tensor product $A \otimes_{R[\pi]} S_*^{\nu}(X;R)$. We may define a boundary map by $1 \otimes \partial$ which is well defined upon verifying the equality $\partial(\alpha \cdot \xi) = \alpha \cdot (\partial \xi)$. We say a chain $\xi \in A \otimes_{R[\pi]} S_*^{\nu}(X;R)$ is $\overline{p}$-allowable if ξ may be expressed as a sum $\xi = \sum_j a_j \otimes (\widetilde{\sigma}_j, \sigma_j)$, where $a_j \in A$, with each σ_j $\overline{p}$-allowable. As usual, we say ξ is a $\overline{p}$-intersection chain if both ξ and $\partial\xi$ are $\overline{p}$-allowable. With the above notation, we make the following definition.

Definition 2.2.7 (Intersection homology with twisted coefficients)**.** Let ν be a connected regular cover of X_{reg} with deck transformation group π and let A be a right $R[\pi]$-module. The $\overline{p}$-*intersection chain complex with twisted coefficients* is defined to be the $\overline{p}$-intersection chains of $A \otimes_{R[\pi]} S_*^{\nu}(X;R)$ and is denoted by $I^{\overline{p}}\widetilde{S}_*^{\nu}(X;A)$. The homology of this chain complex is denoted $I^{\overline{p}}\widetilde{H}_*^{\nu}(X;A)$.

We also let $I^{\overline{p}}\widetilde{S}_*^{\nu|_U}(U;A) \subset I^{\overline{p}}\widetilde{S}_*^{\nu}(X;A)$ denote the sub complex of $\overline{p}$-intersection chains that have base support in U and $I^{\overline{p}}\widetilde{S}_*^{\nu}(X,U;A)$ to denote the relative chain complex.

As in ordinary homology with twisted coefficients, if the right $R[\pi]$-module A is given by a representation $\rho : \pi \to \mathrm{Aut}_R(A)$, then the intersection homology with twisted coefficients may also be denoted by $I^{\overline{p}}\widetilde{H}_*^{\nu}(X;A_\rho)$.

Remark 2.2.8. If $i : X' \hookrightarrow X$ is an inclusion of a stratified pseudomanifold X', then we may also consider $I^{\overline{p}}\widetilde{S}_*^{\nu|_{X'}}(X';A)$ to be intersection chains in $I^{\overline{p}}\widetilde{S}_*^{\nu}(X;A)$ which have base support in X'.

Suppose further that $X' \hookrightarrow X$ is a stratified inclusion and $\nu|_{X'}$ is a connected cover so that $\nu|_{X'}$ is also regular and also has deck transformation group π. Then there is an alternative description of $I^{\overline{p}}\widetilde{S}_*^{\nu|_{X'}}(X';A)$. Define a perversity $\overline{p}_{X'}$ on X' by $\overline{p}_{X'}(S) = \overline{p}(T)$ if the singular stratum S of X' is contained in a singular stratum T of X. We may consider $I^{\overline{p}_{X'}}\widetilde{S}_*^{\nu|_{X'}}(X';A)$, that is, $\overline{p}_{X'}$-intersection chains of $A \otimes_{F[\pi]} S_*^{\nu|_{X'}}(X';A)$. However, the completely analogous argument as in (5, Lemma 4.16) shows that $I^{\overline{p}}\widetilde{S}_*^{\nu|_{X'}}(X';A) = I^{\overline{p}_{X'}}\widetilde{S}_*^{\nu|_{X'}}(X';A)$.

Remark 2.2.9. In general, we will not have equalities of $I^{\bar{p}}\widetilde{S}^{\nu}_{*}(X;A)$ and $A\otimes_{R[\pi]} I^{\bar{p}}S^{\nu}_{*}(X;R)$. The former may allow for more $\bar{p}$-intersection chains because cancellations may occur after applying the boundary map due to twisted coefficients and the result may be allowable; whereas, in the untwisted case, non-allowable simplices may not cancel after applying the boundary map.

The next lemma may be thought of as a "meta" lemma. In the following sections, we will show various intersection chain complexes are isomorphic. Instead of making numerous identical arguments, the lemma summarizes our technique.

Lemma 2.2.10. *Let (C_*, ∂) and (C'_*, ∂') be chain complexes of R-modules (R a commutative ring with unity) and assume we have submodules $A_* \subset B_* \subset C_*$ and $A'_* \subset B'_* \subset C'_*$ such that the boundary maps induce maps $\partial : A_* \to B_*$ and $\partial' : A'_* \to B'_*$. Let $IA_* = \{a \in A : \partial a \in A\}$ and similarly define IA'_*. Then if there is an isomorphism $B' \cong B$ which restricts to an isomorphism $A' \cong A$ such that the boundary maps respect these isomorphisms, then $IA_* \cong IA'_*$ as chain complexes.*

Proof. Let $\Phi : B_* \to B'_*$ be an isomorphism such that the diagram below commutes.

$$\begin{tikzcd} A_* \arrow[r, "\Phi|_{A_*}"] \arrow[d, "\partial"'] & A'_* \arrow[d, "\partial'"] \\ B_* \arrow[r, "\Phi"] & B'_* \end{tikzcd}$$

Suppose $a \in IA_*$. Then $a \in A_*$ and $\partial a \in A_*$. Now $\Phi|_{A_*}(a) \in A'_*$ and also $\partial'\Phi|_{A_*}(a) = \Phi(\partial a)$. However, $\partial a \in A_*$ which means $\Phi(\partial a) = \Phi|_{A_*}(\partial a) \in A'_*$. Thus, $\partial'\Phi|_{A_*}(a) \in A'_*$

which means $\Phi|_{A_*}(a) \in IA'_*$. Thus, Φ restricts to a map $\widehat{\Phi} : IA_* \to IA'_*$ and by our above computation we see the boundary map also commutes with this map. The map is injective because Φ is injective. We need only show the map is also surjective. Let $a' \in IA'_*$. Then there exists $a \in A_*$ such that $\Phi|_{A_*}(a) = a'$ because $\Phi|_{A_*}$ is surjective. We want to show that $a \in IA_*$. Now $\partial' a' \in A'_*$ so there also exists $x \in A_*$ such that $\Phi|_{A_*}(x) = \partial a'$. However, $\Phi(\partial a) = \partial' \Phi|_{A_*}(a) = \partial' a' = \Phi|_{A_*}(x) = \Phi(x)$ so that $\Phi(\partial a) = \Phi(x)$ and by injectivity of Φ this means $\partial a = x \in A_*$. Thus, $a \in A_*$ and $\partial a \in A_*$ so that $a \in IA_*$. Hence, $IA_* \cong IA'_*$ as chain complexes. $\square$

We now move to the main theorem of this subsection, the equivalence of intersection homology with twisted coefficients and intersection homology with local coefficient systems. The outline of our proof follows that of (14, Theorem 3 H.4) modified to our situation.

Theorem 2.2.11. *Let X be a stratified pseudomanifold with X_{reg} connected and with perversity $\bar{p} \leq \bar{t}$. Let ν be a regular cover of X_{reg} with deck transformation group π and let A be a right $R[\pi]$-module (R a commutative ring with unity). Let $p : \mathcal{E} \to X_{reg}$ denote the system of local coefficient R-modules over X_{reg} associated to the $R[\pi]$-module A and the cover ν. Then there is an isomorphism of chain complexes between the intersection chain complex with twisted coefficients $I^{\bar{p}}\widetilde{S}^{\nu}_*(X; A)$ and the intersection chain complex of local coefficient system R-modules $I^{\bar{p}}S_*(X; \mathcal{E})$.*

Proof. Recall $\mathcal{E}$ is the quotient of $A \times E(\nu)$ by the relations $(a, \widetilde{x}) \sim (a \cdot \alpha, \alpha^{-1} \cdot \widetilde{x})$ for all $\widetilde{x} \in E(\nu)$, $a \in A$, and $\alpha \in \pi$.

To prove the theorem we will consider two sub R-modules of $S_*(X; \mathcal{E})$. Let $\widehat{S}_k(X; \mathcal{E}) \subset S_k(X; \mathcal{E})$ be the sub R-module of sums $\sum_i e_i \sigma_i$ where $\sigma_i : \Delta^k \to X$, $e_i : \sigma_i^{-1}(X_{reg}) \to \mathcal{E}$ is a lift of σ_i, and $\mathrm{int}(\Delta^k) \subset \sigma_i^{-1}(X_{reg})$. Similarly, let $S_k^{\overline{p}}(X;\mathcal{E})$ be sums of $\overline{p}$-allowable simplices. Notice that because $\overline{p} \leq \bar{t}$ we have that $S_*^{\overline{p}}(X;\mathcal{E}) \subset \widehat{S}_*(X;\mathcal{E})$ and that $\partial : S_*^{\overline{p}}(X;\mathcal{E}) \to \widehat{S}_*(X;\mathcal{E})$.

Next, take $\xi \in R[A] \otimes_R \widehat{S}_*^{\nu}(X;R)$. Here $R[A]$ is the free R-module on elements of A (in other words $R[A] \cong \bigoplus_{a\in A} R$). Then we can write $\xi = \sum_k \Big((\sum_j s_{jk}a_{jk}) \otimes (\sum_i r_{ik}(\widetilde{\sigma_{ik}}, \sigma_{ik})) \Big) = \sum_k \Big(\sum_{i,j} (r_{ik}s_{jk}a_{jk} \otimes (\widetilde{\sigma_{ik}}, \sigma_{ik})) \Big) = \sum_{i,j,k} r_{ik}s_{jk}a_{jk} \otimes (\widetilde{\sigma_{ik}}, \sigma_{ik})$ where $r_{ik}, s_{jk} \in R$ and $a_{jk} \in A$. The second equality follows from bilinearity of tensor product and that the tensor product is over R and the third equality is elementary. Thus, every element in $R[A] \otimes_R \widehat{S}_*^{\nu}(X;R)$ can be written in the form $\sum r_i a_i \otimes (\widetilde{\sigma}_i, \sigma_i)$

Let $\phi : R[A] \to A$ be the quotient map which takes a formal sum in $R[A]$ to a sum in A. Define $\Phi : R[A] \otimes_R \widehat{S}_*^{\nu}(X;R) \to \widehat{S}_*(X;\mathcal{E})$ by $\Phi(\sum r_i a_i \otimes (\widetilde{\sigma}_i, \sigma_i) = \sum r_i [\phi(a_i), \widetilde{\sigma}_i]\sigma_i$, where $[\phi(a_i), \widetilde{\sigma}_i] : \sigma_i^{-1}(X_{reg}) \to \mathcal{E}$ is the map $[\phi(a_i), \widetilde{\sigma}_i](x) = [\phi(a_i), \widetilde{\sigma}_i(x)]$. By $[\widetilde{\sigma}_i(x), \phi(a_i)]$ we mean the equivalence class of the ordered pair $(\phi(a_i), \widetilde{\sigma}_i(x))$ in $\mathcal{E} = (A \times E(\nu))/\sim$.

We will identify $\ker(\Phi)$ to later show that
$\Big(R[A] \otimes_R \widehat{S}_*^{\nu}(X;R)\Big)/(\ker(\Phi)) \cong A \otimes_{R[\pi]} \widehat{S}_*^{\nu}(X;R)$. Now we have the commutative diagram

$$\begin{tikzcd}
R[A] \otimes_R \widehat{S}_*^{\nu}(X;R) \arrow[r, "\Psi"] \arrow[d, "\phi\otimes \mathrm{id}"'] & A \otimes_{R[\pi]} \widehat{S}_*^{\nu}(X;R) \\
A \otimes_R \widehat{S}_*^{\nu}(X;R) \arrow[ur, "q"] &
\end{tikzcd}$$

where the diagonal map q is the quotient by the module generated by all elements of the form $\sum_i a_i \otimes \alpha_i \cdot (\widetilde{\sigma}_i, \sigma_i) - \alpha_i \cdot a_i \otimes (\widetilde{\sigma}_i, \sigma_i)$ where $\alpha_i \in \pi$ and $a_i \in A$. The horizontal map Ψ is the composition of the other two maps. Notice that $\phi \otimes \mathrm{id}$ and q are clearly surjective because tensor products are right exact. Thus, $A \otimes_{R[\pi]} \widehat{S}^{\nu}_*(X;R) \cong \left(R[A] \otimes_R \widehat{S}^{\nu}_*(X;R)\right) / \ker(\Psi)$. We will prove that $\ker(\Phi) = \ker(\Psi)$.

To see $\ker(\Phi) \subset \ker(\Psi)$ let $\xi \in R[A] \otimes_R \widehat{S}^{\nu}_*(X;R)$ and write $\xi = \sum_{i=1}^{k} r_i a_i \otimes (\widetilde{\sigma}_i, \sigma_i)$ where $a_i \in A, r_i \in R$, and suppose $\Phi(\xi) = 0$. We will assume k is minimal such that ξ can be written in the form $\sum_{i=1}^{k} r_i a_i \otimes (\widetilde{\sigma}_i, \sigma_i)$. We need to show that $\xi \in \ker(\Psi)$. To this end we will induct on k. The case the sum is empty is trivial because then this just says $\Phi(0) = 0$. So assume $\Phi(\xi) = 0$ and $\xi \neq 0$. If $k = 1$, then we have $\Phi(ra \otimes (\widetilde{\sigma}, \sigma)) = 0$ which means $r[\phi(a), \widetilde{\sigma}] = [r\phi(a), \widetilde{\sigma}] = 0$ which means $r\phi(a) = 0$. Thus, $ra \otimes (\widetilde{\sigma}, \sigma) \in \ker(\phi \otimes \mathrm{id})$ which means that $\xi = ra \otimes (\widetilde{\sigma}, \sigma) \in \ker(\Psi)$.

Next, we consider the case $k = 2$. Suppose $\Phi(ra \otimes (\widetilde{\sigma}, \sigma) - sa' \otimes (\widetilde{\beta}, \beta)) = 0$ where $a, a' \in R[A]$ and $r, s \in R$. Then, $r[\phi(a), \widetilde{\sigma}]\sigma = s[\phi(a'), \widetilde{\beta}]\beta$. This means that $\sigma = \beta$ and that $r[\phi(a), \widetilde{\sigma}] = s[\phi(a'), \widetilde{\beta}]$. Thus, $[r\phi(a), \widetilde{\sigma}(x)] = [s\phi(a'), \widetilde{\beta}(x)]$ for each $x \in \sigma^{-1}(X_{reg})$. Thus, by definition of the relations of $\mathcal{E}$ we have $\widetilde{\sigma}(x) = \alpha_x \cdot \widetilde{\beta}(x)$ where $\alpha_x \in \pi$ and $r\phi(a) = s\phi(a') \cdot \alpha_x^{-1}$ for each $x \in \sigma^{-1}(X_{reg}) = \beta^{-1}(X_{reg})$. But by definition of $\widehat{S}^{\nu}_*(X;R)$ we have that $\mathrm{int}(\Delta) \subset \sigma^{-1}(X_{reg})$. Hence we have that $\sigma^{-1}(X_{reg})$ is path connected so any lift is determined by where a single point maps. Thus by uniqueness of lifts we must have that $\alpha_x = \alpha_{x'}$ for each x, x'. Hence, we simply have $\widetilde{\sigma} = \alpha \cdot \widetilde{\beta}$ and $r\phi(a) = s\phi(a') \cdot \alpha^{-1}$ for some $\alpha \in \pi$.

Then,

$$
\begin{aligned}
\Psi\left(ra\otimes(\widetilde{\sigma},\sigma)-sa'\otimes(\widetilde{\beta},\beta)\right) &= (q\circ(\phi\otimes\mathrm{id}))\left(ra\otimes(\widetilde{\sigma},\sigma)-sa'\otimes(\widetilde{\beta},\beta)\right)\\
&= q\left(r\phi(a)\otimes(\widetilde{\sigma},\sigma)-s\phi(a')\otimes(\widetilde{\beta},\beta)\otimes\right)\\
&= r\phi(a)\otimes_{R[\pi]}(\widetilde{\sigma},\sigma)-s\phi(a')\otimes_{R[\pi]}(\widetilde{\beta},\beta)\\
&= r\phi(a)\otimes_{R[\pi]}(\alpha\cdot\widetilde{\beta},\beta)-s\phi(a')\otimes_{R[\pi]}(\widetilde{\beta},\beta)\\
&= r\phi(a)\otimes_{R[\pi]}\alpha\cdot(\widetilde{\beta},\beta)-s\phi(a')\otimes_{R[\pi]}(\widetilde{\beta},\beta)\\
&= r\phi(a)\cdot\alpha\otimes_{R[\pi]}(\widetilde{\beta},\beta)\otimes_{R[\pi]}-s\phi(a')\otimes_{R[\pi]}(\widetilde{\beta},\beta)\\
&= s\phi(a')\otimes_{R[\pi]}(\widetilde{\beta},\beta)-(\widetilde{\beta},\beta)\otimes_{R[\pi]}s\phi(a')\\
&= 0.
\end{aligned}
$$

Hence, $\xi=\left(ra\otimes(\widetilde{\sigma},\sigma)-sa'\otimes(\widetilde{\beta},\beta)\right)\in\ker(\Psi)$ as was to be shown.

Now for the inductive step let us write $\Phi(\xi)=\Phi(\sum_{i=1}^{k}r_ia_i\otimes(\widetilde{\sigma}_i,\sigma_i))=\sum_{i=1}^{k}r_i[\phi(a_i),\widetilde{\sigma}_i]\sigma_i$ where $k>2$ and we assume inductively that if $\xi'=\sum_{i=1}^{k'}s_ia_i'\otimes(\widetilde{\tau}_i,\tau_i)$ where $s_i\in R$, $a_i'\in A$, $\Phi(\xi')=0$, and $k'<k$ is the minimal integer such that ξ' may be written in this form, then $\Psi(\xi')=0$. Since $k>2$ and is minimal we have that $\xi\neq 0$ and so there is some $r_\ell a_\ell\otimes(\widetilde{\sigma}_\ell,\sigma_\ell)\neq 0$. Then we have $r_\ell[\phi(a_\ell),\widetilde{\sigma}_\ell]\sigma_\ell=-\sum_{i\neq\ell}r_i[\phi(a_i),\widetilde{\sigma}_i]\sigma_i$. Then this equation says that there exists $i_1,\ldots,i_j$ such that $\sigma_\ell=\sigma_{i_p}$ for $p=1,\ldots,j$ and $r_\ell[\phi(a_\ell),\widetilde{\sigma}_\ell]=-\sum_{p=1}^{j}r_{i_p}[\phi(a_{i_p}),\widetilde{\sigma_{i_p}}]$.

This means that for each p, we can find $\alpha_{i_p} \in \pi$ such that the following equations hold:

$$\widetilde{\sigma}_\ell = \alpha_{i_p} \cdot \widetilde{\sigma}_{i_p} \tag{3}$$

$$r_\ell \phi(a_\ell) = -\sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \cdot \alpha_{i_p}^{-1}. \tag{4}$$

Let $\xi' = (\widetilde{\sigma}_\ell, \sigma_\ell) \otimes a_\ell + \sum_{p=1}^{j} (\widetilde{\sigma_{i_p}}, \sigma_{i_p}) \otimes a_{i_p}$. Then $\Phi(\xi') = 0$ by construction of ξ'. We now show that $\Psi(\xi') = 0$.

$$\begin{aligned}
\Psi(\xi') &= (q \circ (\phi \otimes \mathrm{id})) \left(r_\ell a_\ell \otimes (\widetilde{\sigma_\ell}, \sigma_\ell) + \sum_{p=1}^{j} r_{i_p} a_{i_p} \otimes (\widetilde{\sigma_{i_p}}, \sigma_{i_p}) \right) \\
&= q \left(r_\ell \phi(a_\ell) \otimes (\widetilde{\sigma_\ell}, \sigma_\ell) \otimes) + \sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \otimes (\widetilde{\sigma_{i_p}}, \sigma_{i_p}) \right) \\
&= r_\ell a_\ell \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell)) + \sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \otimes_{R[\pi]} (\widetilde{\sigma_{i_p}}, \sigma_{i_p}) \\
&= r_\ell \phi(a_\ell) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell)) + \sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \otimes_{R[\pi]} (\alpha_{i_p}^{-1} \alpha_{i_p} \cdot \widetilde{\sigma_{i_p}}, \sigma_{i_p}) \\
&= r_\ell \phi(a_\ell) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell)) + \sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \otimes_{R[\pi]} \alpha_{i_p}^{-1} \cdot (\alpha_{i_p} \cdot \widetilde{\sigma_{i_p}}, \sigma_{i_p})) \\
&= r_\ell \phi(a_\ell) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell) + \sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \cdot \alpha_{i_p}^{-1} \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell) && \text{by (1.3)}
\end{aligned}$$

$$\begin{aligned}
&= r_\ell \phi(a_\ell) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell) + \left(\sum_{p=1}^{j} r_{i_p} \phi(a_{i_p}) \cdot \alpha_{i_p}^{-1} \right) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell) \\
&= (\widetilde{\sigma_\ell}, \sigma_\ell) \otimes_{R[\pi]} r_\ell \phi(a_\ell) + -r_\ell \phi(a_\ell) \otimes_{R[\pi]} (\widetilde{\sigma_\ell}, \sigma_\ell) && \text{by (1.4)} \\
&= 0
\end{aligned}$$

Now $\xi - \xi'$ has $k - (j+1) < k$ terms when written as a sum and this must be minimal since we could otherwise we could write ξ with fewer terms contradicting minimality of k. Therefore, by the inductive hypothesis we have that $\Psi(\xi - \xi') = 0$. Combining this with the above computation, we have that $0 = \Psi(\xi - \xi') = \Psi(\xi) - \Psi(\xi') = \Psi(\xi) - 0 = \Psi(\xi)$. Hence, $\Psi(\xi) = 0$ as was to be shown.

Conversely, we need to show that $\ker(\Psi) \subset \ker(\Phi)$. Before proving this we first prove an algebraic fact that will allow us to write down generators for $\ker(\Psi)$. Suppose we have a commutative diagram of R-modules

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \\ B \arrow[ur, "g"] & \end{tikzcd}$$

with f surjective and $\ker(f)$ generated by $\{a_\alpha\}_{\alpha\in J}$ and $\ker(g)$ generated by $\{b_\beta\}_{\beta\in J'}$. Because f is surjective, for each $\beta \in J'$ there exists $b'_\beta \in A$ such that $f(b'_\beta) = b_\beta$. Then we claim that $\ker(h)$ is generated by $\{a_\alpha\}_{\alpha\in J} \cup \{b'_\beta\}_{\beta\in J'}$. To see this assume $h(a) = 0$. Then, $g(f(a)) = 0$ so that $f(a) \in \ker(g)$. Thus, there exists $r_i \in R$ and $\beta_i \in J'$ such that $f(a) = \sum_i r_i b_{\beta_i}$. On the other hand, $\sum_i r_i b_{\beta_i} = \sum_i r_i f(b'_{\beta_i}) = f(\sum_i r_i b'_{\beta_i})$. Hence, $f(a - \sum_i r_i b'_{\beta_i}) = 0$ which means there exists $s_j \in R$ and $\alpha_j \in J$ such that $a - \sum_i r_i b'_{\beta_i} = \sum s_j a_{\alpha_j}$. Therefore, $a = \sum_j s_j a_{\alpha_j} + \sum_i r_i b'_{\beta_i}$ which is what we wanted to show.

Recall in our situation we have the commutative diagram:

$$\begin{tikzcd} R[A] \otimes_R \widehat{S}^\nu_*(X;R) \arrow[r, "\Psi"] \arrow[d, "\phi\otimes\mathrm{id}"'] & A \otimes_{R[\pi]} \widehat{S}^\nu_*(X;R) \\ A \otimes_R \widehat{S}^\nu_*(X;R) \arrow[ur, "q"] & \end{tikzcd}$$

Each map is surjective so the preceding claim applies to our case. Now $\ker(\phi\otimes\mathrm{id})$ is generated by elements of the form $(\sum r_i a_i) \otimes \xi$ with $\phi(\sum r_i a_i) = 0$. Moreover, $\ker(q)$ is generated by elements of the form $\sum_i a_i \otimes (\alpha_i \cdot \widetilde{\sigma}_i, \sigma_i) - a_i \cdot \alpha_i \otimes (\widetilde{\sigma}_i, \sigma_i)$ where $\alpha_i \in \pi$ and $a_i \in A$. Thus, by the claim above we have that $\ker(\Psi)$ is generated by elements of the form $(\sum r_i a_i) \otimes \xi$ with $\phi(\sum r_i a_i) = 0$ and by elements of the form $\sum_i a_i \otimes (\alpha_i \cdot \widetilde{\sigma}_i, \sigma_i) - a_i \cdot \alpha_i \otimes (\widetilde{\sigma}_i, \sigma_i)$ where

$\alpha_i \in \pi$ and $a_i \in A \subset R[A]$. Therefore, to prove $\ker(\Psi) \subset \ker(\Phi)$ it suffices to prove the elements which generate $\ker(\Psi)$ are mapped to 0 under Φ. So assume first we have an element $(\sum r_i a_i) \otimes \xi$ with $\phi(\sum r_i a_i) = 0$. Let $\xi = \sum_j s_j(\widetilde{\sigma_j}, \sigma_j)$. Then, $\Phi\left((\sum r_i a_i) \otimes \xi\right) = \Phi\left((\sum_i r_i a_i) \otimes (\sum_j s_j(\widetilde{\sigma_j}, \sigma_j))\right) = \sum_j s_j[\phi(\sum_i r_i a_i), \widetilde{\sigma_j}]\sigma_j = \sum_j s_j[0, \widetilde{\sigma_j}]\sigma_j = 0$ as desired. Next consider

$$\begin{aligned}
\Phi\left(\sum_i \left(a_i \otimes (\alpha_i \cdot \widetilde{\sigma_i}, \sigma_i) - a_i \cdot \alpha_i \otimes (\widetilde{\sigma_i}, \sigma_i)\right)\right) &= \sum_i \left([\phi(a_i), \alpha_i \cdot \widetilde{\sigma_i}]\sigma_i - [\phi(a_i) \cdot \alpha_i, \widetilde{\sigma_i}]\sigma_i\right) \\
&= \sum_i \left(\left([\phi(a_i) \cdot \alpha_i, \alpha_i^{-1}\alpha_i \cdot \widetilde{\sigma_i}] - [\phi(a_i) \cdot \alpha_i, \widetilde{\sigma_i}]\right)\sigma_i\right) \\
&\quad \text{by definition of } \mathcal{E} \\
&= \sum_i \left(\left([\phi(a_i) \cdot \alpha_i, \widetilde{\sigma_i}] - [\phi(a_i) \cdot \alpha_i, \widetilde{\sigma_i}]\right)\sigma\right) \\
&= \sum_i 0\sigma_i \\
&= 0
\end{aligned}$$

as desired. Thus, $\ker(\Phi) = \ker(\Psi)$ so that we have an injective map $\widehat{\Phi} : A \otimes_{R[\pi]} \widehat{S}_*^{\nu}(X; R) \to \widehat{S}_*(X; \mathcal{E})$. Let us show this map is also surjective. Let $\xi = \sum e_i \sigma_i \in \widehat{S}_j(X; \mathcal{E})$ where $e_i : \sigma^{-1}(X_{reg}) \to \mathcal{E}$ is a lift of $\sigma_i : \Delta^j \to X$ and $\mathrm{int}(\Delta^j) \subset \sigma_i^{-1}(X_{reg})$ for each i. Let $x_0 \in \mathrm{int}(\Delta^j)$. Then we can write $e_i(x_0) = [a_i, \widetilde{x_0}] \in \mathcal{E}$ where $\widetilde{x_0} \in E(\nu)$ and $a_i \in A$. Notice $A \times E(\nu)$ is a covering of X_{reg} where A is given the discrete topology. Because $\sigma_i^{-1}(X_{reg})$ is simply connected we have a lift, say $\widetilde{\sigma_i}$, of σ_i to $A \times E(\nu)$ with $\widetilde{\sigma_i}(x_0) = (a_i, \widetilde{x_0})$. But then we have that $[a_i, \widetilde{\sigma_i}]$ is a lift of σ_i to E with $[a_i, \widetilde{\sigma_i}](x_0) = [a_i, \widetilde{x_0}] = e_i(x_0)$. Thus, by uniqueness of lifts we must have that $[a_i, \widetilde{\sigma_i}] = e_i$. Let $\xi' = \sum_i a_i \otimes (\widetilde{\sigma_i}, \sigma_i) \in A \otimes_{R[\pi]} \widehat{S}_*^{\nu}(X; R)$.

Then $\widehat{\Phi}(\xi') = \sum_i [a_i, \widetilde{\sigma}_i]\sigma_i = \sum_i e_i\sigma_i = \xi$. Thus, $\widehat{\Phi} : A \otimes_{R[\pi]} \widehat{S}^{\nu}_*(X; R) \to \widehat{S}_*(X; \mathcal{E})$ is an isomorphism of R-modules.

The same argument shows we also have an isomorphism of R-modules $\Phi^{\overline{p}} : {}^{\overline{p}}\widetilde{S}^{\nu}_*(X; A) \to S^{\overline{p}}(X; \mathcal{E})$, where $\Phi^{\overline{p}}$ is the restriction of $\widehat{\Phi}$. Moreover, a simple computation shows we have a commutative diagram

$$\begin{array}{ccc} {}^{\overline{p}}\widetilde{S}^{\nu}(X; A) & \xrightarrow{\Phi^{\overline{p}}} & S^{\overline{p}}_*(X; \mathcal{E}) \\ \Big\downarrow \partial & & \Big\downarrow \partial \\ A \otimes_{R[\pi]} \widehat{S}^{\nu}_*(X; R) & \xrightarrow{\widetilde{\Phi}} & \widehat{S}_*(X; \mathcal{E}). \end{array}$$

Thus, by Lemma 2.2.10 we have that the induced map $I^{\overline{p}}\widetilde{S}^{\nu}_*(X; A) \to I^{\overline{p}}S_*(X; \mathcal{E})$ given by restricting $\Phi^{\overline{p}}$ is an isomorphism of R-module chain complexes. □

Remark 2.2.12. The construction of Theorem 2.2.11 is natural with respect to subspace inclusion. That is, we have the commutative diagram

$$\begin{array}{ccc} I^{\overline{p}}\widetilde{S}^{\nu|_U}_*(U; A) & \longrightarrow & I^{\overline{p}}S_*(U; i^*\mathcal{E}) \\ \Big\downarrow & & \Big\downarrow \\ I^{\overline{p}}\widetilde{S}^{\nu}_*(X; A) & \longrightarrow & I^{\overline{p}}S_*(X; \mathcal{E}) \end{array}$$

where the vertical maps are inclusions and the top horizontal map is the restriction of the bottom horizontal map which is the isomorphism of chain complexes coming from Theorem 2.2.11 which evidently preserves base support of chains. Thus, the top map is an isomorphism since these are by definition intersection chains with base support in U.

More generally, suppose $X' \hookrightarrow X$ is an inclusion of a stratified pseudomanifold with stratification induced by the stratification of X. Then there is also a diagram as above with U replaced by X'.

To end the subsection, we show that the intersection homology of a cover of the regular stratum is the intersection homology with some twisted coefficients. For the ordinary homology of a covering space and ordinary homology with twisted coefficients this is a standard result referred to as Shapiro's Lemma (see for instance (2, Exercise 76)). As a corollary we combine Lemma 2.2.13 and Theorem 2.2.11 which will allow us to carry over other theorems of intersection homology such as cone formulas and Mayer-Vietoris sequences.

Lemma 2.2.13. *Let X be a stratified pseudomanifold with perversity $\overline{p} \leq \overline{t}$ and assume X_{reg} is connected. Let ν denote the data of a connected covering for X_{reg} and let $\mathfrak{u}$ denote the universal cover of X_{reg}. Let $H = p_*(\pi_1(E(\nu)))$ and let π/H denote the set of right cosets. Let R be a commutative ring with unity and let $R[\pi/H] := \bigoplus_{\alpha\in\pi/H} R$ and give $R[\pi/H]$ the structure of a right $R[\pi]$-module in the obvious way. Then $I^{\overline{p}}S^{\nu}_*(X;R)$, the intersection chain complex with coefficients in R of the cover ν, is isomorphic to the intersection chain complex with twisted coefficients $I^{\overline{p}}\widetilde{S}^{\mathfrak{u}}_*(X;R[\pi/H])$ (as R-modules).*

Proof. We first prove $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X;R) \cong \widehat{S}^{\nu}_*(X;R)$.

Recall we can identity $E(\nu)$ with the quotient of $(\pi/H) \times E(\mathfrak{u})$ by the relations $(H\beta, \widetilde{x}) \sim (H\beta\alpha^{-1}, \alpha \cdot \widetilde{x})$ for all $\widetilde{x} \in E(\mathfrak{u}), \alpha, \beta \in \pi$.

Note every element in $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X;R)$ can be written as $\sum_i r_i H\alpha_i \otimes (\widetilde{\sigma}_i, \sigma_i)$ by using bilinearity of the tensor product where $r_i \in R$ and $\alpha_i \in \pi$. So define $\widehat{\Phi} : R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X;R) \to \widehat{S}^{\nu}_*(X;R)$ by $\widehat{\Phi}(rH\alpha \otimes (\widetilde{\sigma}, \sigma)) = r([H\alpha, \widetilde{\sigma}], \sigma)$ and extended linearly. The map $\widehat{\Phi}$ is well-defined since if $rH\beta \otimes (\widetilde{\sigma}, \sigma)$ is equivalent in $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X;R)$ to $rH\beta\alpha^{-1} \otimes \alpha \cdot (\widetilde{\sigma} \cdot \alpha, \sigma)$, then $\widehat{\Phi}(rH\beta\alpha^{-1} \otimes \alpha \cdot (\widetilde{\sigma} \cdot \alpha, \sigma)) = r([H\beta\alpha^{-1}, \alpha \cdot \widetilde{\sigma}], \sigma) = r([H\beta, \widetilde{\sigma}], \sigma) = \widehat{\Phi}(rH\beta \otimes (\widetilde{\sigma}, \sigma))$.

Consider the partition $\mathcal{T}$ of the set of extended simplices into orbit classes induced by the action of the deck transformation group π. For each distinct orbit choose a single extended simplex in that orbit and denote the set of these extended simplices by T. We claim that $T_\pi = \{H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma) : (\widetilde{\sigma}, \sigma) \in T \text{ and } H\alpha \in \pi/H\}$ is in bijective correspondence with extended simplices in $\widehat{S}^{\nu}_*(X;R)$ with the bijection given by restricting $\widehat{\Phi}$ to S_π.

To see it is surjective, let $([H, \widetilde{\tau}], \tau) \in \widehat{S}^{\nu}_*(X;R)$ be an extended simplex so that $\widetilde{\tau}$ is a lift of τ and so there exists $(\widetilde{\sigma}, \sigma) \in T$ with $\sigma = \tau$ and $\widetilde{\sigma}$ a lift of σ and therefore τ. Thus, there exists $\alpha \in \pi$ with $\widetilde{\tau} = \alpha \cdot \widetilde{\sigma}$. Hence, $\widehat{\Phi}\left(H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma)\right) = \widehat{\Phi}\left(H \otimes_{R[\pi]} \alpha \cdot (\widetilde{\sigma}, \sigma)\right) = \widehat{\Phi}\left(H \otimes_{R[\pi]} (\widetilde{\tau}, \tau)\right) = ([H, \widetilde{\tau}], \tau)$. Since $H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma) \in T_\pi$ this shows $\widehat{\Phi}$ restricted to T_π is surjective onto extended simplices in $\widehat{S}^{\nu}_*(X;R)$.

Next, we show $\widehat{\Phi}$ restricted to T_π is injective. Suppose $H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma)$ and $H\beta \otimes_{R[\pi]} (\widetilde{\tau}, \tau)$ are in T_π with $\widehat{\Phi}\left(H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma)\right) = \widehat{\Phi}\left(H\beta \otimes_{R[\pi]} (\widetilde{\tau}, \tau)\right)$. Then we have that $([H\alpha, \widetilde{\sigma}], \sigma) = ([H\beta, \widetilde{\tau}], \tau)$ which means that $\sigma = \tau$ and $[H\alpha, \widetilde{\sigma}] = [H\beta, \widetilde{\tau}]$. Hence, $\widetilde{\sigma}$ and $\widetilde{\tau}$ are both lifts of σ.

However, because $[H\alpha, \widetilde{\sigma}] = [H\beta, \widetilde{\tau}]$ we have that $\widetilde{\sigma} = \gamma \cdot \widetilde{\tau}$ and $H\alpha = H\beta\gamma^{-1}$ for some $\gamma \in \pi$. Thus,

$$\begin{aligned} H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma) &= H\beta\gamma^{-1} \otimes_{R[\pi]} \gamma \cdot (\widetilde{\tau}, \tau) \\ &= H\beta\gamma^{-1}\gamma \otimes_{R[\pi]} (\widetilde{\tau}, \tau) \\ &= H\beta \otimes_{R[\pi]} (\widetilde{\tau}, \tau) \end{aligned}$$

as was to be shown.

Next, we show T_π generates $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R)$. As already noted every element of $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R)$ may be written in the form $\sum_i r_i H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i)$. Hence, we need only show every element of the form $H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma)$ corresponds to an element of T_π. Now $(\widetilde{\sigma}, \sigma) = (\beta \cdot \widetilde{\tau}, \tau)$ for some $(\widetilde{\tau}, \tau) \in T$ and $\beta \in \pi$ so that $H\alpha \otimes_{R[\pi]} (\widetilde{\sigma}, \sigma) = H\alpha \otimes_{R[\pi]} (\beta \cdot \widetilde{\tau}, \tau) = H\alpha\beta \otimes_{R[\pi]} (\widetilde{\tau}, \tau) \in T_\pi$.

So we have shown that the homomorphism of R-modules, $\widehat{\Phi}$, restricted to T_π is bijective onto the set of extended simplices of $\widehat{S}^{\nu}_*(X; R)$ and that T_π generates $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R)$. We claim that this is enough to show $\widehat{\Phi}$ is actually an isomorphism. Since extended simplices provide a basis for $\widehat{S}^{\nu}_*(X; R)$ we clearly have that $\widehat{\Phi}$ is surjective. To see $\widehat{\Phi}$ is injective let $x \in R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R)$ be such that $\widehat{\Phi}(x) = 0$. Since T_π generates $R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R)$ we may write $x = \sum_i r_i H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i)$ where $H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i) \in T_\pi$. We may assume without loss of generality that $H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i) \neq H\alpha_j \otimes_{R[\pi]} (\widetilde{\sigma_j}, \sigma_j)$ for $i \neq j$ (otherwise combine like terms). Then $\widehat{\Phi}\left(\sum_i r_i H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i)\right) = \sum_i r_i \widehat{\Phi}(H\alpha_i \otimes_{R[\pi]} (\widetilde{\sigma_i}, \sigma_i)) = \sum_i r_i([H\alpha_i, \widetilde{\sigma_i}], \sigma_i) = 0$. However, as noted above $\widehat{\Phi}$ restricted to T_π is injective which means $([H\alpha_i, \widetilde{\sigma_i}], \sigma_i) \neq ([H\alpha_j, \widetilde{\sigma_j}], \sigma_j)$ for $i \neq j$. But extended simplices provide a basis of $\widehat{S}^{\nu}_*(X; R)$ which means in the sum we must have $r_i = 0$ for all i so that $x = 0$. So $\widehat{\Phi}$ is in fact injective as a map of R-modules; and therefore, we have shown $\widehat{\Phi}$ is an isomorphism of R-modules.

The same argument, and a tautological verification of $\overline{p}$-allowability, shows we also have an isomorphism $\Phi^{\overline{p}} : R[\pi/H] \otimes_{R[\pi]} {}^{\overline{p}}S^{\mathfrak{u}}_*(X; R) \to {}^{\overline{p}}S^{\nu}_*(X; R)$ with $\Phi^{\overline{p}}$ the restriction of $\widehat{\Phi}$. Moreover, a simple computation shows we have the commutative diagram below.

$$\begin{tikzcd}
R[\pi/H] \otimes_{R[\pi]} {}^{\overline{p}}S^{\mathfrak{u}}_*(X; R) \arrow[r, "\Phi^{\overline{p}}"] \arrow[d, "\partial"'] & {}^{\overline{p}}S^{\nu}_*(X; R) \arrow[d, "\partial"'] \\
R[\pi/H] \otimes_{R[\pi]} \widehat{S}^{\mathfrak{u}}_*(X; R) \arrow[r, "\widehat{\Phi}"] & \widehat{S}^{\nu}_*(X; R)
\end{tikzcd}$$

Thus, by Lemma 2.2.10 we have that the map induced by restricting $\Phi^{\overline{p}}$, $I^{\overline{p}}\widetilde{S}^{\mathfrak{u}}_*(X; R[\pi/H]) \to I^{\overline{p}}S^{\nu}_*(X; R)$, is an isomorphism of chain complexes.

□

Corollary 2.2.14. *Let X be a stratified pseudomanifold and let ν denote the data associated to a cover of X_{reg}. Let R be a commutative ring with unity. There exists a system of local coefficients $\mathcal{E}_\nu$ over X_{reg} such that $I^{\overline{p}}S^{\nu}_*(X; R) \cong I^{\overline{p}}S_*(X; \mathcal{E}_\nu)$. Moreover, the isomorphism restricts to isomorphisms $I^{\overline{p}}S^{i^*\nu}_*(U; R) \cong I^{\overline{p}}S_*(U; i^*\mathcal{E}_\nu)$ for every open $U \subset X$, where $i : U \hookrightarrow X$ is the inclusion map and where $i^*\nu$ and $i^*\mathcal{E}_\nu$ are restrictions to U.*

Proof. We prove the theorem first for special cases with each becoming more general. We begin with the case that X is a normal pseudomanifold, X is connected, and ν is a connected covering. However, because X is connected and normal we have that X_{reg} is connected (5, Lemma 2.63). Thus, this case follows immediately by combining Lemma 2.2.13 and Theorem 2.2.11. Next, we assume ν is not necessarily a connected covering. Let ν_i denote the connected components of ν. Then we have

$$\begin{aligned} I^{\bar{p}}S^{\nu}_*(X;R) &\cong \bigoplus_i I^{\bar{p}}S^{\nu_i}_*(X;R) \\ &\cong \bigoplus_i I^{\bar{p}}S_*(X;\mathcal{E}_{\nu_i}) \\ &\cong I^{\bar{p}}S_*(X;\bigoplus_i \mathcal{E}_{\nu_i}) \end{aligned}$$

The first isomorphism is by Proposition 2.1.6, the second is by applying the previous case, and the third is elementary. Next, we consider the case X is normal, but not necessarily connected. We can write $X = \coprod X_i$ with each X_i normal and connected. Let ν_i denote the restriction of ν to X_i. Then we have

$$\begin{aligned} I^{\bar{p}}S^{\nu}_*(X;R) &\cong \bigoplus_i I^{\bar{p}}S^{\nu_i}_*(X_i;R) \\ &\cong \bigoplus_i I^{\bar{p}}S_*(X_i;\mathcal{E}_{\nu_i}) \\ &\cong I^{\bar{p}}S_*(X;\coprod_i \mathcal{E}_{\nu_i}) \end{aligned}$$

The first isomorphism is by Proposition 2.1.7, the second is by applying the previous case, and the third is elementary.

Finally, we consider the case that X is not necessarily a normal pseudomanifold. Let $\mathbf{n} : X^N \to X$ be a normalization of X. Then we have the following diagram

$$\begin{tikzcd}
I^{\bar{p}}S^{\nu}_{*}(X^N;R) \arrow[r] \arrow[d, "\mathbf{n}"'] & I^{\bar{p}}S_{*}(X^N;\mathcal{E}_\nu) \arrow[d, "\mathbf{n}"'] \\
I^{\bar{p}}S^{\nu}_{*}(X;R) \arrow[r] & I^{\bar{p}}S_{*}(X;\mathcal{E}_\nu).
\end{tikzcd}$$

The top map is an isomorphism by the previous case, the two vertical maps are isomorphism by Proposition 2.1.8 and (16) (although the author in (16) only proves this for ordinary intersection homology the proof for the local coefficients case is similar). Thus, we may take the bottom horizontal map to be the composition around the square which is therefore an isomorphism. The last part of the theorem follows by observing that the bottom horizontal map will preserve the base support of extended simplices and that $I^{\bar{p}}S^{i^*\nu}_{*}(U;R)$ and $I^{\bar{p}}S_{*}(U;i^*\mathcal{E}_\nu)$ correspond respectively to chains in $I^{\bar{p}}S^{\nu}_{*}(X;R)$ and $I^{\bar{p}}S_{*}(X;\mathcal{E}_\nu)$ which have base support in U.

$\square$

2.3 Local computations

In this subsection we apply Corollary 2.2.14 to carry over the local computations of intersection homology such as the cone formula. These will be useful when we make local to global arguments later in the paper. Recall for a compact stratified pseudomanifold X we give cX the filtration $(cX)^i = cX^{i-1}$ and $(cX)^0 =$ cone vertex.

Proposition 2.3.1. *Let X be a stratified $(k-1)$-dimensional pseudomanifold with X_{reg} connected. Let $\overline{p} \leq \overline{t}$ be a perversity defined on cX. Define a perversity $\overline{p}_X$ on X by $\overline{p}_X(S) = \overline{p}(S \times (0,1))$. Let ν be a connected regular cover $(cX)_{reg} = X_{reg} \times (0,1)$ with deck transformation group π and let A be a right $R[\pi]$-module (R a commutative ring with unity). Let $i : X \times \{t_0\} \hookrightarrow cX$, $0 < t_0 < 1$ denote the inclusion map. Then,*

$$I^{\overline{p}}\widetilde{H}^{\nu}_j(cX;A) \cong \begin{cases} 0, & \text{if } j \geq k-1-\overline{p}(\{v\}), \\ I^{\overline{p}_X}\widetilde{H}^{\nu|_X}_j(X;A), & \text{if } j < k-1-\overline{p}(\{v\}) \end{cases}$$

where v is the cone vertex.

Proof. By Theorem 2.2.11 and Remark 2.2.12 there is a system of local coefficients $\mathcal{E}$ on cX such that $I^{\overline{p}}H_*(cX;\mathcal{E}) \cong I^{\overline{p}}\widetilde{H}^{\nu}_*(cX;A)$; and moreover, $I^{\overline{p}}H_*(X;i^*\mathcal{E}) \cong I^{\overline{p}}\widetilde{H}^{\nu|_X}_*(X;A)$. However, notice that $\nu|_X$ is a connected regular cover of X_{reg} and thus also has deck transformation group π. Therefore, we have that $I^{\overline{p}}\widetilde{S}^{\nu|_X}_*(X;A) = I^{\overline{p}_X}\widetilde{S}^{\nu|_X}_*(X;A)$ by Remark 2.2.8.

By the cone formula for intersection homology with local coefficients (8, Proposition 2.18) we have that $I^{\overline{p}}H_j(cX;\mathcal{E}) = 0$ whenever $j \geq k-1-\overline{p}(\{v\})$. Thus, $I^{\overline{p}}\widetilde{H}^{\nu}_j(cX;A) = 0$ in this dimension range as well. Next, consider the case $j < k-1-\overline{p}(\{v\})$. Then we have the commutative diagram

$$\begin{array}{ccc} I^{\overline{p}_X}\widetilde{H}_j^{\nu|_X}(X;A) & \cong & I^{\overline{p}_X}H_j(X;i^*\mathcal{E}) \\ \downarrow & & \downarrow \\ I^{\overline{p}}\widetilde{H}_j^{\nu}(cX;A) & \cong & I^{\overline{p}}H_j(cX;\mathcal{E}) \end{array}$$

where the vertical maps are induced by inclusion. The right vertical map is an isomorphism in this dimension range by the cone formula for intersection homology with local coefficients. Hence, by commutativity the left vertical map is an isomorphism in this dimension range as well. □

Recall for a stratified pseudomanifold X and a trivially stratified m-manifold M we have a standard filtration for $X \times M$ given by $(X \times M)^i = X^{i-m} \times M$.

Proposition 2.3.2. *Let X be a stratified pseudomanifold with X_{reg} connected.Let ν be a connected regular cover of $X \times \mathbb{R}^m$ with deck transformation group π and let A be a right $R[\pi]$-module (R a commutative ring with unity). Let $\overline{p}$ be a perversity defined on $X \times \mathbb{R}^m$ and define a perversity on X by $\overline{p}_X(S) = \overline{p}(S \times \mathbb{R}^m)$ for each stratum S of X. Then the inclusion map $i : X \times \{0\} \hookrightarrow X \times \mathbb{R}^m$ induces an isomorphism*

$$I^{\overline{p}}\widetilde{H}_*^{\nu}(X \times \mathbb{R}^m; A) \cong I^{\overline{p}_X}\widetilde{H}_*^{\nu|_X}(X; A).$$

Proof. By Theorem 2.2.11 there is a system of local coefficients $\mathcal{E}$ over $(X \times \mathbb{R}^m)_{reg} = X_{reg} \times \mathbb{R}^m$ such that $I^{\bar{p}}\widetilde{H}_*^{\nu}(X \times \mathbb{R}^m; A) \cong I^{\bar{p}}H_*(X \times \mathbb{R}^m; \mathcal{E})$ and $I^{\bar{p}}\widetilde{H}_*^{\nu|_X}(X; A) \cong I^{\bar{p}}H_*(X; i^*\mathcal{E})$. Notice that $\nu|_X$ is a connected regular cover of X_{reg} and also has deck transformation group π. Thus, by Remark 2.2.8 $I^{\bar{p}}\widetilde{H}_*^{\nu|_X}(X; A) = I^{\bar{p}_X}\widetilde{H}_*^{\nu|_X}(X; A)$.

Consider the diagram below

$$\begin{tikzcd}
I^{\bar{p}}\widetilde{H}_*^{\nu}(X \times \mathbb{R}^m; A) \arrow[r, phantom, "\cong"] & I^{\bar{p}}H_*(X \times \mathbb{R}^m; \mathcal{E}) \\
I^{\bar{p}_X}\widetilde{H}_*^{\nu|_X}(X; A) \arrow[u] \arrow[r, phantom, "\cong"] & I^{\bar{p}}H_*(X; i^*\mathcal{E}) \arrow[u]
\end{tikzcd}$$

where the vertical maps are induced by inclusion. The right vertical map is an isomorphism by invariance of intersection homology with local coefficients under stratified homotopy equivalence ((7, Corollary 2.3)). Thus, by commutativity of the diagram the left vertical map is also an isomorphism. $\square$

Proposition 2.3.3 (Cone Formula)**.** *Let X be a stratified pseudomanifold of dimension $k-1$ with perversity $\overline{p} \leq \overline{t}$ defined on cX. Let ν be any covering of $(cX)_{reg} = X_{reg} \times (0,1)$. Let R be a commutative ring with unity and $i : X \times \{t_0\} \hookrightarrow cX$, $0 < t_0 < 1$, be the inclusion map. Then*

$$I^{\overline{p}}H_j^{\nu}(cX;R) \cong \begin{cases} 0, & \text{if } j \geq n-1-\overline{p}(\{v\}), \\ I^{\overline{p}_X}H_j^{i^*\nu}(X;R), & \text{if } j < n-1-\overline{p}(\{v\}) \end{cases}$$

where v is the cone vertex of cX.

Proof. This follows immediately by Lemma 2.2.13 and Proposition 2.3.1.

□

Proposition 2.3.4. *Let R be a commutative ring with unity, X be a stratified pseudomanifold with perversity $\overline{p} \leq \overline{t}$, and let ν be the data associated to a covering of X_{reg}. Then we have an isomorphism*

$$I^{\overline{p}}H_*^{\nu \times id_{\mathbb{R}^k}}(X \times \mathbb{R}^k;R) \cong I^{\overline{p}}H_*^{\nu}(X;R)$$

which is induced by the inclusion map $i : X \times \{0\} \hookrightarrow X \times \mathbb{R}^k$.

Proof. Immediate as in Proposition 2.3.3. □

2.4 Excision and Mayer-Vietoris sequences

In this section we show excision holds and there is a Mayer-Vietoris long exact sequence for intersection homology of coverings spaces of the regular set.

Open covers and excision

The crucial result we will need this section is stated below. Applying Corollary 2.2.14 will allow us to carry over this result to the intersection homology of covers of the regular set. We refer the reader to (8, Proposition 2.9) for a proof. We note the author there considers local coefficient systems which are more general than our setting. There singular maps may dip into codimension one strata, however, as the author notes when perversities are bounded above by the top perversity the more general construction is the same as intersection homology with local coefficients which we use.

Although we simply state the result below, we should pause to note that carrying over subdivision arguments from ordinary homology theory to intersection homology is not as completely straightforward as one might expect. The issue is if we subdivide an intersection chain, say $\widehat{\xi} = \xi_U + \xi_V$ where ξ_U has support in U and ξ_V has support in V, then we also need ξ_U and ξ_V to be intersection chains in order to achieve Mayer-Vietoris long exact sequences. However, this requires $\partial\xi_U$ to be allowable, and there is no guarantee this happens since the cancellations which occur in $\partial\widehat{\xi}$ to make $\partial\widehat{\xi}$ allowable may come from $\partial\xi_V$. For more on Mayer-Vietoris sequences for intersection homology we also refer the reader to (5, Mayer-Vietoris and Excision).

Proposition 2.4.1. *Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover for X. Let $I^{\overline{p}}_{\mathcal{U}}S_*(X;\mathcal{E}) \subset I^{\overline{p}}S_*(X;\mathcal{E})$ be the subcomplex of intersection chains ξ which can be written as a sum of intersection chains $\xi = \sum_j \xi_j$ where ξ_j is an intersection chain with support in some $U_{\alpha_j} \in \mathcal{U}$. Then the inclusion map $\iota : I^{\overline{p}}_{\mathcal{U}}S_*(X;\mathcal{E}) \to I^{\overline{p}}S_*(X;\mathcal{E})$ is a quasi-isomorphism.*

Next, we state the excision theorem for intersection homology with local coefficients (8, Lemma 2.11).

Proposition 2.4.2. *Let X be a stratified space and let $U \subset X$ be open and let $C \subset U$ be closed (as a subspace of U). Let $\mathcal{E}$ be a system of local coefficients defined on X_{reg}. Then $I^{\overline{p}}H_*(X, U; \mathcal{E}) \cong I^{\overline{p}}H_*(X - C, U - C; \mathcal{E})$.*

By Lemma 2.2.13 and Proposition 2.4.2 we immediately obtain the following corollary. We remind the reader that within the context below, $I^{\overline{p}}H_*^{\nu}(X - C, U - C; R)$ is implicitly the restriction of ν to $X - C$, that is, $I^{\overline{p}}H_*^{i^*\nu}(X - C, U - C; R)$, where $i : X - C \hookrightarrow X$ is the inclusion map and $i^*\nu$ is the restriction of ν to $X - C$.

Corollary 2.4.3 (Excision)**.** *Let R be a commutative ring with unity. Let X be a stratified pseudomanifold and let $U \subset X$ be open and $C \subset U$ be closed (as a subspace of U). Let ν be the data associated to any cover of X_{reg}. Then $I^{\overline{p}}H_*^{\nu}(X, U; R) \cong I^{\overline{p}}H_*^{\nu}(X - C, U - C; R)$ with the isomorphism induced by inclusion $(X - C, U - C) \hookrightarrow (X, U)$.*

Mayer-Vietoris sequences

Although not stated in (8) we can easily obtain Mayer-Vietoris sequences from Proposition 2.4.1. Suppose $\mathcal{U} = \{U, V\}$ is an open cover of X. For an open set $W \subset X$ let $i_W : W \hookrightarrow X$ denote the inclusion map. We note that

$I^{\overline{p}}S_*(U; i_U^*\mathcal{E}) \cap I^{\overline{p}}S_*(V; i_V^*\mathcal{E}) = I^{\overline{p}}S_*(U \cap V; i_{U\cap V}^*\mathcal{E})$. This is because

$$
\begin{aligned}
\xi \in I^{\overline{p}}S_*(U; i_U^*\mathcal{E}) \cap I^{\overline{p}}S_*(V; i_V^*\mathcal{E}) &\iff \xi \in I^{\overline{p}}S_*(U; i_U^*\mathcal{E}) \text{ and } \xi \in I^{\overline{p}}S_*(V; i_V^*\mathcal{E}) \\
&\iff \xi \text{ is an intersection chain and has support in } U \text{ and } V \\
&\iff \xi \text{ is an intersection chain and has support in } U \cap V \\
&\iff \xi \in I^{\overline{p}}S_*(U \cap V; i_{U\cap V}^*\mathcal{E}).
\end{aligned}
$$

Standard algebra gives us the short exact sequence

$$0 \to I^{\overline{p}}S_*(U \cap V; i_{U\cap V}^*\mathcal{E}) \to I^{\overline{p}}S_*(U; i_U^*\mathcal{E}) \oplus I^{\overline{p}}S_*(V; i_V^*\mathcal{E}) \to I^{\overline{p}}_{\mathcal{U}}S_*(X; \mathcal{E}) \to 0.$$

which induces a long exact sequence of homology, but because $\iota : I^{\overline{p}}_{\mathcal{U}}S_*(X; \mathcal{E}) \to I^{\overline{p}}S_*(X; \mathcal{E})$ is a quasi-isomorphism we arrive at the following.

Proposition 2.4.4. *There is a long exact sequence*

$$\longrightarrow I^{\overline{p}}H_i(U \cap V; i_{U\cap V}^*\mathcal{E}) \longrightarrow I^{\overline{p}}H_i(U; i_U^*\mathcal{E}) \oplus I^{\overline{p}}H_i(V; i_V^*\mathcal{E}) \longrightarrow I^{\overline{p}}H_i(X; \mathcal{E}) \longrightarrow I^{\overline{p}}H_{i-1}(U \cap V; i_{U\cap V}^*\mathcal{E}) \longrightarrow$$

Combining Corollary 2.2.14 and Proposition 2.4.1 we arrive at Mayer-Vietoris sequences for intersection homology of coverings of the regular set.

Corollary 2.4.5. *Let U, V be open subsets of a stratified pseudomanifold X (not necessarily a cover of X). Let $\overline{p}$ be a perversity with $\overline{p} \leq \overline{t}$. Let ν be the data associated to a cover of X_{reg}. There is a long exact sequence*

$$\longrightarrow I^{\overline{p}}H_i^{\nu}(U \cap V; R) \longrightarrow I^{\overline{p}}H_i^{\nu}(U; R) \oplus I^{\overline{p}}H_i^{\nu}(V; R) \longrightarrow I^{\overline{p}}H_i^{\nu}(U \cup V; R) \longrightarrow I^{\overline{p}}H_{i-1}^{\nu}(U \cap V; R) \longrightarrow$$

There will be multiple arguments in this paper which will use typical Mayer-Vietoris local to global arguments. In order to reduce redundancy, we state the structure of the arguments below as meta-theorems. We refer the reader to (5) for a proof of these results, though, the basic idea of the arguments may also be found in (14) where the author proves Poincaré duality.

Theorem 2.4.6 (Meta-Theorem 1)**.** *Let $\mathcal{M}_M$ be the category whose objects are homeomorphic to open subsets of a given manifold M and whose morphisms are inclusions. Let Ab_* be the category of graded abelian groups. Let $F_*, G_* : \mathcal{M}_M \to Ab_*$ be functors and let $\Phi : F_* \to G_*$ be a natural transformation such that F_*, G_*, Φ satisfy the following conditions:*

1. *$\Phi : F_*(U) \to G_*(U)$ is an isomorphism for each U homeomorphic to $\mathbb{R}^n$*
2. *F_* and G_* admit Mayer-Vietoris long exact sequences. That is, for open $U, V \subset M$ there is a Mayer-Vietoris sequence*

$$\to F_i(U \cap V) \to F_i(U) \oplus F_i(V) \to F_i(U \cup V) \to F_{i-1}(U \cap V) \to$$

   and similarly for G_*. Moreover, Φ induces a commutative diagram between the two sequences.

3. *If $\{U_\alpha\}$ is a totally ordered set of open subsets of M and $\Phi : F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each α, then $\Phi : F_*(\cup_\alpha U_\alpha) \to G_*(\cup_\alpha U_\alpha)$ is also an isomorphism.*

Then $\Phi : F_*(M) \to G_*(M)$ is an isomorphism.

For the next meta-theorem we recall the definition of a stratified homeomorphism. A map between stratified pseudomanifolds $f : X \to Y$ is said to be a *stratified homeomorphism* if f is a topological homeomorphism and for each stratum S of X, $f(S)$ is a stratum of Y and the codimension of $f(S)$ in Y equals the codimension of S in X.

Theorem 2.4.7 (Meta-Theorem 2)**.** *Let $\mathcal{F}_X$ be the category whose objects are homeomorphic to open subsets of a given stratified pseudomanifold X and whose morphisms are stratified homeomorphisms and inclusions. Let Ab_* be the category of graded abelian groups. Let $F_*, G_* : \mathcal{F}_X \to Ab_*$ be functors and let $\Phi : F_* \to G_*$ be a natural transformation satisfying the following properties:*

1. *F_* and G_* admit Mayer-Vietoris sequences with respect to open subsets of X, and Φ induces a commutative diagram between the two sequences.*

2. *If $\{\mathcal{U}_\alpha\}$ is a totally ordered set of open subsets of X, then the natural maps*

$$\lim_{\rightarrow} F_*(U_\alpha) \to F_*(\cup_\alpha U_\alpha) \text{ and } \lim_{\rightarrow} G_*(U_\alpha) \to G_*(\cup_\alpha U_\alpha)$$

   are isomorphisms.

3. *If L is a compact filtered space such that X has an open subset stratified homeomorphic to $\mathbb{R}^i \times cL$ and $\Phi : F_*(\mathbb{R}^i \times (cL - v)) \to G_*(\mathbb{R}^i \times (cL - v))$ is an isomorphism, then $\Phi : F_*(\mathbb{R}^i \times cL) \to G_*(\mathbb{R}^i \times cL)$ is also an isomorphism.*

4. *If U is an open subset of X homeomorphic to euclidean space, then $\Phi : F_*(U) \to G_*(U)$ is an isomorphism.*

Then $\Phi : F_*(X) \to G_*(X)$ is an isomorphism.

3 Cross product and Künneth theorem

In this section we formulate the cross product for intersection homology for coverings of the regular set and also prove a Künneth theorem.

3.1 Existence of cross product map

First we prove a lemma which will allow us to define our cross product map in terms of the cross product for singular chains.

Lemma 3.1.1. *Let X be any space, let Y be a stratified space, and let ν be a covering space for Y_{reg}. Let $f : X \to Y$ be a map such that there is a lift $\widetilde{f} : f^{-1}(Y_{reg}) \to E(\nu)$. Then there is a chain map $(\widetilde{f}, f)_\# : S_*(X) \to S_*^\nu(Y)$*

Proof. Define $(\widetilde{f}, f)_\# : S_*(X) \to S_*^\nu(Y)$ by $(\widetilde{f}, f)_\#(\sigma) = (\widetilde{f} \circ \sigma, f \circ \sigma)$ where $\widetilde{f} \circ \sigma$ is restricted to $(f \circ \sigma)^{-1}(Y_{reg})$ and we extend $(\widetilde{f}, f)_\#$ linearly. Then,

$$\begin{aligned}
\partial(\widetilde{f}, f)_\#(\sigma) &= \partial(\widetilde{f} \circ \sigma, f \circ \sigma) \\
&= \sum_k (-1)^k (\widetilde{f} \circ \sigma \circ \partial_k, f \circ \sigma \circ \partial_k) \\
&= \sum_k (-1)^k (\widetilde{f}, f)_\#(\sigma \circ \partial_k) \\
&= (\widetilde{f}, f)_\# \left(\sum_k (-1)^k \sigma \circ \partial_k \right) \\
&= (\widetilde{f}, f)_\#(\partial\sigma)
\end{aligned}$$

Thus, $(\widetilde{f}, f)_\#$ is a chain map as claimed. □

The usual way to construct a cross product map $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ is abstractly via acyclic models. This construction, however, is not available to intersection homology theories. An alternative approach is to use shuffle products as in (5). For the convenience of the reader we recall how these maps are defined.

Let p and q be two non-negative integers, then a (p,q)-shuffle is a partition of the set $\{1, 2, \ldots, p+q\}$ into two disjoint ordered sets $\mu = \{\mu_i\}_{i=1}^{p}$ and $\nu = \{\nu_j\}_{j=1}^{q}$ with $\mu_i < \mu_{i+1}$ and $\nu_j < \nu_{j+1}$ for each i and j. The partition (μ, ν) can then be used to shuffle two ordered sets of cardinalities p and q to form a new ordered set of cardinality $p+q$. For example, take the ordered sets (A, B, C) and (α, β, γ) and suppose we have a $(3,3)$-shuffle given by $(\{2,4,5\}, \{1,3,6\})$. We can shuffle our sets using this $(3,3)$-shuffle to obtain the ordered set $\{\alpha, A, \beta, B, C, \gamma\}$.

Let (μ, ν) be a (p,q)-shuffle. Let $\eta^\mu : \Delta^{p+q} \to \Delta^p$ be the map which takes the vertex $w_i \in \Delta^{p+q}$ to the vertex $u_j \in \Delta^p$ if $\mu_j \leq w_i < \mu_{j+1}$ (here we set $\mu_0 = 0$ and $\mu_{p+1} = p+q+1$). Similarly, we can form a map $\eta^\nu : \Delta^{p+q} \to \Delta^q$. From these two maps we can form the product map $\eta_{\mu\nu} = (\eta^\mu, \eta^\nu) : \Delta^{p+q} \to \Delta^p \times \Delta^q$. Let $\mathrm{sgn}(\mu, \nu)$ denote the sign of the permutation from $(1, 2, \ldots, p+q)$ to $(\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q)$. The following then follows by (5, Proposition 5.10) and the comments proceeding (5, Theorem 5.31).

Proposition 3.1.2. *Let R be a commutative principal ideal domain with unity and let $\sigma_1 \in S_p(X;R)$ and $\sigma_2 \in S_q(Y;R)$. The sum over all (p,q)-shuffles (μ,ν)*

$$\epsilon(\sigma_1 \otimes \sigma_2) = \sum sgn(\mu,\nu)(\sigma_1 \times \sigma_2) \circ \eta_{\mu\nu}$$

extends to a chain map

$$\epsilon : S_*(X;R) \otimes_R S_*(Y;R) \to S_*(X \times Y;R).$$

Theorem 3.1.3. *Let R be a commutative principal ideal domain with unity. Let X, Y be stratified spaces with perversities $\overline{p}, \overline{q}$; respectively. Suppose Q is any perversity on $X \times Y$ with $Q(S \times S') \geq \overline{p}(S) + \overline{q}(S')$ for all strata $S \subset X$ and $S' \subset Y$. Let ν and ϑ be covering spaces for X_{reg} and Y_{reg}; respectively. Then there is a cross product map*

$$\widetilde{\epsilon} : I^{\overline{p}}S_*^{\nu}(X;R) \otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R) \to I^{Q}S_*^{\nu\times\vartheta}(X \times Y;R)$$

Proof. We first define

$$\widetilde{\epsilon} : S_*^{\nu}(X;R) \otimes_R S_*^{\vartheta}(Y;R) \to S_*^{\nu\times\vartheta}(X \times Y;R)$$

by

$$\widetilde{\epsilon}((\widetilde{\sigma},\sigma) \otimes (\widetilde{\tau},\tau)) = (\widetilde{\sigma} \times \widetilde{\tau}, \sigma \times \tau)_{\#}\left(\epsilon(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})\right)$$

where $p = \dim(\sigma)$ and $q = \dim(\tau)$ and we extend linearly to all of $S_*^{\nu}(X;R) \otimes_R S_*^{\vartheta}(Y;R)$.

Let us show that $\widetilde{\epsilon}$ is a chain map. To simplify notation we let $\phi = \sigma \times \tau$ and $\widetilde{\phi} = \widetilde{\sigma} \times \widetilde{\tau}$

$$
\begin{aligned}
\partial\widetilde{\epsilon}((\widetilde{\sigma},\sigma)\otimes(\widetilde{\tau},\tau)) &= \partial(\widetilde{\phi},\phi)_{\#}\left(\epsilon(\mathrm{id}_{\Delta^p}\otimes\mathrm{id}_{\Delta^q})\right) \\
&= (\widetilde{\phi},\phi)_{\#}\left(\partial\epsilon(\mathrm{id}_{\Delta^p}\otimes\mathrm{id}_{\Delta^q})\right) \\
&= (\widetilde{\phi},\phi)_{\#}\left(\epsilon(\partial(\mathrm{id}_{\Delta^p}\otimes\mathrm{id}_{\Delta^q}))\right) \\
&= (\widetilde{\phi},\phi)_{\#}\left(\epsilon((\partial\mathrm{id}_{\Delta^p})\otimes\mathrm{id}_{\Delta^q} + (-1)^p\mathrm{id}_{\Delta^p}\otimes(\partial\mathrm{id}_{\Delta^q}))\right) \\
&= (\widetilde{\phi},\phi)_{\#}\left(\epsilon\left(\sum_i(-1)^i(\mathrm{id}_{\Delta^p}\circ\partial_i)\otimes\mathrm{id}_{\Delta^q} + (-1)^p\sum_j(-1)^j\mathrm{id}_{\Delta^p}\otimes(\mathrm{id}_{\Delta^q}\circ\partial_j)\right)\right) \\
&= (\widetilde{\phi},\phi)_{\#}\left(\sum_{\substack{(p-1,q)\\ \text{shuffles}}}\sum_i \mathrm{sgn}(\mu,\omega)(-1)^i((\mathrm{id}_{\Delta^p}\circ\partial_i)\times\mathrm{id}_{\Delta^q})\circ\eta_{\mu\omega}\right) \\
&\quad + (\widetilde{\phi},\phi)_{\#}\left((-1)^p\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\sum_j \mathrm{sgn}(\mu',\omega')(-1)^j(\mathrm{id}_{\Delta^p}\times(\mathrm{id}_{\Delta^q}\circ\partial_j))\circ\eta_{\mu'\omega'}\right) \\
&= \sum_{\substack{(p-1,q)\\ \text{shuffles}}}\sum_i \mathrm{sgn}(\mu,\omega)(-1)^i(\widetilde{\phi}\circ((\mathrm{id}_{\Delta^p}\circ\partial_i)\times\mathrm{id}_{\Delta^q})\circ\eta_{\mu\omega},\phi\circ((\mathrm{id}_{\Delta^p}\circ\partial_i)\times\mathrm{id}_{\Delta^q})\circ\eta_{\mu\omega}) \\
&\quad + (-1)^p\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\sum_j \mathrm{sgn}(\mu',\omega')(-1)^j(\widetilde{\phi}\circ(\mathrm{id}_{\Delta^p}\times(\mathrm{id}_{\Delta^q}\circ\partial_j))\circ\eta_{\mu'\omega'},\phi\circ(\mathrm{id}_{\Delta^p}\times(\mathrm{id}_{\Delta^q}\circ\partial_j))\circ\eta_{\mu'\omega'}) \\
&= \sum_{\substack{(p-1,q)\\ \text{shuffles}}}\sum_i \mathrm{sgn}(\mu,\omega)(-1)^i(((\widetilde{\sigma}\circ\partial_i)\times\widetilde{\tau})\circ\eta_{\mu\omega},((\sigma\circ\partial_i)\times\tau)\circ\eta_{\mu\omega}) \\
&\quad + (-1)^p\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\sum_j \mathrm{sgn}(\mu',\omega')(-1)^j((\widetilde{\sigma}\times(\widetilde{\tau}\circ\partial_j))\circ\eta_{\mu'\omega'},(\sigma\times(\tau\circ\partial_j))\circ\eta_{\mu'\omega'})
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\widetilde{\epsilon}\,(\partial(\widetilde{\sigma},\sigma)\otimes(\widetilde{\tau},\tau))) &= \widetilde{\epsilon}((\partial(\widetilde{\sigma},\sigma))\otimes(\widetilde{\tau},\tau)+(-1)^p(\widetilde{\sigma},\sigma)\otimes(\partial(\widetilde{\tau},\tau)))\\
&= \widetilde{\epsilon}\left(\sum_i(-1)^i(\widetilde{\sigma}\circ\partial_i,\sigma\circ\partial_i)\otimes(\widetilde{\tau},\tau)+(-1)^p\sum_j(-1)^j(\widetilde{\sigma},\sigma)\otimes(\widetilde{\tau}\circ\partial_j,\tau\circ\partial_j)\right)\\
&= \sum_i(-1)^i((\widetilde{\sigma}\circ\partial_i)\times\tau,(\sigma\circ\partial_i)\times\tau)_{\#}(\epsilon(\mathrm{id}_{\Delta^{p-1}}\otimes\mathrm{id}_{\Delta^q}))\\
&\quad+(-1)^p\sum_j(-1)^j(\widetilde{\sigma}\times(\widetilde{\tau}\circ\partial_j),\sigma\times(\tau\circ\partial_j))_{\#}(\epsilon(\mathrm{id}_{\Delta^p}\otimes\mathrm{id}_{\Delta^{q-1}}))\\
&= \sum_i\left((-1)^i((\widetilde{\sigma}\circ\partial_i)\times\widetilde{\tau},(\sigma\circ\partial_i)\times\tau)_{\#}\left(\sum_{\substack{(p-1,q)\\ \text{shuffles}}}\mathrm{sgn}(\mu,\omega)(\mathrm{id}_{\Delta^{p-1}}\times\mathrm{id}_{\Delta^q})\circ\eta_{\mu\omega}\right)\right)\\
&\quad+(-1)^p\sum_j\left((-1)^j(\widetilde{\sigma}\times(\widetilde{\tau}\circ\partial_j),\sigma\times(\tau\circ\partial_j))_{\#}\left(\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\mathrm{sgn}(\mu',\omega')(\mathrm{id}_{\Delta^p}\times\mathrm{id}_{\Delta^{q-1}})\circ\eta_{\mu'\omega'}\right)\right)\\
&= \sum_i\sum_{\substack{(p-1,q)\\ \text{shuffles}}}\mathrm{sgn}(\mu,\omega)(-1)^i((\widetilde{\sigma}\circ\partial_i)\times\widetilde{\tau},(\sigma\circ\partial_i)\times\tau)_{\#}((\mathrm{id}_{\Delta^{p-1}}\times\mathrm{id}_{\Delta^q})\circ\eta_{\mu\omega})\\
&\quad+(-1)^p\sum_j\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\mathrm{sgn}(\mu',\omega')(-1)^j(\widetilde{\sigma}\times(\widetilde{\tau}\circ\partial_j),\sigma\times(\tau\circ\partial_j))_{\#}((\mathrm{id}_{\Delta^p}\times\mathrm{id}_{\Delta^{q-1}})\circ\eta_{\mu'\omega'})\\
&= \sum_{\substack{(p-1,q)\\ \text{shuffles}}}\sum_i\mathrm{sgn}(\mu,\omega)(-1)^i(((\widetilde{\sigma}\circ\partial_i)\times\widetilde{\tau})\circ\eta_{\mu\omega},((\sigma\circ\partial_i)\times\tau)\circ\eta_{\mu\omega})\\
&\quad+(-1)^p\sum_{\substack{(p,q-1)\\ \text{shuffles}}}\sum_j\mathrm{sgn}(\mu',\omega')(-1)^j((\widetilde{\sigma}\times(\widetilde{\tau}\circ\partial_j))\circ\eta_{\mu'\omega'},(\sigma\times\tau\circ\partial_j)\circ\eta_{\mu'\omega'})
\end{aligned}
$$

Hence, we see that $\widetilde{\epsilon}\partial=\partial\widetilde{\epsilon}$. Next, we must justify that we can restrict $\widetilde{\epsilon}$ to $I^{\overline{p}}S_*^{\nu}(X;R)\otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R)$. This can be done because $I^{\overline{p}}S_*^{\nu}(X;R)$ and $I^{\overline{q}}S_*^{\vartheta}(Y;R)$ are submodules of the free R-modules $S_*^{\nu}(X;R)$ and $S_*^{\vartheta}(Y;R)$; respectively. Thus, because R is a principal ideal domain $I^{\overline{p}}S_*^{\nu}(X;R)$ and $I^{\overline{q}}S_*^{\vartheta}(Y;R)$ are also free R-modules. Thus, the functors $I^{\overline{p}}S_*^{\nu}(X;R)\otimes_R$ and $\otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R)$ are both left exact so we have an exact sequence

$$0\to I^{\overline{p}}S_*^{\nu}(X;R)\otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R)\to S_*^{\nu}(X;R)\otimes_R S_*^{\vartheta}(Y;R)$$

Therefore, we are justified in restricting $\widetilde{\epsilon}$ to $I^{\overline{p}}S_*^{\nu}(X;R) \otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R)$. To complete the theorem we need only show that $\widetilde{\epsilon}\left(I^{\overline{p}}S_*^{\nu}(X;R) \otimes_R I^{\overline{q}}S_*^{\vartheta}(Y;R)\right) \subset I^{Q}S_*^{\nu\times\vartheta}(X \times Y;R)$.

To this end if $(\widetilde{\sigma_1}, \sigma_1)$ is $\overline{p}$-allowable and an extended i-simplex and $(\widetilde{\sigma_2}, \sigma_2)$ is $\overline{q}$-allowable and an extended j-simplex, then

$$\widetilde{\epsilon}((\widetilde{\sigma_1}, \sigma_1) \otimes (\widetilde{\sigma_2}, \sigma_2)) = \sum_{\substack{(i,j)\\ \text{shuffles}}} \operatorname{sgn}(\mu, \omega)(\widetilde{\sigma_1} \times \widetilde{\sigma_2} \circ \eta_{\mu\omega}, \sigma_1 \times \sigma_2 \circ \eta_{\mu\omega})$$

However, from the proof of (5, Lemma 5.12) each $(\sigma_1 \times \sigma_2) \circ \eta_{\mu\omega}$ is Q-allowable, hence, $\widetilde{\epsilon}((\widetilde{\sigma_1}, \sigma_1) \otimes (\widetilde{\sigma_2}, \sigma_2))$ is a Q-allowable chain.

For a $\overline{p}$-allowable chain ξ_1 and a $\overline{q}$-allowable chain ξ_2 we have that $\widetilde{\epsilon}(\xi_1 \otimes \xi_2)$ is Q-allowable as $\xi_1 \otimes \xi_2$ can be written as a sum of terms having the form $r(\widetilde{\sigma_1}, \sigma_1) \otimes (\widetilde{\sigma_2}, \sigma_2)$ by using bilinearity of tensor products, where σ_1 is $\overline{p}$-allowable, σ_2 is $\overline{q}$-allowable, and $r \in R$. So our above proof shows that $\widetilde{\epsilon}(\xi_1 \otimes \xi_2)$ is also Q-allowable.

Finally, take chains $\xi_1 \in I^{\overline{p}}S_*^{\nu}(X;R)$ and $\xi_2 \in I^{\overline{q}}S_*^{\vartheta}(Y;R)$. We must show now that $\partial(\widetilde{\epsilon}(\xi_1 \otimes \xi_2))$ is Q-allowable. However, using that $\widetilde{\epsilon}$ is a chain map we have $\partial(\widetilde{\epsilon}(\xi_1 \otimes \xi_2)) = \widetilde{\epsilon}((\partial\xi_1) \otimes \xi_2 \pm \xi_1 \otimes (\partial\xi_2))$. But because ξ_1 is a $\overline{p}$-intersection chain we have ξ_1 and $\partial\xi_1$ are both $\overline{p}$-allowable. Similarly, because ξ_2 is a $\overline{q}$-intersection chain we have ξ_2 and $\partial\xi_2$ are both $\overline{q}$ allowable. Thus, our above proof of allowablility applies to show $\widetilde{\epsilon}((\partial\xi_1) \otimes \xi_2 \pm \xi_1 \otimes (\partial\xi_2))$ is Q-allowable, Thus, $\widetilde{\epsilon}(\xi_1 \otimes \xi_2)$ and $\partial(\widetilde{\epsilon}(\xi_1 \otimes \xi_2))$ are both Q-allowable so that $\widetilde{\epsilon}(\xi_1 \otimes \xi_2)$ is a Q-intersection chain which is what we wanted to show. $\square$

Remark 3.1.4 (Notation)**.** Instead of writing the cross product as $\widetilde{\epsilon}(\xi_1 \otimes \xi_2)$, we will follow custom and often use the notation of writing the cross product of ξ_1 and ξ_2 as $\xi_1 \times \xi_2$

3.2 Künneth theorem where one term is a manifold

Next, we prove a Künneth theorem where one term in the product is a manifold.

Theorem 3.2.1. *Let F be a field and let X be a stratified pseudomanifold with perversity $\overline{p}$ and M an n-manifold with trivial stratification. Filter $X \times M$ as $(X \times M)^i = X^{i-n} \times M$. Let ν and ϑ be covers for X_{reg} and M; respectively. Then the cross product induces an isomorphism*

$$\times : H_*(I^{\overline{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(M;F)) \to H_*(I^{\overline{p}}S_*^{\nu\times\vartheta}(X \times M;F))$$

Proof. We will apply Theorem 2.4.6 to prove the theorem. Let $\mathbf{F}_*$ be the functor $\mathbf{F}_*(U) = H_*(I^{\overline{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F))$ for $U \subset M$ open. Also let $\mathbf{G}_*$ be the functor $\mathbf{G}_*(U) = I^{\overline{p}}H_*^{\nu\times\vartheta}(X \times U;F)$. The natural transformation $\mathbf{F}_* \to \mathbf{G}_*$ is given by the cross product.

We first consider the case U is an open subset of M homeomorphic to $\mathbb{R}^n$ and $\vartheta|U$ is isomorphic to the trivial covering $\mathrm{id}_{\mathbb{R}^n}$. By Proposition 2.3.4 we have that $I^{\overline{p}}H_*^{\nu\times\mathrm{id}_{\mathbb{R}^n}}(X \times \mathbb{R}^n;F)) \cong I^{\overline{p}}H_*^{\nu}(X;F)$. Moreover, this isomorphism is induced by the inclusion $X \cong X \times \{0\} \hookrightarrow X \times \mathbb{R}^n$.

Obviously we have that $S_*^{\mathrm{id}_{\mathbb{R}^n}}(\mathbb{R}^n;F) \cong S_*(\mathbb{R}^n;F)$. From this and the algebraic Künneth theorem we have $H_*(I^{\overline{p}}S_*^{\nu}(X;F)\otimes_F S_*^{\mathrm{id}_{\mathbb{R}^n}}(\mathbb{R}^n;F)) \cong I^{\overline{p}}H_*^{\nu}(X;F)\otimes_F H_*(\mathbb{R}^n;F)) \cong I^{\overline{p}}H_*^{\nu}(X;F)\otimes_F H_0(\mathbb{R}^n;F)$. If we take $H_0(\mathbb{R}^n;F)$ to be generated by the vertex at the origin then the cross product of a simplex with the vertex at the origin is the same as the composition of the simplex with the inclusion map $X \cong X \times \{0\} \hookrightarrow X \times \mathbb{R}^n$. Thus, the cross product $\times : H_*(I^{\overline{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\mathrm{id}_{\mathbb{R}^n}}(\mathbb{R}^n)) \to I^{\overline{p}}H_*^{\nu\times\mathrm{id}_{\mathbb{R}^n}}(X \times \mathbb{R}^n;F))$ is an isomorphism.

Now consider the case U is homeomorphic to $\mathbb{R}^n$ and $\vartheta|U$ is isomorphic to a disjoint union of trivial covers of $\mathbb{R}^n$. In other words, $\vartheta|U$ is isomorphic to the cover $p : \coprod_{\alpha\in J} \mathbb{R}^n \to \mathbb{R}^n$ for some index set J and where p maps each copy of $\mathbb{R}^n$ in the disjoint union $\coprod_{\alpha\in J} \mathbb{R}^n$ trivially onto the base $\mathbb{R}^n$. Then $\nu \times \vartheta$ restricted to $X \times U$ is isomorphic to $\coprod_{\alpha\in J} X \times \mathbb{R}^n$.

In this case $H_*(I^{\bar{p}}S_*^{\nu\times\vartheta}(X \times \mathbb{R}^n; F)) \cong \bigoplus_{\alpha\in J} I^{\bar{p}}H_*^{\nu}(X; F)$ by Proposition 2.1.6 and Proposition 2.3.4. Again, each term in the summand is induced by the inclusion map which as noted above is compatible with the cross product map. On the other hand, we have by the algebraic Künneth theorem and Proposition 2.1.6 that $H_*(I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F S_*^{\vartheta}(\mathbb{R}^n; F)) \cong H_*(I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F \bigoplus_{\alpha\in J} S_*^{\mathrm{id}_{\mathbb{R}^n}}(\mathbb{R}^n; F)) \cong I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F \bigoplus_{\alpha\in J} H_0(\mathbb{R}^n; F) \cong \bigoplus_{\alpha\in J} I^{\bar{p}}H_*^{\nu}(X; F)$. Since cross product is compatible with the inclusion map we again have that the cross product induces an isomorphism $\times : H_*(I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F S_*^{\vartheta}(U; F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U; F)$ in the special case U is homeomorphic to $\mathbb{R}^n$ and $\vartheta|U$ is isomorphic to a trivial (not necessarily connected) cover of $\mathbb{R}^n$.

Next, let $\{U_\alpha\} \subset M$ be a totally ordered set with the ordering given by inclusion and with each U_α open and such that $\mathbf{F}_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism. We will show that $\mathbf{F}_*(\cup_\alpha U_\alpha) \to \mathbf{G}_*(\cup_\alpha U_\alpha)$ is also an isomorphism. Let $U = \cup_\alpha U_\alpha$. We have the following commutative diagram.

$$\begin{tikzcd}
H_*(I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F S_*^{\vartheta}(U_\beta; F)) \arrow[r, "\times"] \arrow[d] & I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U_\beta; F) \arrow[d] \\
H_*(I^{\bar{p}}S_*^{\nu}(X; F) \otimes_F S_*^{\vartheta}(U; F)) \arrow[r, "\times"] & I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U; F)
\end{tikzcd}$$

Let us first show that $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U;F)$ is surjective. Let $\xi \in H_*(I^{\bar{p}}S_*^{\nu\times\vartheta}(X \times U;F))$. Write $\xi = \sum_i f_i(\widetilde{\sigma}_i, \sigma_i)$ where $f_i \in F$. Then since the image of each σ_i is compact it is contained in a finite subcover of $\{X \times U_\alpha\}$ which means because ξ is a finite sum we then have that ξ has base support (the union of the supports $|\sigma_i|$ in the sum above) in a finite subcover of $\{X \times U_\alpha\}$. However, since $\{U_\alpha\}$ is totally ordered this means we can find U_β such that ξ has base support in $X \times U_\beta$. But we have $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U_\beta;F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U_\beta;F)$ is an isomorphism, in particular surjective. Thus, from the above commutative diagram there is also an element of $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F))$ mapping to ξ.

Similarly, we can show $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U;F)$ is injective. Suppose $\xi \in H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F))$ maps to 0. Let us write $\xi = \sum \xi_i \otimes x_i$ where each $x_i \in S_*^{\vartheta}(U;F)$ and $\xi_i \in I^{\bar{p}}S_*^{\nu}(X;F)$. We are assuming that $\sum \xi_i \times x_i = \partial\zeta$ for some $\zeta \in I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U;F)$. Again, by compactness there is some U_β such that each x_i has base support in U_β and ζ has base support in $X \times U_\beta$. Thus, ξ is an element of $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U_\beta;F))$ and ζ is an element of $I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U_\beta;F))$. However, since $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U_\beta;F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U_\beta;F)$ is an isomorphism there is some $\mu \in H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U_\beta;F))$ such that $\partial\mu = \xi$. Thus, from the above commutative diagram ξ must also represent zero as an element of $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F))$.

Hence, we have shown that $H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes_F S_*^{\vartheta}(U;F)) \to I^{\bar{p}}H_*^{\nu\times\vartheta}(X \times U;F)$ is an isomorphism.

Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I^{\overline{p}}S^{\nu}_{*}S_{*}(X;F) \otimes_F S^{\vartheta}_{*}(U \cap V;F) \arrow[r, "\times"] \arrow[d] & I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times (U \cap V);F) \arrow[d] \\
\left(I^{\overline{p}}\widetilde{S}^{\nu}_{*}(X;F) \otimes_F S^{\vartheta}_{*}(U;F)\right) \oplus \left(I^{\overline{p}}S^{\nu}_{*}(X;F) \otimes_F S^{\vartheta}_{*}(V;F)\right) \arrow[r, "\times"] \arrow[d] & I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times U;F) \oplus I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times V;F) \arrow[d] \\
I^{\overline{p}}S^{\nu}_{*}(X;F) \otimes_F \left(S^{\vartheta}_{*}(U;F) + S^{\vartheta}_{*}(V;F)\right) \arrow[r, "\times"] \arrow[d] & I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times U;F) + I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times V;F) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The right column is a Mayer-Vietoris short exact sequence, while the left column is the Mayer-Vietoris sequence

$$0 \to S^{\vartheta}_{*}(U \cap V;F) \to S^{\vartheta}_{*}(U;F) \oplus S^{\vartheta}_{*}(V;F) \to S^{\vartheta}_{*}(U;F) + S^{\vartheta}_{*}(V;F) \to$$

tensored with $I^{\overline{p}}S^{\nu}_{*}(X;F)$. For each open set W, the complexes $S^{\vartheta}_{*}(W;F)$ are free F-modules because F is a field so that exactness is preserved upon tensoring with $I^{\overline{p}}S^{\nu}_{*}(X;F)$. From the proof of Corollary 2.4.5 we have that $H_*(I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times U;F) + I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times V;F)) \cong H_*(I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times (U \cup V);F))$. So we may substitute $H_*(I^{\overline{p}}S^{\nu\times\vartheta}_{*}(X \times (U \cup V);F))$ into the long exact sequence induced by the short exact sequence of the bottom row. Hence, the long exact sequence induced by the bottom row will have the form of a Mayer-Vietoris long exact sequence. As for the top row, note that because M is trivially stratified, we have that for any perversity $\bar{r}$ on M, $I^{\bar{r}}S^{\vartheta}_*(W;F) = S^{\vartheta}_*(W : F)$ (the allowability condition is trivial if there are no singular strata) for each open set $W \subset M$. Thus, we can apply Proposition 2.4.1 so that $S^{\vartheta}_*(U;F) + S^{\vartheta}_*(V;F)$ is quasi-isomorphic to $S^{\vartheta}_*(U \cup V;F)$. Furthermore, because F is a field, we may appeal to the algebraic Künneth theorem and naturality to see that $H_*(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F (S^{\vartheta}_*(U;F) + S^{\vartheta}_*(V;F))) \cong H_*(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F S^{\vartheta}_*(U \cup V;F))$. Thus, the long exact sequence induced by the short exact sequence of the top row also has the form of a Mayer-Vietoris sequence. With these substitutions the cross product induces a commutative diagram of Mayer-Vietoris sequences as can be seen from the commutative diagrams

$$\begin{tikzcd}
H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F S^{\vartheta}_*(U;F)) \oplus H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F S^{\vartheta}_*(V;F)) \arrow[r, "\times \oplus \times"] \arrow[d] & I^{\bar{p}}H^{\nu\times\vartheta}_i(X \times U;F) \oplus I^{\bar{p}}H^{\nu\times\vartheta}_i(X \times V;F) \arrow[d] \\
H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F (S^{\vartheta}_*(U;F) + S^{\vartheta}_*(V;F))) \arrow[r, "\times"] \arrow[d, "\cong"'] & H_i(I^{\bar{p}}S^{\nu\times\vartheta}_*(X \times U;F) + I^{\bar{p}}S^{\nu\times\vartheta}_*(X \times V;F)) \arrow[d, "\cong"'] \\
H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F S^{\vartheta}_*(U \cup V;F)) \arrow[r, "\times"] & I^{\bar{p}}H^{\nu\times\vartheta}_i(X \times (U \cup V);F)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F S^{\vartheta}_*(U \cup V;F)) \arrow[r, "\times"] \arrow[d, "\cong"'] & I^{\bar{p}}H^{\nu\times\vartheta}_i(X \times (U \cup V);F) \arrow[d, "\cong"'] \\
H_i(I^{\bar{p}}S^{\nu}_*(X;F) \otimes_F (S^{\vartheta}_*(U;F) + S^{\vartheta}_*(V;F))) \arrow[r, "\times"] \arrow[d, "\partial_*"'] & H_i(I^{\bar{p}}S^{\nu\times\vartheta}_*(X \times U;F) + I^{\bar{p}}S^{\nu\times\vartheta}_*(X \times V;F)) \arrow[d, "\partial_*"'] \\
H_{i-1}(I^{\bar{p}}S_*(X;F) \otimes_F S^{\vartheta}_*(U \cap V;F)) \arrow[r, "\times"] & I^{\bar{p}}H^{\nu\times\vartheta}_{i-1}(X \times (U \cap V);F)
\end{tikzcd}$$

with the bottom square commuting because the cross product inducing a commutative diagram of short exact sequences above means that the cross product will induce a commutative diagram of long exact sequences. So finally we may appeal to Theorem 2.4.6 to finish the proof.

□

3.3 Künneth theorem where both terms are coned spaces

Before proceeding to our general Künneth theorem, we prove another special case when both terms in the product are cones. Our version of the Künneth theorem follows the argument as in (5, Section 6.4). We will make the stronger assumption of field coefficients as that is all we will need for our purposes. The idea in (5) is to define a perversity $Q_{\overline{p},\overline{q}}$ on $X \times Y$ when we are given perversities $\overline{p}$ on X and $\overline{q}$ on Y by

$$Q_{\overline{p},\overline{q}}(S \times S') = \begin{cases} \overline{p}(S) + \overline{q}(S') + 2 & \text{if } S, S' \text{ are both singular} \\ \overline{p}(S) & \text{if } S \text{ is singular and } S' \text{is regular} \\ \overline{q}(S') & \text{if } S \text{ is regular and } S' \text{ is singular} \\ 0 & \text{if } S, S' \text{ are both regular} \end{cases}$$

Remark 3.3.1. To achieve a Künneth theorem we could also set $Q(S \times S') = \overline{p}(S) + \overline{q}(S')$ or $\overline{p}(S) + \overline{q}(S') + 1$ in the case S and S' are both singular. We will want the above definition though for the purposes of comaptibility with the diagonal map.

Lemma 3.3.2. *Let F be a field and let X, Y be stratified psuedomanifolds of dimension $n-1$ and $m-1$, respectively, and with X_{reg} and Y_{reg} connected. Let $\overline{p}$ and $\overline{q}$ be perversities on cX and cY;respectively. Let ν be a covering for X_{reg} and let ϑ be a covering for Y_{reg}. Let $c\nu$ denote the cover $\nu \times id_{(0,1)}$ of $(cX)_{reg} = X_{reg} \times (0,1)$ and similarly define $c\vartheta$. Assume that for any open $U \subset cX$ and $V \subset cY$ with depth(U) + depth(V) < depth(cX) + depth(cY) that the cross product induces an isomorphism*

$$\times : H_*(I^{\overline{p}}S_*^{c\nu}(U;F) \otimes_F I^{\overline{q}}S_*^{c\vartheta}(V;F)) \to I^{Q_{\overline{p},\overline{q}}}H_*^{c\nu\times c\vartheta}(U \times V;F).$$

Then the cross product induces an isomorphism

$$\times : H_*(I^{\overline{p}}S_*^{c\nu}(cX;F) \otimes_F I^{\overline{q}}S_*^{c\vartheta}(cY;F)) \to I^{Q_{\overline{p},\overline{q}}}H_*^{c\nu\times c\vartheta}(cX \times cY;F).$$

Proof. Throughout the proof we will assume field coefficients; however we suppress them from notation. We will also use write Q to mean $Q_{\overline{p},\overline{q}}$. To compute $I^Q H_*^{c\nu\times c\vartheta}(cX \times cY)$ we will need the construction of the join of two spaces. The join is given by $X*Y = (X\times cY)\cup_{X\times Y}(cX\times Y)$ and from this formula we can filter $X*Y$ in an obvious way. From this filtration we have a filtered homeomorphism of $c(X*Y)$ with $cX \times cY$ (for details see (5, Section 2.9) for more on the join of stratified spaces). Thus, $c\nu \times c\vartheta$ will also be a cover for $(c(X*Y))_{reg}$ because the regular stratum of $c(X*Y)$ is homeomorphic to the regular stratum of $cX \times cY$. Let v be the cone vertex of cX and w be the cone vertex for cY. Let $U_X = cX - \{v\}$ and $V_Y = cY - \{w\}$ and let $U = U_X \times cY$ and $V = cX \times V_Y$. Notice that under the stratified homeomorphism $cX \times cY \cong c(X*Y)$, $v \times w$ maps to the cone vertex of $c(X*Y)$. Hence, by the cone formula Proposition 2.3.3 we have that

$$I^Q H_i^{c\nu\times c\vartheta}(cX\times cY)\cong\begin{cases}0, & \text{if } i\geq m+n-Q(\{v\}\times\{w\})-1,\\ \\ I^Q H_i^{c\nu\times c\vartheta}(U\cup V) & \text{if } i< m+n-Q(\{v\}\times\{w\})-1\end{cases}\tag{5}$$

where we have used that $U\cup V=((cX-\{v\})\times cY)\cup(cX\times(cY-\{w\}))=(cX\times cY)-\{v\}\times\{w\}$.

This suggests breaking our proof of the theorem up into the cases that $i\geq m+n-Q(\{v\}\times\{w\})-1$ and $i<m+n-Q(\{v\}\times\{w\})-1$ and using a Mayer-Vietoris sequence applied to the open sets U and V. Before descending into these cases separately, we first make a few computations. Notice that

$$\begin{aligned}\mathrm{depth}(U)&=\mathrm{depth}(U_X\times cY)\\&=\mathrm{depth}(U_X)+\mathrm{depth}(cY)\\&=\mathrm{depth}(cX-\{v\})+\mathrm{depth}(cY)\\&=\mathrm{depth}(X\times(0,1))+\mathrm{depth}(cY)\\&=\mathrm{depth}(X)+\mathrm{depth}(cY)\\&<\mathrm{depth}(cX)+\mathrm{depth}(cY)\end{aligned}$$

Thus, we have by our assumption in the statement of the theorem that the cross product induces an isomorphism

$$\times : H_*(I^{\overline{p}}S_*^{c\nu}(U_X) \otimes I^{\overline{q}}S_*^{c\vartheta}(cY)) \to I^Q H_*^{c\nu \times c\vartheta}(U). \tag{6}$$

Similarly, we have that $\text{depth}(V) < \text{depth}(cX) + \text{depth}(cY)$ and therefore an isomorphism

$$\times : H_*(I^{\overline{p}}S_*^{c\nu}(cX) \otimes I^{\overline{q}}S_*^{c\vartheta}(V_Y)) \to I^Q H_*^{c\nu \times c\vartheta}(V). \tag{7}$$

We also have that $\text{depth}(U \cap V) = \text{depth}(U_X \times V_Y) < \text{depth}(cX) + \text{depth}(cY)$. So again we have an isomorphism

$$\times : H_*(I^{\overline{p}}S_*^{c\nu}(U_X) \otimes I^{\overline{q}}S_*^{c\vartheta}(V_Y)) \to I^Q H_*^{c\nu \times c\vartheta}(U \cap V). \tag{8}$$

Next, let us apply the algebraic Künneth theorem and the cone formula to the above three isomorphisms. First, we work with (6).

$$\begin{aligned}
I^Q H_i^{c\nu \times c\vartheta}(U) &\cong H_i(I^{\overline{p}}S_*^{c\nu}(U_X) \otimes I^{\overline{q}}S_*^{c\vartheta}(cY)) \\
&\quad \text{by (6)} \\
&\cong \bigoplus_{i=j+k} I^{\overline{p}}H_j^{c\nu}(U_X) \otimes_F I^{\overline{q}}H_k^{c\vartheta}(cY) \\
&\quad \text{by the algebraic Künneth theorem} \\
&\cong \bigoplus_{\substack{i=j+k \\ k<n-1-\overline{q}(w)}} I^{\overline{p}}H_j^{c\nu}(U_X) \otimes_F I^{\overline{q}}H_k^{c\vartheta}(V_Y) \qquad (9) \\
&\quad \text{by Proposition 2.3.3.}
\end{aligned}$$

Similarly, we have that

$$
\begin{aligned}
I^Q H_i^{c\nu\times c\vartheta}(V) &\cong H_i(I^{\overline{p}} S_*^{c\nu}(cX) \otimes I^{\overline{q}} S_*^{c\vartheta}(V_Y)) \\
&\text{by (7)} \\
&\cong \bigoplus_{i=j+k} I^{\overline{p}} H_j^{c\nu}(cX) \otimes_F I^{\overline{q}} H_k^{c\vartheta}(V_Y) \\
&\text{by the algebraic Künneth theorem} \\
&\cong \bigoplus_{\substack{i=j+k \\ j<m-1-\overline{p}(\{v\})}} I^{\overline{p}} H_j^{c\nu}(U_X) \otimes_F I^{\overline{q}} H_k^{c\vartheta}(V_Y) \qquad (10) \\
&\text{by Proposition 2.3.3.}
\end{aligned}
$$

and we also have that

$$
\begin{aligned}
I^Q H_*^{c\nu\times c\vartheta}(U\cap V) &\cong H_i(I^{\overline{p}} S_*^{c\nu}(U_X) \otimes I^{\overline{q}} S_*^{c\vartheta}(V_Y)) \\
&\text{by (8)} \\
&\cong \bigoplus_{i=j+k} I^{\overline{p}} H_j^{c\nu}(U_X) \otimes_F I^{\overline{q}} H_k^{c\vartheta}(V_Y) \qquad (11) \\
&\text{by the algebraic Künneth theorem.}
\end{aligned}
$$

The final computation we will need for our argument is that

$$
\begin{aligned}
H_i(I^{\overline{p}}S_*^{c\nu}(cX)\otimes I^{\overline{q}}S_*^{c\vartheta}(cY)) &\cong \bigoplus_{i=j+k} I^{\overline{p}}H_j^{c\nu}(cX)\otimes I^{\overline{q}}H_k^{c\vartheta}(cY) \\
&\quad \text{by the algebraic Künneth theorem} \\
&\cong \bigoplus_{\substack{i=j+k \\ j<n-\overline{p}(\{v\})-1 \\ k<m-\overline{q}(w)-1}} H_j(I^{\overline{p}}S_*^{\nu}(U_X))\otimes H_k(I^{\overline{q}}S_*^{\vartheta}(V_Y)) \qquad (12) \\
&\quad \text{by Proposition 2.3.3.}
\end{aligned}
$$

We want to show that the cross product

$$
\times : \bigoplus_{i=j+k} I^{\overline{p}}H_j^{c\nu}(cX)\otimes I^{\overline{q}}H_k^{c\vartheta}(cY)\to I^{Q}H_i^{c\nu\times c\vartheta}(cX\times cY) \qquad (13)
$$

induces an isomorphism for all i. As we suggested above we will break this up into the cases $i\geq m+n-Q(\{v\}\times\{w\})-1$ and $i<m+n-Q(\{v\}\times\{w\})-1$. We first deal with the case $i\geq m+n-Q(\{v\}\times\{w\})-1$. From the cone formula (5) above we see that $I^{Q}H_i^{c\nu\times c\vartheta}(cX\times cY)=0$ in this case.

On the other hand, we see that in the case $i \geq m+n-Q(\{v\}\times\{w\})-1$ the sum index in (12) is empty because if $i = j+k$ and $j < n-\overline{p}(\{v\})-1$ and $k < m-\overline{q}(w)-1$ so that $j \leq n-\overline{p}(\{v\})-2$ and $k \leq m-\overline{q}(w)-2$, then

$$\begin{aligned}
i &= j+k \\
&\leq n-\overline{p}(\{v\})-2+m-\overline{q}(w)-2 \\
&= n+m-(\overline{p}(\{v\})+\overline{q}(w)+2)-2 \\
&= n+m-Q(\{v\}\times\{w\})-2.
\end{aligned}$$

Hence, $i < n+m-Q(\{v\}\times\{w\})-1$ which of course can't happen in the case $i \geq m+n-Q(\{v\}\times\{w\})-1$. Thus, the sum in (12) is 0 so that the cross product (13) is trivially an isomorphism in this case.

Next, consider the case $i < m+n-Q(\{v\}\times\{w\})-1$. We will show that in this dimension range the map

$$I^Q H_i^{c\nu\times c\vartheta}(U\cap V) \to I^Q H_i^{c\nu\times c\vartheta}(U) \oplus I^Q H_i^{c\nu\times c\vartheta}(V)$$

coming from the Mayer-Vietoris long exact sequence is injective. However, we have by (9), (10), and (11) that this corresponds to a map

$$\bigoplus_{i=j+k} I^{\overline{p}}H_j^{c\nu}(U_X)\otimes_F I^{\overline{q}}H_k^{c\vartheta}(V_Y) \longrightarrow \left(\bigoplus_{\substack{i=j+k \\ k<n-1-\overline{q}(w)}} I^{\overline{p}}H_j^{c\nu}(U_X)\otimes_F I^{\overline{q}}H_k^{c\vartheta}(V_Y)\right) \oplus \left(\bigoplus_{\substack{i=j+k \\ j<m-1-\overline{p}(\{v\})}} I^{\overline{p}}H_j^{c\nu}(U_X)\otimes_F I^{\overline{q}}H_k^{c\vartheta}(V_Y)\right)$$

Recall we are in the case $i < m+n-Q(\{v\}\times\{w\})-1 = m+n-\overline{p}(\{v\})-\overline{q}(w)-3 < m+n-\overline{p}(\{v\})-\overline{q}(w)-2$. Therefore, if we write $i = j+k$ as in the map above, we must have that $j < n-\overline{p}(\{v\})-1$ or $k < m-\overline{q}(w)-1$. Let G_{jk} denote a summand in (11). By naturality of the cross product and the algebraic Künneth theorem in the case of field coefficients, we have that the map above on the level of direct summands is given by

$$\begin{cases} G_{jk} \xrightarrow{\mathrm{id}} G_{jk}, & \text{if } j < n-\overline{p}(\{v\})-1 \text{ and } k \geq m-\overline{q}(w)-1 \\ G_{jk} \xrightarrow{-\mathrm{id}} G_{jk}, & \text{if } j \geq n-\overline{p}(\{v\})-1 \text{ and } k < m-\overline{q}(w)-1 \\ G_{jk} \xrightarrow{\mathrm{id}\oplus-\mathrm{id}} G_{jk}\oplus G_{jk}, & \text{if } j < n-\overline{p}(\{v\})-1 \text{ and } k < m-\overline{q}(w)-1. \end{cases} \tag{14}$$

Because these are all the possible cases for a summand in (11) in the case that $i < m+n-Q(\{v\}\times\{w\})-1$ we see that in each possible case the map is injective. Hence, we have shown that $I^QH_i^{c\nu\times c\vartheta}(U\cap V) \to I^QH_i^{c\nu\times c\vartheta}(U)\oplus I^QH_i^{c\nu\times c\vartheta}(V)$ is injective whenever $i < m+n-Q(\{v\}\times\{w\})-1$. Moreover, we have the diagram below whose top row is exact by standard Mayer-Vietoris and whose bottom row is exact by the above computation (14). Also, as we are in the case $i < m+n-Q(\{v\}\times\{w\})-1$ from the cone formula (5) we may replace $I^QH_i^{c\nu\times c\vartheta}(U\cup V)$ with $I^QH_i^{c\nu\times c\vartheta}(cX\times cY)$ for these values of i in the Mayer-Vietoris long exact sequence in the top row below.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I^QH_i^{c\nu\times c\vartheta}(U\cap V) \arrow[r] & I^QH_i^{c\nu\times c\vartheta}(U)\oplus I^QH_i^{c\nu\times c\vartheta}(V) \arrow[r] & I^QH_i^{c\nu\times c\vartheta}(cX\times cY) \arrow[r, "0"] & I^QH_{i-1}^{c\nu\times c\vartheta}(U\cap V) \arrow[r] & I^QH_{i-1}^{c\nu\times c\vartheta}(U)\oplus I^QH_{i-1}^{c\nu\times c\vartheta}(V) \\
0 \arrow[r] & \bigoplus\limits_{i=j+k} G_{jk} \arrow[r] \arrow[u, "\times", "\cong"'] & \bigoplus\limits_{\substack{i=j+k\\ j<n-\overline{p}(\{v\})-1}} G_{jk} \oplus \bigoplus\limits_{\substack{i=j+k\\ j<m-\overline{q}(w)-1}} G_{jk} \arrow[r] \arrow[u, "\times\oplus\times", "\cong"'] & \bigoplus\limits_{\substack{i=j+k\\ j<m-\overline{p}(\{v\})-1\\ k<n-\overline{q}(w)-1}} G_{jk} \arrow[r, "0"] \arrow[u, "\times"] & \bigoplus\limits_{i-1=j+k} G_{jk} \arrow[r] \arrow[u, "\times", "\cong"'] & \bigoplus\limits_{\substack{i-1=j+k\\ j<n-\overline{p}(\{v\})-1}} G_{jk} \oplus \bigoplus\limits_{\substack{i=j+k\\ j<m-\overline{q}(w)-1}} G_{jk} \arrow[u, "\times\oplus\times", "\cong"']
\end{tikzcd}$$

The squares not containing the connecting homomorphism commute by naturality of the cross product and naturality of the algebraic Künneth theorem for field coefficientes and the square with the connecting homomorphisms trivially commutes. Therefore, by the five lemma we have

$$\times : \bigoplus_{\substack{i=j+k \\ j<n-\overline{p}(\{v\})-1 \\ k<m-\overline{q}(w)}} G_{jk} \to I^{Q}H_{i}^{c\nu\times c\vartheta}(cX \times cY)$$

is an isomorphism whenever $i < m+n-Q(\{v\}\times\{w\})-1$. But by (13) this isomorphism factors as

$$\bigoplus_{\substack{i=j+k \\ j<n-\overline{p}(\{v\})-1 \\ k<m-\overline{q}(w)}} G_{jk} \cong H_i(I^{\overline{p}}S_*^{c\nu}(cX)\otimes I^{\overline{q}}S_*^{c\vartheta}(cY)) \xrightarrow{\times} I^{Q}H_{i}^{c\nu\times c\vartheta}(cX \times cY).$$

Therefore, because the composition is an isomorphism we have that in the case $i < m+n-Q(\{v\}\times\{w\})-1$

$$H_i(I^{\overline{p}}S_*^{c\nu}(cX)\otimes I^{\overline{q}}S_*^{c\vartheta}(cY)) \xrightarrow{\times} I^{Q}H_{i}^{c\nu\times c\vartheta}(cX \times cY)$$

is an isomorphism. Thus, we have shown the cross product induces an isomorphism in all possible cases. □

Remark 3.3.3. We note that because any cover ν of $(cX)_{reg} = X \times (0,1)$ can be written as $\nu = c\nu_X$ where ν_X is a cover of X_{reg}, we have that the above Lemma holds if we just assumed we had any coverings for $(cX)_{reg}$ and $(cY)_{reg}$.

3.4 Künneth theorem for general products of pseudomanifolds

Finally, we come to our main result of the section.

Theorem 3.4.1 (Künneth)**.** *Let F be a field and let X and Y be stratified pseudomanifolds of dimension n and m with perversities $\bar{p} \leq \bar{t}$ and $\bar{q} \leq \bar{t}$; respectively. Let ν be a cover for X_{reg} and ϑ a cover for Y_{reg}. Then the cross product induces an isomorphism*

$$\times : H_*(I^{\bar{p}}S_*^{\nu}(X;F) \otimes I^{\bar{q}}S_*^{\vartheta}(Y;F)) \to I^{Q}H_*^{\nu\times\vartheta}(X \times Y;F).$$

Proof. We will induct on the depth of $X \times Y$. If $\mathrm{depth}(X \times Y) = 0$, then both X and Y are manifolds and the result follows from Theorem 3.2.1. If $\mathrm{depth}(X \times Y) = 1$ then either X or Y is a manifold so that this case also reduces to Theorem 3.2.1.

So assume $\mathrm{depth}(X \times Y) = N$ and the theorem has been proven for all products whose depth is less than N. We will argue by applying Theorem 2.4.7.

We first prove the special case when $Y = cL \times \mathbb{R}^j$ for some compact stratified pseudomanifold L. Define functors $\mathbf{F}_*$ and $\mathbf{G}_*$ on open subsets of X by $\mathbf{F}_*(U) = H_*(I^{\bar{p}}S_*^{\nu}(U) \otimes I^{\bar{q}}S_*^{\vartheta}(Y))$ and $\mathbf{G}_*(U) = I^{Q}H_*^{\nu\times\vartheta}(U \times Y)$. The cross product induces a natural transformation $\Phi : \mathbf{F}_* \to \mathbf{G}_*$. Now $\mathbf{G}_*$ has a Mayer-Vietoris sequences by Corollary 2.4.5 and $\mathbf{F}_*$ has Mayer-Vietoris sequences from the induced long exact sequence of the short exact sequence obtained from the short exact sequence of $I^{\bar{p}}S_*(\,\cdot\,;F)$ for subsets of X and tensoring with $I^{\bar{q}}S_*^{\vartheta}(Y;F)$. The resulting sequence after tensoring is still exact because F is a field. As in the proof of Theorem 3.2.1 Φ induces a map of Mayer-Vietoris short exact sequences and hence a map of long exact sequences on homology.

The part of the Theorem 2.4.7 concerning direct limits is also satisfied using the same argument of compactness of chains as in Theorem 3.2.1.

Next, we need to verify that whenever $\Phi : \mathbf{F}_*((cL' - \{v\}) \times \mathbb{R}^i) \to \mathbf{G}_*((cL' - \{v\}) \times \mathbb{R}^i)$ is an isomorphism, so is $\Phi : \mathbf{F}_*(cL' \times \mathbb{R}^i) \to \mathbf{G}_*(cL' \times \mathbb{R}^i)$ where L' is a compact stratified pseudomanifold and v is the cone vertex of cL'. Consider open subsets of the form $U_X = U \times \mathbb{R}^i \subset cL \times \mathbb{R}^i$ and $V_Y = V \times \mathbb{R}^j \subset cL \times \mathbb{R}^j$ where U and V are open and with $\text{depth}(U) + \text{depth}(V) < \text{depth}(cL') + \text{depth}(cL)$. Then, we have that

$$\begin{aligned}\text{depth}(U_X \times V_Y) &= \text{depth}(U_X) + \text{depth}(V_Y)\\ &= \text{depth}(U) + \text{depth}(V)\\ &< \text{depth}(cL') + \text{depth}(cL)\\ &\leq N\end{aligned}$$

Thus, we have by our inductive assumption that the cross product induces an isomorphism

$$\times : H_*(I^{\bar{p}}S_*^{\nu}(U_X) \otimes I^{\bar{q}}S_*^{\vartheta}(V_Y)) \to I^Q H_*^{\nu\times\vartheta}(U_X \times V_Y).$$

However, using that U_X is stratified homotopy equivalent to U and similarly V_Y is stratified homotopy equivalent to V, the above isomorphism then gives us an isomorphism

$$\times : H_*(I^{\bar{p}}S_*^{\nu}(U) \otimes I^{\bar{q}}S_*^{\vartheta}(V)) \to I^Q H_*^{\nu\times\vartheta}(U \times V).$$

Therefore, the hypothesis of Lemma 3.3.2 are satisfied so that the cross product induces an isomorphism

$$\times : H_*(I^{\overline{p}}S_*^{\nu}(cL') \otimes I^{\overline{q}}S_*^{\vartheta}(cL)) \to I^Q H_*^{\nu\times\vartheta}(cL' \times cL).$$

However, once again we have that $cL' \times \mathbb{R}^i$ is stratified homotopy equivalent to cL' and similarly $cL \times \mathbb{R}^i$ is stratified homotopy equivalent to cL. Thus, the above isomorphism gives us the desired isomorphism induced by the cross product

$$\times : H_*(I^{\overline{p}}S_*^{\nu}(cL' \times \mathbb{R}^i) \otimes I^{\overline{q}}S_*^{\vartheta}(cL \times \mathbb{R}^j)) \to I^Q H_*^{\nu\times\vartheta}(cL' \times \mathbb{R}^i \times cL \times \mathbb{R}^j).$$

In other words, $\Phi : \mathbf{F}_*(cL' \times \mathbb{R}^i) \to \mathbf{G}_*(cL' \times \mathbb{R}^i)$ is an isomorphism.

Finally, to complete the proof of the special case $Y = cL \times \mathbb{R}^j$, we need to prove that whenever $U \subset X$ is an open subset that is empty or contained in a single stratum of X, $\Phi : \mathbf{F}_*(U) \to \mathbf{G}_*(U)$ is an isomorphism. However, if $U = \emptyset$, then we trivially have an isomorphism $\Phi : \mathbf{F}_*(U) \to \mathbf{G}_*(U)$. Otherwise, U is homeomorphic to a manifold that inherits a trivial stratification from X. Thus, this case follows from Theorem 3.2.1 and the proof of the theorem in this special case follows by Theorem 2.4.7.

Finally, we can prove the general case. We again apply Theorem 2.4.7. Now let $\mathbf{F}_*$ and $\mathbf{G}_*$ functors defined on open subsets of Y that are given by $\mathbf{F}_*(U) = H_*(I^{\overline{p}}S_*^{\nu}(X) \otimes I^{\overline{q}}S_*^{\vartheta}(U))$ and $\mathbf{G}_*(U) = I^Q H_*^{\nu\times\vartheta}(X \times U)$ and again we have a natural transformation Φ given by the cross product. The same reasoning above gives us a commutative diagram of Mayer-Vietoris sequences and the statement concerning direct limits. The condition that Φ : $\mathbf{F}_*((cL - \{v\}) \times \mathbb{R}^i) \to \mathbf{G}_*((cL - \{v\}) \times \mathbb{R}^i)$ is an isomorphism is fulfilled by the inductive hypothesis. We must show that $\Phi : \mathbf{F}_*(cL \times \mathbb{R}^i) \to \mathbf{G}_*(cL \times \mathbb{R}^i)$ is also an isomorphism. However, this is just the special case of the theorem we proved above. The final item we must prove is that if $U \subset Y$ is homeomorphic to Euclidean space and contained within a single stratum, then $\Phi : \mathbf{F}_*(U) \to \mathbf{G}_*(U)$ is an isomorphism. However, this is holds by Theorem 3.2.1. Thus, by Theorem 2.4.7 we have that $\Phi(Y) : F_*(Y) \to G_*(Y)$ is an isomorphism which is what we wanted to show. $\square$

Without too much further effort we can prove a relative version of the Künneth theorem.

Theorem 3.4.2 (Relative Künneth). *Let X and Y be stratified pseudomanifolds with perversities $\overline{p}$ and $\overline{q}$ and open subsets A, B; respectively. Let ν be a covering for X_{reg} and ϑ be a covering for Y_{reg}. Then the cross product induces a quasi-isomorphism*

$$\times : I^{\overline{p}}S^{\nu}_*(X, A; F) \otimes_F I^{\overline{q}}S^{\vartheta}_*(Y, B; F)$$
$$\to I^Q S^{\nu\times\vartheta}_*(X \times Y, A \times Y \cup X \times B; F)$$

Proof. Consider the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I^{\overline{p}}S^{\nu}_*(A;F) \otimes_F I^{\overline{q}}S^{\vartheta}_*(Y;F) \arrow[r] \arrow[d, "\times"'] & I^{\overline{p}}S^{\nu}_*(X;F) \otimes_F I^{\overline{q}}S^{\vartheta}_*(Y;F) \arrow[r] \arrow[d, "\times"'] & I^{\overline{p}}S^{\nu}_*(X,A;F) \otimes_F I^{\overline{q}}S^{\vartheta}_*(Y;F) \arrow[r] \arrow[d, "\times"'] & 0 \\
0 \arrow[r] & I^Q S^{\nu\times\vartheta}_*(A \times Y;F) \arrow[r] & I^Q S^{\nu\times\vartheta}_*(X \times Y;F) \arrow[r] & I^Q S^{\nu\times\vartheta}_*(X \times Y, A \times Y;F) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact since F is a field and the bottom row is the short exact sequence induced by the pair $(X \times Y, A \times Y)$. It follows from the five lemma that $\times : I^{\overline{p}}S^{\nu}_*(X, A; F) \otimes_F I^{\overline{q}}S^{\vartheta}_*(Y; F) \to I^Q S^{\nu\times\vartheta}_*(X \times Y, A \times Y; F)$ is a quasi-isomorphism.

To prove the general case, we first need some homological algebra. Let A_*, B_*, C_*, D_* be chain complexes of left R-modules with $C_* \subset A_* \subset B_*$, $C_* \subset D_* \subset B_*$, and $A_* \cap D_* = C_*$. Then we will show there is a short exact sequence

$$0 \to A_*/C_* \xrightarrow{f} B_*/D_* \xrightarrow{g} B_*/(A_* + D_*) \to 0$$

where f and g are the obvious maps. The fact that these are well defined is elementary (just use that $C_* \subset A_* \subset B_*$ and $C_* \subset D_* \subset B_*$).

To see f is injective we have that if $f(a + C_*) = 0 \in C_*/D_*$ for some $(a + C_*) \in A_*/C_*$, then $f(a + C_*) = a + D_* = D_*$ so that $a \in D_*$. Thus, $a \in A_* \cap D_* = C_*$. Hence, $a + C_* = 0 \in A_*/C_*$.

Now we clearly have $g \circ f = 0$ so that $\mathrm{im}(f) \subset \ker(g)$. Suppose now that $g(b+D_*) = 0 \in B_*/(A_*+D_*)$ for some $(b+D_*) \in B_*/D_*$. Then, $b+D_* = A_*+D_*$ which means that $b = a+d$ for some $a \in A_*$ and $d \in D_*$. Thus, $b+D_* = a+D_*$. Hence, $f(a+C_*) = a+D_* = b+D_*$ so that $\ker(g) \subset \mathrm{im}(f)$. Thus, $\mathrm{im}(f) = \ker(g)$. Finally, the proof that $B_*/D_* \to B_*/(A_* + D_*)$ is surjective is trivial.

Recall that $I^Q S_*^{\nu\times\vartheta}(U;F) \cap I^Q S_*^{\nu\times\vartheta}(V;F) = I^Q S_*^{\nu\times\vartheta}(U \cap V;F)$ by observing the base support of extended simplices. Thus, our above work shows we have an exact sequence

$$0 \longrightarrow I^Q S_*^{\nu\times\vartheta}(X\times B, A\times B;F) \longrightarrow I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y;F) \longrightarrow I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y + X\times B;F) \longrightarrow 0$$

where we define

$$I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y + X\times B;F) := I^Q S_*^{\nu\times\vartheta}(X\times Y;F)/(I^Q S_*^{\nu\times\vartheta}(A\times Y;F) + I^Q S_*^{\nu\times\vartheta}(X\times B;F))$$

Next, we show that we have a quasi-isomorphism $I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y + X\times B;F) \cong I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y \cup X\times B;F)$.

Consider the following commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I^Q S_*^{\nu\times\vartheta}(A\times Y \cup X\times B;F) \arrow[r] & I^Q S_*^{\nu\times\vartheta}(X\times Y;F) \arrow[r] & I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y \cup X\times B;F) \arrow[r] & 0 \\
0 \arrow[r] & I^Q S_*^{\nu\times\vartheta}(A\times Y;F) + I^Q S_*^{\nu\times\vartheta}(X\times B;F) \arrow[r] \arrow[u] & I^Q S_*^{\nu\times\vartheta}(X\times Y;F) \arrow[r] \arrow[u] & I^Q S_*^{\nu\times\vartheta}(X\times Y, A\times Y + X\times B;F) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Now the inclusion map

$$I^Q S_*^{\nu\times\vartheta}(A\times Y;F) + I^Q S_*^{\nu\times\vartheta}(X\times B;F) \hookrightarrow I^Q S_*^{\nu\times\vartheta}(A\times Y \cup X\times B;F)$$

is a quasi-isomorphism from the proof of Corollary 2.4.5.

Thus, from the commutative diagram of long exact sequences induced by the commutative diagram of short exact sequences above and the five lemma, we have that the map $I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y+X\times B;F)\hookrightarrow I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y\cup X\times B;F)$ is a quasi-isomorphism.

Next, notice we have the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I^QS_*^{\nu\times\vartheta}(X\times B, A\times B;F) \arrow[r] & I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y;F) \arrow[r] & I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y+X\times B;F) \arrow[r] & 0 \\
0 \arrow[r] & I^{\bar{p}}S_*^{\nu}(X,A;F)\otimes_F I^{\bar{q}}S_*^{\vartheta}(B;F) \arrow[r] \arrow[u, "\times"] & I^{\bar{p}}S_*^{\nu\times\vartheta}(X,A;F)\otimes_F I^{\bar{q}}S_*^{\vartheta}(Y;F) \arrow[r] \arrow[u, "\times"] & I^{\bar{p}}S_*^{\nu}(X,A;F)\otimes_F I^{\bar{q}}S_*^{\vartheta}(Y,B;F) \arrow[r] \arrow[u, "\times"] & 0.
\end{tikzcd}$$

The top row is exact by above and the bottom row is exact because it is the tensor product of exact sequences of the pair (Y,B) with $I^{\bar{p}}S_*^{\nu}(X,A;F)$ and because F is a field. However, the first two vertical cross product maps are quasi-isomorphisms by our above work, and therefore, the third vertical cross product map is also a quasi-isomorphism from the induced commutative diagram of long exact sequences of homology and the five lemma. Thus, by above we have a composition of quasi-isomorphisms

$$I^{\bar{p}}S_*^{\nu}(X,A;F)\otimes_F I^{\bar{q}}S_*^{\vartheta}(Y,B;F)\xrightarrow{\times} I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y+X\times B;F)\hookrightarrow I^QS_*^{\nu\times\vartheta}(X\times Y, A\times Y\cup X\times B;F)$$

which is our desired quasi-isomorphism. $\square$

3.5 Properties of the cross product

Finally, to end the section we have that the cross product satisfies commutativity, associativity, and stability under boundary maps. All of these come down to properties of shuffle products to which we refer the reader to [4] where these results are proven for ordinary intersection homology

Theorem 3.5.1 (Commutativity). *Let F be a field and let (X, A) and (Y, B) be stratified pseudomanifold pairs with ν and ϑ coverings for X_{reg} and Y_{reg}; respectively. Let $\overline{p}$ and $\overline{q}$ be perversities. Then the diagram below commutes:*

$$\begin{array}{ccc} I^{\overline{p}}S^{\nu}_*(X,A;F) \otimes I^{\overline{q}}S^{\vartheta}_*(Y,B;F) & \xrightarrow{\times} & I^{Q_{\overline{p},\overline{q}}}S^{\nu\times\vartheta}_*((X,A)\times(Y,B);F) \\ \Big\downarrow \mathbf{t} & & \Big\downarrow t \\ I^{\overline{q}}S^{\vartheta}_*(Y,B;F) \otimes I^{\overline{p}}S^{\nu}_*(X,A;F) & \xrightarrow{\times} & I^{Q_{\overline{q},\overline{p}}}S^{\vartheta\times\nu}_*((Y,B)\times(X,A);F) \end{array}$$

where $\mathbf{t}$ is the signed map swapping factors and t is the map swapping coordinates. More explicitly, let $x \in I^{\overline{p}}S^{\nu}_i(X, A; F)$ and let $y \in I^{\overline{q}}S^{\vartheta}_j(Y, B; F)$. Then,

$$t(x \times y) = (-1)^{i \cdot j} y \times x$$

Proof. From bilinearity of the cross product, it suffices to prove the theorem on the level of extended simplices. To this end, let $(\widetilde{\sigma}, \sigma)$ be an i-dimensional extended simplex in X and let $(\widetilde{\tau}, \tau)$ be an j-dimensional extended simplex in Y. Notice that $t \circ (\sigma \times \tau) = (\tau \times \sigma) \circ t$ as functions (here we abuse notation and use t to also denote the coordinate swap map $\Delta^i \times \Delta^j \to \Delta^j \times \Delta^i$).

Thus we have the equalities

$$
\begin{aligned}
t(\widetilde{\epsilon}((\widetilde{\sigma},\sigma)\otimes(\widetilde{\tau},\tau))) &= (t\circ(\widetilde{\sigma}\times\widetilde{\tau}), t\circ(\sigma\times\tau))_{\#}\epsilon(\mathrm{id}_{\Delta^i}\otimes\mathrm{id}_{\Delta^j}) \\
&= (\widetilde{\tau}\times\widetilde{\sigma}, \tau\times\sigma)_{\#}(t_{\#}\epsilon(\mathrm{id}_{\Delta^i}\otimes\mathrm{id}_{\Delta^j})) \\
&= (\widetilde{\tau}\times\widetilde{\sigma}, \tau\times\sigma)_{\#}((-1)^{i\cdot j}\epsilon(\mathrm{id}_{\Delta^j}\otimes\mathrm{id}_{\Delta^i})) \\
&= (-1)^{i\cdot j}(\widetilde{\tau}\times\widetilde{\sigma}, \tau\times\sigma)_{\#}\epsilon(\mathrm{id}_{\Delta^j}\otimes\mathrm{id}_{\Delta^i})) \\
&= (-1)^{i\cdot j}\widetilde{\epsilon}((\widetilde{\tau},\tau)\otimes(\widetilde{\sigma},\sigma))
\end{aligned}
$$

where the equality $t_{\#}\epsilon(\mathrm{id}_{\Delta^i}\otimes\mathrm{id}_{\Delta^j}) = (-1)^{i\cdot j}\epsilon(\mathrm{id}_{\Delta^j}\otimes\mathrm{id}_{\Delta^i})$ follows by closely looking at orientations and triangulations from the shuffle products. For a proof see (5, Lemma 5.20) where commutativity of the cross product is proven for ordinary intersection homology. □

In a similar fashion to the proof above, we can also prove the associativity of the cross product. The proof again is reduced to the level of extended simplices and then appealing to the associativity of the cross product of ordinary singular simplices and shuffle products. See (5, Lemma 5.19) for these details.

Theorem 3.5.2 (Associativity). *Let $(X,A),(Y,B),(Z,C)$ be ordered triples of stratified pseudomanifolds with coverings ν,ϑ,γ of their regular points; respectively. Let $\bar{p},\bar{q}$, and $\bar{r}$ also be respective perversities. We will denote $Q_{\bar{p},Q_{\bar{q},\bar{r}}}$ (which is the same as $Q_{Q_{\bar{p},\bar{q}},\bar{r}}$ as may be verified directly from the definition in Section 3.3) by $Q_{\bar{p},\bar{q},\bar{r}}$. Then we have the commutative diagram*

$$\begin{tikzcd}
I^{\overline{p}}S_*^{\nu}(X,A;F)\otimes I^{\overline{q}}S_*^{\vartheta}(Y,B;F)\otimes I^{\overline{r}}S_*^{\gamma}(Z,C;F) \arrow[r, "\epsilon\otimes 1"] \arrow[d, "1\otimes\epsilon"'] & I^{Q_{\overline{p},\overline{q}}}S_*^{\nu\times\vartheta}((X,A)\times(Y,B);F)\otimes I^{\overline{r}}S_*^{\gamma}(Z,C;F) \arrow[d, "\epsilon"'] \\
I^{\overline{p}}S_*^{\nu}(X,A;F)\otimes I^{Q_{\overline{q},\overline{r}}}S_*^{\vartheta\times\gamma}((Y,B)\times(Z,C);F) \arrow[r, "\epsilon"] & I^{Q_{\overline{p},\overline{q},\overline{r}}}S_*^{\nu\times\vartheta\times\gamma}((X,A)\times(Y,B)\times(Z,C);F)
\end{tikzcd}$$

On the level of homology, if $x \in I^{\overline{p}}H_*^{\nu}(X,A;F)$, $y \in I^{\overline{q}}H_*^{\vartheta}(Y,B;F)$, and $z \in I^{\overline{r}}H_*^{\gamma}(Z,C;F)$, then we have the equality

$$x \times (y \times z) = (x \times y) \times z.$$

For the next property of cross products, we remark that under our sign conventions we define $(1 \otimes \partial_*)(x \otimes y) = (-1)^{|x|} x \otimes (\partial_* y)$. We refer the reader to (5, Lemma 5.2.1) for the proof of stability in the ordinary intersection homology case.

Theorem 3.5.3 (Stability)**.** *Let (X, A) and (Y, B) be pairs of stratified pseudomanifolds with perversities $\overline{p}$ and $\overline{q}$; respectively. Let ν and ϑ be covers for the respective regular subsets of X and Y. Then we have the following commutative diagram*

$$\begin{tikzcd}
H_*(I^{\overline{p}}S_*^{\nu}(X,A;F) \otimes I^{\overline{q}}S_*^{\vartheta}(Y,B;F)) \arrow[r, "\times"] \arrow[ddd, "1 \otimes \partial_*"'] & I^{Q_{\overline{p},\overline{q}}}H_*^{\nu\times\vartheta}((X,A)\times(Y,B);F) \arrow[d, "\partial_*"'] \\
& I^{Q_{\overline{p},\overline{q}}}H_*^{\nu\times\vartheta}((X\times B)\cup(A\times Y);F) \arrow[d] \\
& I^{Q_{\overline{p},\overline{q}}}H_*^{\nu\times\vartheta}((X\times B)\cup(A\times Y), A\times Y;F) \\
H_*(I^{\overline{p}}S_*^{\nu}(X,A;F) \otimes I^{\overline{q}}S_*^{\vartheta}(B;F)) \arrow[r, "\times"] & I^{Q_{\overline{p},\overline{q}}}H_*^{\nu\times\vartheta}(X\times B, A\times B;F). \arrow[u]
\end{tikzcd}$$

4 Finitely branched coverings of pseudomanifolds

The main result of this section will be that every finitely branched covering of a pseudomanifold is again a pseudomanifold. We will consider branched coverings in the topological sense of Fox (4).

4.1 Spreads and the existence and uniqueness of their completions

In this subsection we recall a few definitions and results in (4) we will need. The fundamental concept is that of a spread, and the definition below is originally due to Fox (4, Section 1).

Definition 4.1.1. A map $g : Y \to Z$ between T_1 spaces is a *spread* if and only if the connected components of pre-images of open sets of Z form a basis for the topology of Y. A point $z \in Z$ which is evenly covered by g is called an *ordinary point*. The set of ordinary points will be denoted Z_o (which is clearly an open subset of Z). The points which are not ordinary are called *singular points*. If $X \subset Y$ is locally connected and $f = g|_X$, then we will say that $g : Y \to Z$ is an *extension* of the spread $f : X \to Z$.

Remark 4.1.2. If we wish to emphasize that the ordinary points Z_o above are ordinary with respect to the map f, we will write Z_o^f for the set of ordinary points under a spread $f : Y \to Z$.

We note that in Fox's original paper (4) he assumes that the space Z above is locally connected, however, as the author in (1) points out, this assumption is unnecessary.

Example 4.1.3. (a) Any covering space is a spread

(b) If X is a stratified pseudomanifold and ν is the data associated to a covering of X_{reg}, then $E(\nu) \to X$ is a spread.

Definition 4.1.4. A spread $g : Y \to Z$ is said to be *complete* if and only if for every point $z \in Z$, the following condition is satisfied: If to every open neighborhood W of z there is a selected component V of $g^{-1}(W)$ in such a way that $V \subset V'$ whenever $W \subset W'$, then $\cap_W V$ is non-empty (and consequently a point because Y is locally connected and T_1). If the condition above is satisfied for a particular point $z_0 \in Z$ we say that g is *complete over* z_0. Thus, a spread is complete if it is complete over every point $z_0 \in Z$.

Remark 4.1.5. In the above definition the condition of being complete over a point $z_0 \in Z$ may be trivially satisfied. That is, if there is no way to form a collection of connected components V of $g^{-1}(W)$ so that $V \subset V'$ whenever $W \subset W'$, then the condition that every intersection $\cap_W V$ is non-empty is vacuously true.

Definition 4.1.6. An extension $g : Y \to Z$ of a spread $f : X \to Z$ will be called a *completion* of f if g is complete and X is dense and locally connected in Y[6].

.

Example 4.1.7. If X is a stratified pseudomanifold, then its normalization $\mathbf{n} : X^N \to X$ is the completion of $X_{reg} \hookrightarrow X$. To see that $X^N \to X$ is a complete spread we proceed by induction on the depth of X. If $\mathrm{depth}(X) = 0$, then X is a manifold and the normalization is given by the identity map $\mathrm{id}_X : X \to X$ which is clearly a complete spread. If $\mathrm{depth}(X) > 0$, we recall the construction of normalizations (16, Definition 2.2).

[6]A space X is locally connected in Y if there is a basis $\mathcal{V}$ of Y such that $V \cap X$ is connected for every basic open set $V \in \mathcal{V}$

There Padilla considers a family of normalizations of links $\{\mathbf{n}_L : \mathbf{L}^N \to L\}$ of some fixed atlas of X which satisfies the commutative diagram

$$\begin{tikzcd} \mathbf{n}^{-1}(V) \arrow[r, "\mathbf{n}"] \arrow[d, "\varphi^N"'] & V \arrow[d, "\varphi"'] \\ (cL)^N \times \mathbb{R}^i \arrow[r, "\mathbf{n}_0"] & cL \times \mathbb{R}^i \end{tikzcd}$$

where

- $V \subset X$ is open and $\varphi : V \to cL \times \mathbb{R}^i$ is a chart
- $(cL)^N = \coprod_j cK_j$ where $K_1, \ldots, K_m$ are the connected components of L^N
- φ^N is a homeomorphism
- $\mathbf{n}_0([p,r]_j, u) = ([\mathbf{n}_L(p), r], u)$ where $[p,r]_j \in cK_j$.

By induction on depth, the maps $\mathbf{n}_L : L^N \to L$ are complete spreads. Moreover, by a similar proof as in Lemma 4.3.1 we have that $(cL)^N \times \mathbb{R}^i \to cL \times \mathbb{R}^i$ is a complete spread. Thus, $\mathbf{n}|_{\mathbf{n}^{-1}(V)} : \mathbf{n}^{-1}(V) \to V$ is a complete spread. Since the open sets V form a basis of X we have that $\mathbf{n} : X^N \to X$ will also be a complete spread. Thus, by Proposition 4.1.8 normalizations may be seen as the unique completion of $X_{reg} \hookrightarrow X$.

Fox (4) shows that every spread has a completion and that it is unique up to an appropriate equivalence. As pointed out in (1), the original assumption that Z be locally connected is unnecessary. Existence and uniqueness is also shown in (1, Theorem 6.2, Corollary 7.4). We state this as a proposition below.

Proposition 4.1.8. *Let $f : X \to Z$ be a spread. Then f has a completion $g : Y \to Z$. Moreover, the completion is unique in the following sense. If $g' : Y' \to Z$ is any other completion of $f : X \to Z$, then there exists a homeomorphism $\phi : Y \to Y'$ such that $g'\phi = g$ and $\phi|_X = id_X$.*

We will also need an extension result for maps between spreads and their completions. The proof may be found in (1, Theorem 7.2).

Proposition 4.1.9. *Let $g_i : Y_i \to Z_i$ be completions of spreads $f_i : X_i \to Z_i$, $i = 1, 2$. Let $h : X_1 \to X_2$ and $\ell : Z_1 \to Z_2$ be maps such that $f_2 \circ h = \ell \circ f_1$. Then h extends uniquely to a map $k : Y_1 \to Y_2$ such that $g_2 \circ k = \ell \circ g_1$.*

4.2 Branched covers as completions of pre-branched covers

In this subsection we show how branched coverings may be formulated in the language of spreads. We first define the term *unbranched cover* below. We have borrowed this language from Fox's use of the term in his paper (4) (although he is not the first to use the phrase either).

Definition 4.2.1. A surjective spread $f : X \to Z$ with Z connected is called an *unbranched covering* if X is connected and there are no singular points ($Z = Z_o$). In other words, $f : X \to Z$ is just a covering in the usual sense.

Definition 4.2.2. We define a *pre-branched covering* to be a spread $f : X \to Z$ with X connected and such that $f(X)$ satisfies the following properties.

(a) $f(X) = Z_o$

(b) Z_o is connected, dense, and locally connected in Z

Notice from the definition above that a pre-branched cover $f : X \to Z$ factors as $X \twoheadrightarrow Z_o \hookrightarrow Z$ with $X \twoheadrightarrow Z_o$ an unbranched cover. We have incorporated pre-branched covers into our treatment because we wish to emphasize that the completion of a spread $f : X \to Z$ is sensitive to the target space Z.

Next, we give the definition of a branched cover. The definition is due to Fox (4, Section 5). Although Fox does not use the phrase pre-branched in his definition of a branched cover as we do, the definition is exactly the same. As mentioned above, we have chosen to use the term pre-branched cover merely to emphasize the codomain under which the completion takes place.

Definition 4.2.3. Let $g : Y \to Z$ be a spread. Let Z_o denote the set of ordinary points and let $X = g^{-1}(Z_o)$. We say $g : Y \to Z$ is a *branched covering* if

(i) Z_o is connected, dense, and locally connected in Z

(ii) X is connected

(iii) g is the completion of the associated pre-branched covering $g|_X : X \to Z$

The following proposition motivates the terminology "pre-branched covering". Notice the difference between this proposition and the definition of a branched covering is that the definition of a branched covering required the completion of a specific pre-branched covering. This proposition says the completion of any pre-branched covering is a branched covering.

Proposition 4.2.4. *Let $g : Y \to Z$ be the completion of a pre-branched covering $f : X \to Z$. Then $g : Y \to Z$ is a branched covering.*

Proof. Recall that $Z_o^f \subset Z$ denotes the set of ordinary points with respect to the map $f : X \to Z$, and Z_o^g the set of ordinary points with respect to the map $g : Y \to Z$. Since g is an extension of f it is evident that $Z_o^f \subset Z_o^g$.

Because $Z_o^f \subset Z_o^g$ and Z_o^f is dense, connected, and locally connected in Z we have that Z_o^g is also dense, connected, and locally connected in Z (1, Lemma 9.5). Thus, condition (i) of Definition 4.2.3 is satisfied.

Next, let $X' = g^{-1}(Z_0)$. Then because $X \subset X' \subset Y$ and X is dense, connected, and locally connected in Y we have that X' is also dense, connected, and locally connected in Y (1, Lemma 9.5). In particular, condition (ii) of Definition 4.2.3 is satisfied.

Thus, we have shown that $g|_{X'} : X' \to Z$ is a pre-branched covering. Lastly, we need to show that $g : Y \to Z$ is the completion of the pre-branched covering $g|_{X'} : X' \to Z$. We have already shown that X' is dense and locally connected in Y and the map $g : Y \to Z$ is complete by assumption. Thus, $g : Y \to Z$ is a branched covering. $\square$

We will use the following result in the proof of our main theorem for this section. We refer the reader to (1, Theorem 10.4) for a proof.

Proposition 4.2.5. *Let $g : Y \to Z$ be a branched covering. Let W be a connected open set of Z and let C be a connected component of $g^{-1}(W)$. Then $g : C \to W$ is a branched covering.*

The next definition is given in (4, Section 5).

Definition 4.2.6. Let $g : Y \to Z$ be a branched covering associated to an unbranched covering $g|_X : X \to Z_0$ and let $y \in Y$. Let $z = g(y)$ and W be a connected open set containing the point z such that $W_0 = W \cap Z_0$ is also connected. Let V be the connected component of $g^{-1}(W)$ containing the point y and let $U = V \cap X$. Then $g|_V : V \to W$ is the branched covering associated to the unbranched covering $g|_U : U \to W_o$ by the above proposition. Let $j(y, W)$ denote the index of the unbranched covering $g|_U : U \to W_0$ (that is the cardinality of a fiber which is well defined by connectedness of U and W_0). Notice $j(y, W) \leq j(y, W')$ if $W \subset W'$. This may be seen from the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "g|_U"] \arrow[d, hook] & W_o \arrow[d, hook] \\
U' \arrow[r, "g|_{U'}"] & W'_o
\end{tikzcd}$$

from which we can see that $(g|_U)^{-1}(\{z\}) \subset (g|_{U'})^{-1}(\{z\}$. We then define $j(y)$ to be the minimum of these numbers (if the cardinality of a fiber is always infinite we define $j(y) = \infty$) and say $j(y)$ is the *index of branching* of the point y. A branched cover $g : Y \to Z$ such that for every $y \in Y$, $j(y) < \infty$ is called a *finitely branched covering.*

4.3 Branched covers of stratified pseudomanifolds

The next lemma will provide the inductive step in the proof of our main theorem of this section.

Lemma 4.3.1. *Assume Y and Z are compact spaces. If $g : Y \to Z$ is a branched covering, then $cg : cY \to cZ$ is a branched covering.*

Proof. We first verify that $cg : cY \to cZ$ is a spread. It suffices to show that the connected components of pre-images of a basis of cZ gives a basis for cY. Specifically, we will consider open sets of the form $V \times (a, b)$ where $V \subset Z$ is open and $0 < a < b < 1$ along with open sets of the form $c_\epsilon Z = Z \times [0, \epsilon)/\sim$.

Let $V \subset Z$ be open and consider $V_{a,b} = V \times (a, b)$ where $0 < a < b < 1$. Then $(cg)^{-1}(V_{a,b}) = g^{-1}(V) \times (a, b)$. Thus, a connected component of $g^{-1}(V_{a,b})$ will have the form $C \times (a, b)$ where C is a connected component of $g^{-1}(V)$. Conversely, if C is a connected component of $g^{-1}(V)$ for some open V, then $C \times (a, b)$ will be a connected component of $(cg)^{-1}(V_{a,b})$. Thus, because the connected components of pre-images of open sets of Y give a basis of Y and the collection of open sets of the form (a, b) is a basis of $(0, 1)$, we see that the connected components of pre-images of open sets of the form $V \times (a, b)$ gives a basis of $Y \times (0, 1)$.

Next, consider an open set of the form $c_\epsilon Z$. Then $(cg)^{-1}(c_\epsilon Z) = c_\epsilon Y$, which is connected. Thus, as ϵ varies we see that connected components of pre-images of open sets of the form $c_\epsilon Z$ are cofinal among open sets containing the cone vertex of cY (compactness of Y is used here). This, combined with our work above proves that connected components of pre-images of open sets of cZ provides a basis for the topology of cY. Hence, $cg : cY \to cZ$ is a spread.

Let $Y' = g^{-1}(Z_o)$ and let $g' = g|_{Y'}$. We will show that $g' \times \mathrm{id} : Y' \times (0,1) \to cZ$ is a pre-branched covering. Now $\mathrm{im}(g' \times \mathrm{id}) = Z_o \times (0,1) \subset (cZ)_o^{g' \times \mathrm{id}}$. On the other hand, $(cZ)_o^{g' \times \mathrm{id}} \subset \mathrm{im}(g' \times \mathrm{id})$ since any element that is evenly covered by $g' \times \mathrm{id}$ is necessarily in the image of $g \times \mathrm{id}$. Thus, we have $\mathrm{im}(g' \times \mathrm{id}) = (cZ)_o^{g' \times \mathrm{id}}$ so that condition (a) of Definition 4.2.2 is satisfied. To see condition (b), we have from the equality $(cZ)_o^{g' \times \mathrm{id}} = Z_o \times (0,1)$ that $(cZ)_o^{g' \times \mathrm{id}}$ is a dense subset of cZ since Z_o is dense in Z by our assumption that $g : Y \to Z$ is a branched covering. We also have that $Z_o \times (0,1)$ is connected and locally connected in cZ since Z_o is connected and locally connected in Z. Hence, we have shown that $g' \times \mathrm{id} : Y' \times (0,1) \to cZ$ is a pre-branched covering.

By Proposition 4.2.4 if we can show that $g' \times \mathrm{id} : Y' \times (0,1) \to cZ$ is a pre-branched covering, and that $cg : cY \to cX$ is its completion, we will be done. By our assumption that $g : Y \to Z$ is a branched covering we have that Y' is dense, connected, and locally connected. Thus, $Y' \times (0,1)$ is connected and dense and locally connected in cY.

The only item left to verify in Definition 4.2.3. is that the map $cg : cY \to cZ$ is a complete spread. The map $cg : cY \to cZ$ is complete over each $z \in cZ - \{v\}$ where v is the cone vertex because the map $g \times \mathrm{id} : Y \times (0,1) \to Z \times (0,1)$ is complete. To see that $g \times \mathrm{id} : Y \times (0,1) \to Z \times (0,1)$ is complete, let $(z, r) \in Z \times (0,1)$ and assume that for every open set $W \ni (z, r)$ we have selected a connected component V of $(g \times \mathrm{id})^{-1}(W)$ in such a way that $V \subset V'$ whenever $W \subset W'$. Consider open sets of the form $U \times (-\epsilon + r, r + \epsilon)$ where $U \subset Z$ is open. Then $(g \times \mathrm{id})^{-1}(U \times (-\epsilon + r, r + \epsilon)) = g^{-1}(U) \times (-\epsilon + r, r + \epsilon)$ so that the selected connected component of $(g \times \mathrm{id})^{-1}(U \times (-\epsilon + r, r + \epsilon))$ must have the form $C \times (-\epsilon + r, r + \epsilon)$ where C is a connected component of $g^{-1}(U)$. Moreover, $C \subset C'$ if $U \subset U'$. Thus, since g is a complete spread we have that $\cap_U C \neq \emptyset$. Now open sets of the form $U \times (-\epsilon + r, r + \epsilon)$ are cofinal among open sets of (z, r) so we have that

$$\begin{aligned} \bigcap_W V &= \bigcap_{U \times (-\epsilon + r, r+\epsilon)} C \times (-\epsilon + r, r + \epsilon) \\ &= \left(\bigcap_U C \right) \times \{r\} \end{aligned}$$

which is nonempty.

The only point left to consider is then the cone vertex. Now for $\epsilon > 0$ the collection of open sets $c_\epsilon Z$ is cofinal among open sets containing v. What's more, we have that $(cg)^{-1}(c_\epsilon Z) = c_\epsilon Y$ which is connected. Thus, for each $c_\epsilon Z$ there is only one connected component of $(cg)^{-1}(c_\epsilon Z)$ to choose and we have that

$$\bigcap_{\epsilon > 0} c_\epsilon Y = \{w\}$$

where w is the cone vertex of cY. Thus, $cg : cY \to cZ$ is complete. Hence, we have shown that $cg : cY \to cZ$ is the completion of a pre-branched cover so that it is a branched cover. □

Proposition 4.3.2. *Let Z be a connected normal stratified pseudomanifold and let ν denote the data associated to an unbranched covering of Z_{reg} (so $E(\nu)$ is connected). Then $E(\nu) \to Z$ is a pre-branched covering, and therefore, its completion is a branched covering.*

Proof. We have that $E(\nu)$ is connected by assumption. The only non-trivial item to verify is condition (b) of Definition 4.2.2. We have that Z_{reg} is connected and locally connected in Z because Z is normal and connected and Z_{reg} is dense by definition of stratified pseudomanifolds. Thus, $E(\nu) \to Z$ is a pre-branched covering and by Proposition 4.2.4. its completion is a branched covering. $\square$

Finally, we arrive at the main theorem of this section. The theorem says that in the situation of Proposition 4.3.2, if the branched covering $g : Y \to Z$ is a finitely branched covering, then Y is also a connected normal stratified pseudomanifold.

Theorem 4.3.3. *Let Z be a connected normal stratified n-dimensional pseudomanifold and let ν be the data associated to an unbranched covering of Z_{reg}. Let $g : Y \to Z$ be the branched covering associated to the pre-branched covering $E(\nu) \to Z$. If $g : Y \to Z$ is a finitely branched covering, then Y is a connected normal stratified n-dimensional pseudomanifold with stratification induced by the filtration $Y^i = g^{-1}(Z^i)$ where Z^i is the filtration inducing the stratification of Z.*

Proof. Y is connected because it contains the dense connected subspace $E(\nu)$. Moreover, Y is Hausdorff by (1, Corollary 2.8). We proceed by an induction on depth(Z). If depth$(Z) = 0$ we are done because Z is a manifold and there are no singular points so that $Y = E(\nu)$ is a manifold with trivial stratification. So assume depth$(Z) > 0$ and the theorem holds for all normal connected stratified pseudomanifolds with depth less than depth(Z).

Let

$$Z^n \supset Z^{n-1} = Z^{n-2} \supset \cdots \supset Z^0 \supset Z^{-1} = \emptyset$$

be the filtration inducing the stratification of the stratified pseudomanifold Z. Recall we use the notation $Z_k = Z^k - Z^{k-1}$. We will show that the filtration on Y given by $Y^k = g^{-1}(Z^k)$ gives Y the structure of a stratified pseudomanifold. First, notice that $Y^n - Y^{n-2} = g^{-1}(Z^n - Z^{n-2}) = g^{-1}(Z_{reg}) = E(\nu)$ which is dense because $g : Y \to Z$ is the completion of the pre-branched cover $E(\nu) \to Z$.

Assume that $y \in Y_k$ and let $z = g(y)$. Then $z \in Z_k$ so that there exists an open neighborhood $U \ni z$ and a filtration preserving homeomorphism $\psi : U \to cL \times \mathbb{R}^k$ such that $\psi(z) = (v, 0)$ where v denotes the cone vertex and where L is a connected compact stratified $(n-k-1)$-dimensional pseudomanifold.

Let $W = cL \times \mathbb{R}^k$ and let C denote the connected component of $g^{-1}(W)$ containing the point y. Then by Proposition 4.2.5, $\psi \circ g|_C : C \to W$ is a branched covering. Note that $Y_{reg} \cap C$ is connected, dense in C, and locally connected in C because C is an open connected set and because Y_{reg} is dense and locally connected in Y (1, Lemma 7.1 and Lemma 7.5). Thus, $\psi \circ g|_C ; C \to W$ is the completion of the pre-branched cover $\psi \circ g|_{Y_{reg} \cap C} : Y_{reg} \cap C \to cL \times \mathbb{R}^k$. However, by standard covering space theory the associated unbranched covering $\psi \circ g|_{Y_{reg} \cap C} : Y_{reg} \cap C \to L_{reg} \times (0,1) \times \mathbb{R}^k$ fits into the commutative diagram

$$\begin{tikzcd}
Y_{reg} \cap C \arrow[r, "\psi \circ g|_{Y_{reg} \cap C}"] \arrow[d, "\cong"'] & L_{reg} \times (0,1) \times \mathbb{R}^k \arrow[d, "="'] \\
\widetilde{L_{reg}} \times (0,1) \times \mathbb{R}^k \arrow[r, "q \times \mathrm{id}"] & L_{reg} \times (0,1) \times \mathbb{R}^k
\end{tikzcd}$$

for some unbranched covering map $q : \widetilde{L_{reg}} \to L_{reg}$. Notice that $\widetilde{L_{reg}}$ is connected because $Y_{reg} \cap C$ is connected. Because Z is normal, we have that L is also a connected normal stratified pseudomanifold. Let $\widetilde{q} : \widetilde{L} \to L$ denote the branched covering map associated to the pre-branched covering map $\widetilde{L_{reg}} \to L$ guaranteed to exist by Proposition 4.2.4.

By the assumption that $g : Y \to Z$ is a finitely branched cover, we can assume without loss of generality that $g|_C : C \to W$ has an associated unbranched covering that is finitely fibered. In particular, by our diagram above this means that $\widetilde{q} : \widetilde{L} \to L$ is a finitely branched covering. Now, $\mathrm{depth}(L) < \mathrm{depth}(Z)$ which means by our inductive hypothesis that $\widetilde{L}$ is a connected normal stratified $(n-k-1)$-dimensional pseudomanifold with stratification induced by the filtration $q^{-1}(L^k)$. What's more, because $\widetilde{q} : \widetilde{L} \to L$ has associated unbranched cover that is finitely fibered and because L is compact, we have that $\widetilde{L}$ is also compact.

By Lemma 4.3.1 we have that $cq : c\widetilde{L} \to cL$ is a branched covering. Moreover, the map is filtration preserving by the equalities $(cq)^{-1}((cL)^i) = (cq)^{-1}(cL^{i-1}) = c\widetilde{L}^{i-1} = (c\widetilde{L})^i$. Thus, the map $cq \times \mathrm{id} : c\widetilde{L} \times \mathbb{R}^k \to cL \times \mathbb{R}^k$ is also a filtration preserving branched covering. However, by uniqueness of completions we have that there is a homeomorphism $\phi : c\widetilde{L} \times \mathbb{R}^k \to C$ such that $\psi \circ g|_C \circ \phi = cq \times \mathrm{id}$.

Next, we show that the map ϕ is filtration preserving. This follows from the equalities

$$
\begin{aligned}
\phi^{-1}(C \cap Y^i) &= \phi^{-1}(C \cap g^{-1}(Z^i)) \\
&= \phi^{-1}((g|_C)^{-1}(U \cap Z^i)) \\
&= \phi^{-1}((g|_C)^{-1}(\psi^{-1}(cL^{i-k-1} \times \mathbb{R}^k))) \\
&= (\psi \circ g|_C \circ \phi)^{-1}(cL^{i-k-1} \times \mathbb{R}^k) \\
&= (cq \times \mathrm{id})^{-1}(cL^{i-k-1} \times \mathbb{R}^k) \\
&= c\widetilde{L}^{i-k-1} \times \mathbb{R}^k.
\end{aligned}
$$

Thus, we have shown that Y is a connected stratified pseudomanifold. Moreover, it is normal because the links $\widetilde{L}$ above were connected so that each point in Y has a connected link which is enough to guarantee that any link in Y is connected (5, Remark 2.68).

$\square$

Branched covers for stratified normal pseudomanifolds which are not necessarily connected extend in the obvious way by reducing to connected components.

Proposition 4.3.4. *Let Z be a normal connected stratified pseudomanifold with ν a locally finite unbranched regular cover. Let $g : Y \to Z$ be the completion of the pre-branched cover $E(\nu) \to Z$. If S is a stratum of Z and $y_1, y_2 \in g^{-1}(S)$, then $j(y_1) = j(y_2)$.*

Proof. First we show that if $z \in Z$ and $y_1, y_2 \in g^{-1}(\{z\})$, then $j(y_1) = j(y_2)$. Choose an connected open subset $W \subset Z$ such that the y_1 connected component of $g^{-1}(W)$, call it C_1, does not intersect the y_2 connected component of $g^{-1}(W)$, call it C_2. This may be done because connected components of pre images of open sets form a basis for Y and Y is a pseudomanifold, in particular Hausdorff, by Theorem 4.3.3. Then $C_i \cap g^{-1}(W_{reg}) \to W_{reg}$ are unbranched covers of W_{reg} so that if $z' \in W_{reg}$, there exists $y_i' \in C_i \cap g^{-1}(W_{reg})$, $i = 1, 2$, such that $g(y_i') = z'$. Since ν is a regular cover of Z_{reg}, there exists a deck transformation $\phi : E(\nu) \to E(\nu)$ such that $\phi(y_1') = y_2'$. Because ϕ is a homeomorphism and $C_1 \cap W_{reg}$ is connected, we must have $\phi(C_1 \cap g^{-1}(W_{reg})) = C_2 \cap g^{-1}(W_{reg})$. In particular, this means that $j(y_1, W) = j(y_2, W)$ for all W so that $j(y_1) = j(y_2)$.

Thus, for $z \in Z$, we define $j(z) = j(y)$ where $y \in g^{-1}(\{z\})$. By above this is well defined.

Next, let us show that $j(z_1) = j(z_2)$ for any $z_1, z_2 \in S$ where S is a stratum of Z. Fix $z_0 \in S$ and let $W \subset Z$ be an open set such that $z_0 \in W$ and $j(z_0) = j(z_0, W)$ (such a choice may be made by the well-ordering principle and because we assumed all branching indexes were finite). By perhaps shrinking W we may assume without loss of generality that W is stratified homeomorphic to $cL \times \mathbb{R}^i$ for some compact connected stratified pseudomanifold L. Let C denote a connected component of $g^{-1}(W)$. From the proof of Theorem 4.3.3 we have the commutative diagram below.

$$\begin{tikzcd} C \arrow[r, "g|_C"] \arrow[d, "\widetilde{\psi}"'] & W \arrow[d, "\psi"'] \\ c\widetilde{L} \times \mathbb{R}^i \arrow[r, "cq \times \mathrm{id}"] & cL \times \mathbb{R}^i \end{tikzcd}$$

where ψ and $\widetilde{\psi}$ are stratified homeomorphisms with $\psi(z_0) = (v, 0)$ and $q : \widetilde{L} \to L$ is the branched cover that is the completion of an unbranched cover $\widetilde{L_{reg}} \to L$. Note that

$j(z_0) = |q^{-1}(\{u\})|$ for any $u \in L_{reg}$. Next, assume $z' \in W \cap S$. Let $W' \subset Z$ be an open set such that $j(z') = j(z', W')$. Now from the proof of (5, Lemma 2.39) there exists $\epsilon > 0$ and an open disk $D^i \subset \mathbb{R}^i$ such that $c_\epsilon L \times D^i \subset \psi(W \cap W')$. Let $V_\epsilon = \psi^{-1}(c_\epsilon L \times D^i)$. Then $V_\epsilon \subset W'$ so that we must have $j(z', V_\epsilon) = j(z', W') = j(z')$. We will show that $j(z', V_\epsilon) = j(z_0)$ as well.

Now an easy computation shows that $\widetilde{\psi}\,(g^{-1}(V_\epsilon) \cap C) = c_\epsilon \widetilde{L} \times D^i$. In particular, since ψ is a homeomorphism, $g^{-1}(V_\epsilon) \cap C$ is connected. Moreover, because $g^{-1}(V_\epsilon) \subset g^{-1}(W)$, we have that $g^{-1}(V_\epsilon) \cap C$ is a connected component of $g^{-1}(V_\epsilon)$. Restricting to Y_{reg} then yields the commutative diagram below.

$$\begin{tikzcd}
g^{-1}(V_\epsilon) \cap C \cap Y_{reg} \arrow[r, "g"] \arrow[d, "\widetilde{\psi}"'] & (V_\epsilon)_{reg} \arrow[d, "\psi"'] \\
\widetilde{L}_{reg} \times (0, \epsilon) \times D^i \arrow[r, "q \times \mathrm{id} \times \mathrm{id}"] & L_{reg} \times (0, \epsilon) \times D^i
\end{tikzcd}$$

For the sake of brevity we have omitted the notation that we are restricting the maps from the previous commutative diagram. So we see that $j(z', V_\epsilon) = |q^{-1}(u)|$ for any $u \in L_{reg}$ which means from our previous remarks that we have $j(z', V_\epsilon) = j(z_0)$. Hence, $j(z_0) = j(z')$ for any $z' \in W \cap S$. Thus, we have shown that the branching index j defines a locally constant function on the stratum S so that from connectedness of S we have j is constant on S. This is what we wanted to show. $\square$

Remark 4.3.5. Our proof above may be modified to say a bit more. If $z \in Z$ and W is any distinguished neighborhood of z and if V is an open set containing z such that $j(z) = j(z, V)$, then by our arguments above there is an open set $W' \subset V$ such that $j(z, W') = j(z, V)$. However, since $W' \subset V$ we have that $j(z) \leq j(z, W') \leq j(z, V) = j(z)$. Thus, $j(z) = j(z, W)$.

Corollary 4.3.6. *If Z is a compact connected normal stratified pseudomanifold and $g : Y \to Z$ is finitely branched regular covering, that is, $g|_{Y_{reg}} : Y_{reg} \to Z_{reg}$ is a regular ordinary covering, then $\{j(z) : z \in Z\}$ is a finite set.*

Proof. By Proposition 4.3.4 j is constant along the strata of Z, but if Z is compact then Z has finitely many strata (5, Lemma 2.37). □

4.4 Equivalence of intersection homology of finitely branched coverings with intersection homology for coverings of the regular stratum

In this subsection we show that if $g : Y \to Z$ is a finitely branched covering of a stratified pseudomanifold, then the intersection homology of the normal stratified pseudomanifold Y is isomorphic to the intersection homology for the associated unbranched cover $g|_{g^{-1}}(Z_{reg}) : g^{-1}(Z_{reg}) \to Z_{reg}$ that we defined in Section 1.1.

Definition 4.4.1. For a branched cover $g : Y \to Z$, we define the group of *branched deck transformations* to be the group of homeomorphisms $\gamma : Y \to Y$ such that $g\gamma = g$.

We begin by proving a lemma that the group of branched deck transformations of a branched cover is isomorphic to the group of deck transformations of the associated unbranched cover.

Lemma 4.4.2. *Let Z be a connected normal stratified pseudomanifold and let ν denote the data associated to a connected cover of Z_{reg} such that $E(\nu) \to Z$ is finitely branched. Let $g : Y \to Z$ denote the induced branched cover. We let*

- *$\widetilde{\pi}$ denote the group of branched deck transformations $\gamma : Y \to Y$ such that $g \circ \gamma = g$,*
- *π denote the group of deck transformations $\gamma : E(\nu) \to E(\nu)$ such that $g|_{E(\nu)} \circ \gamma = g|_{E(\nu)}$.*

Then $\widetilde{\pi} \cong \pi$.

Proof. Consider the map $\phi : \pi \to \widetilde{\pi}$ given by $\phi(\gamma) = \widetilde{\gamma}$ where $\widetilde{\gamma} : Y \to Y$ is the unique extension of $\gamma : E(\nu) \to E(\nu)$ which exists by Proposition 4.1.9. This is a homomorphism because we have that $\widetilde{\gamma_1 \circ \gamma_2} = \widetilde{\gamma_1} \circ \widetilde{\gamma_2}$ by uniqueness of extensions of lifts. The map is injective because if $\widetilde{\gamma_1} = \widetilde{\gamma_2}$, then $\gamma_1 = \widetilde{\gamma_1}|_{E(\nu)} = \widetilde{\gamma_2}|_{E(\nu)} = \gamma_2$. Finally, the map is surjective because if $\gamma : Y \to Y$ is a deck transformation, and $\widetilde{\gamma|_{E(\nu)}}$ is the extension of $\gamma|_{E(\nu)}$ to a lift of g, then $\widetilde{\gamma|_{E(\nu)}} = \gamma$ by uniqueness of lifts. Hence, $\phi : \widetilde{\pi} \to \pi$ is an isomorphism. □

If $\overline{p}$ is a perversity defined on the the strata of X and $g : Y \to X$ is a finitely branched covering we will abuse notation and also use $\overline{p}$ to denote a perversity defined on Y in the following way. If S is a stratum of X and $S' \subset g^{-1}(S)$ is a stratum of Y we let $\overline{p}(S') = \overline{p}(S)$.

Theorem 4.4.3. *Let R be a commutative ring with unity. Let Z be a connected normal stratified pseudomanifold with perversity $\overline{p} \leq \bar{t}$ and let ν denote the data associated to a connected cover of Z_{reg} such that $E(\nu) \to Z$ is finitely branched. Let $g : Y \to Z$ denote the induced branched cover. For an open set $W \subset Z$ we use the notation $\widetilde{W} = g^{-1}(W)$.*

Let $V \subset U \subset Z$ be open sets. Then we have chain isomorphisms $I^{\overline{p}}S_*(\widetilde{U}, \widetilde{V}; R) \cong I^{\overline{p}}S_*^{\nu}(U, V; R)$. Moreover, this is an isomorphism of $R[\pi]$-modules (we identify $\widetilde{\pi}$ with π via the isomorphism in Lemma 4.4.2).

Proof. We first consider the case $V = \emptyset$. Let $S_*^{\overline{p}}(\widetilde{U}; R)$ denote the submodule generated by $\overline{p}$-allowable singular simplices and let ${}^{\overline{p}}S_*^{\nu}(U; R) \subset S_*^{\nu}(U; R)$ denote the submodule generated by $\overline{p}$-allowable simplices.

Consider the map $\Phi^{\overline{p}} : S_k^{\overline{p}}(\widetilde{U}; R) \to {}^{\overline{p}}S_k^{\nu}(U; R)$ given by $\tau \mapsto (\tau|_{\tau^{-1}(Y_{reg})}, g \circ \tau)$ and extended linearly. This map is well-defined because $\tau|_{\tau^{-1}(Y_{reg})} = \tau|_{(g\circ\tau)^{-1}(X_{reg})} : (g\circ\tau)^{-1}(Z_{reg}) \to E(\nu)$ which is evidently a lift of $g \circ \tau|_{(g\circ\tau)^{-1}(Z_{reg})} : (g \circ \tau)^{-1}(Z_{reg}) \to Z_{reg}$. We also need to verify that $(\tau|_{\tau^{-1}(Y_{reg})}, g \circ \tau)$ is $\overline{p}$-allowable whenever τ is $\overline{p}$-allowable for the map to be well-defined. To see this notice we can write $g^{-1}(S) = \cup_\alpha S'_\alpha$ where S'_α is a stratum of Y. Then we have that

$$
\begin{aligned}
(g \circ \tau)^{-1}(S) &= \tau^{-1}(g^{-1}(S)) \\
&= \tau^{-1}(\cup_\alpha S'_\alpha) \\
&= \cup_\alpha \tau^{-1}(S'_\alpha) \\
&\subset \cup_\alpha (k - \operatorname{codim}(S'_\alpha) + \overline{p}(S'_\alpha) \text{ skeleton of } \Delta^k)
\end{aligned}
$$

However, from our proof that $g : Y \to Z$ being a finitely branched covering implies Y is a stratified pseudomanifold we see that $\mathrm{codim}(S'_\alpha) = \mathrm{codim}(S)$ and we have that $\overline{p}(S'_\alpha) = \overline{p}(S)$ by definition. Hence, we see that

$$(g \circ \tau)^{-1}(S) \subset k - \mathrm{codim}(S) + \overline{p}(S) \text{ skeleton of } \Delta^k$$

so that $\Phi(\tau)$ is $\overline{p}$-allowable.

To see the map $\Phi^{\overline{p}}$ is injective suppose $\Phi^{\overline{p}}(\tau) = \Phi^{\overline{p}}(\tau')$. Then $\tau|_{\tau^{-1}(Y_{reg})} = \tau'|_{\tau'^{-1}(Y_{reg})}$. However, because $\overline{p} \leq \overline{t}$ we have that $\tau^{-1}(Y_{reg}) \supset \mathrm{int}(\Delta^k)$ and $\tau'^{-1}(Y_{reg}) \supset \mathrm{int}(\Delta^k)$ which means τ and τ' agree on the dense subset $\mathrm{int}(\Delta^k)$, and hence, $\tau = \tau'$ by continuity.

Next, we show $\Phi^{\overline{p}}$ is also surjective. Consider a $\overline{p}$-allowable extended k-simplex $(\widetilde{\sigma}, \sigma)$ where $\widetilde{\sigma} : \sigma^{-1}(Z_{reg}) \to Y_{reg}$ is a lift of $\sigma : \Delta^k \to Z$ and σ is $\overline{p}$-allowable. Because $\overline{p} \leq \overline{t}$ we have that $\sigma^{-1}(X_{reg}) \supset \mathrm{int}(\Delta^k)$. Notice that the spread $\sigma^{-1}(X_{reg}) \hookrightarrow \Delta^k$ has a unique completion to $\mathrm{id} : \Delta^k \to \Delta^k$ because $\sigma^{-1}(Z_{reg}) \supset \mathrm{int}(\Delta^k)$. Thus, we may apply Proposition 4.1.9 with $Y_1 = \Delta^k$, $Y_2 = Y$, $X_1 = \sigma^{-1}(Z_{reg})$, $X_2 = Z_{reg}$, $h = \widetilde{\sigma}$, and $\ell = \sigma$. Thus, $\widetilde{\sigma}$ extends uniquely to a map $\tau_\sigma : \Delta^k \to Y$ which is a lift of σ. If we can show τ_σ is $\overline{p}$-allowable we will be done as in this case we obviously have $\Phi(\tau_\sigma) = (\widetilde{\sigma}, \sigma)$. To this end, let S' be a stratum of Y and let S be the stratum of X such that $S' \subset g^{-1}(S)$. Then we have

$$
\begin{aligned}
\tau_\sigma^{-1}(S') &\subset \tau_\sigma^{-1}(g^{-1}(S)) \\
&= (g \circ \tau_\sigma)^{-1}(S) \\
&= \sigma^{-1}(S) \\
&\subset k - \operatorname{codim}(S) + \overline{p}(S) \text{ skeleton of } \Delta^k \\
&= k - \operatorname{codim}(S') + \overline{p}(S') \text{ skeleton of } \Delta^k
\end{aligned}
$$

where the third line follows because τ_σ is a lift of σ, the fourth line follows because σ is $\overline{p}$-allowable, and the last line follows as before from our proof of Theorem 4.3.3. Thus, we have shown $\Phi^{\overline{p}}$ induces a bijection between groups of $\overline{p}$-allowable simplices, and therefore, is an isomorphism.

In a similar fashion if we let $\widehat{S}_k(\widetilde{U}; R)$ denote the submodule generated by k-simplices $\tau : \Delta^k \to \widetilde{U}$ such that $\tau^{-1}(Y_{reg}) \supset \operatorname{int}(\Delta^k)$, then we have an isomorphism of R-modules $\widehat{\Phi} : \widehat{S}_k(\widetilde{U}; R) \to \widehat{S}_k^\nu(U; R)$. Consider the diagram below

$$
\begin{tikzcd}
S_*^{\overline{p}}(\widetilde{U}; R) \arrow[r, "\Phi"] \arrow[d, "\partial"'] & {}^{\overline{p}}S_*^\nu(U; R) \arrow[d, "\partial"'] \\
\widehat{S}_*(\widetilde{U}; R) \arrow[r, "\widehat{\Phi}"] & \widehat{S}_*^\nu(U; R).
\end{tikzcd}
$$

An easy verification as in previous proofs we made shows the diagram commutes. Thus Lemma 2.2.10 implies $\Phi^{\overline{p}}$ restricts to an isomorphism of chain complexes : $I^{\overline{p}}S_*(\widetilde{U}; R) \to I^{\overline{p}}S_*^\nu(U; R)$.

Next, we show that $\Phi^{\overline{p}}$ is an isomorphism of $R[\pi]$-modules. Let $\gamma \in \pi$, $\xi \in I^{\overline{p}}S_*(\widetilde{U};R)$, and write $\xi = \sum_i r_i\tau_i$ where $r_i \in R$. Recall that $\gamma \cdot \xi := \widetilde{\gamma} \cdot \xi$ where $\widetilde{\gamma}$ is the extension of γ.We also note that because $\widetilde{\gamma}$ is an extension of γ and a lift of g we have that $\widetilde{\gamma}$ is filtration preserving, and therefore, $(\widetilde{\gamma}\circ\tau_i)^{-1}(Y_{reg}) = \tau_i^{-1}(\widetilde{\gamma}^{-1}(Y_{reg})) = \tau_i^{-1}(\gamma^{-1}(Y_{reg})) = (\gamma\tau_i)^{-1}(Y_{reg})$. Moreover, if $x \in (\gamma\tau)^{-1}(Y_{reg})$, then $(\widetilde{\gamma} \circ \tau_i)(x) = \widetilde{\gamma}(\tau_i(x))$. But $\tau_i(x) \in Y_{reg}$, which means that $\widetilde{\gamma}(\tau_i(x)) = \gamma(\tau_i(x))$ because $\widetilde{\gamma}$ is an extension of γ. Thus, we have shown that

$$
\begin{aligned}
\Phi^{\overline{p}}(\gamma \cdot \xi) &= \Phi^{\overline{p}}\left(\sum_i r_i\widetilde{\gamma} \circ \tau_i\right) \\
&= \sum_i r_i(\widetilde{\gamma} \circ \tau_i|_{(\widetilde{\gamma}\circ\tau_i)^{-1}(Y_{reg})}, g \circ \widetilde{\gamma} \circ \tau_i) \\
&= \sum_i r_i(\gamma \circ \tau_i|_{(\gamma\circ\tau_i)^{-1}(Y_{reg})}, g \circ \tau_i) \\
&= \sum_i r_i\gamma \cdot (\tau_i|_{\tau_i^{-1}(Y_{reg})}, g \circ \tau_i) \\
&= \gamma \cdot \Phi^{\overline{p}}(\xi)
\end{aligned}
$$

where the third to last line follows because $\widetilde{\gamma}$ is a deck transformation so that $g \circ \widetilde{\gamma} = g$.

Finally, if $V \neq \emptyset$ we have that $I^{\bar{p}}S_*(\widetilde{U}, \widetilde{V}; R) \cong I^{\bar{p}}S^{\nu}_*(U, V; R)$ because

$$\begin{aligned} I^{\bar{p}}S_*(\widetilde{U}, \widetilde{V}; R) &= I^{\bar{p}}S_*(\widetilde{U}; R)/I^{\bar{p}}S_*(\widetilde{V}; R) \\ &\cong \Phi^{\bar{p}}(I^{\bar{p}}S_*(\widetilde{U}; R))/\Phi^{\bar{p}}(I^{\bar{p}}S_*(\widetilde{V}; R)) \\ &= I^{\bar{p}}S^{\nu}_*(U; R)/I^{\bar{p}}S_*(V; R) \\ &= I^{\bar{p}}S^{\nu}_*(U, V; R). \end{aligned}$$

□

5 Fundamental classes with twisted coefficients

In this section we show how to construct fundamental classes with twisted coefficients. Our result generalizes (11, Theorem 5.8) to cases when the regular set of a stratified pseudomanifold is possibly non-orientable.

5.1 Intersection homology with coefficients twisted by the orientation character

For ordinary manifolds there are two equivalent approaches to defining twisted homology. One approach is to use the chain complex $R^w \otimes_{R[\pi]} S_*(\widetilde{M}; R)$ where R is commutative ring with unity, $\widetilde{M}$ is the universal cover of the manifold M, $\pi = \pi_1(M)$, and R^w is the left $R[\pi]$-module induced by the orientation character $w : \pi \to \mathrm{Aut}(\mathbb{Z})$. Another approach is to use the orientation double cover $\widehat{M} \to M$ and the chain complex $R^\tau \otimes_{R[\mathbb{Z}_2]} S_*(\widehat{M}; R)$ where

R^τ is the right $R[\mathbb{Z}_2]$-module induced by the isomorphism $\tau : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$. We also use τ to denote the non-trivial deck transformation involution $\widehat{M} \to \widehat{M}$. One may then show these two chain complexes are isomorphic (14, Examples 3H.2, 3H.3). We opt to generalize the latter approach to pseudomanifolds.

Let $\mathfrak{o}$ denote the data corresponding to the orientation cover of X_{reg}. We denote the $\overline{p}$-intersection chains of $R^\tau \otimes_{R[\mathbb{Z}_2]} S_*^{\mathfrak{o}}(X;R)$ by $I^{\overline{p}}S_*(X;R^\tau)$. We will also be concerned with open subsets $U \subset X$. The $\overline{p}$-intersection chains in $I^{\overline{p}}S_*(X;R^\tau)$ which have base support in U correspond to $\overline{p}$-intersection chains of $R^\tau \otimes_{R[\mathbb{Z}_2]} S_*^{i^*\mathfrak{o}}(U;R)$. However, for an open set U we have that $i^*\mathfrak{o} \cong \mathfrak{o}_U$, where $\mathfrak{o}_U$ is the orientation covering of $U_{reg} = X_{reg} \cap U$. This is true because any two sheeted covering of a manifold with orientation reversing deck transformation is equivalent to the orientation cover. Hence, $\overline{p}$-intersection chains of $R^\tau \otimes_{R[\mathbb{Z}_2]} S_*^{i^*\mathfrak{o}}(U;R)$ are $\overline{p}$-intersection chains of $R^\tau \otimes_{R[\mathbb{Z}_2]} S_*^{\mathfrak{o}_U}(U;R)$. Thus, our notation is well-defined upon restriction to open subsets.

The benefit of working with $I^{\overline{p}}S_*(X;R^\tau)$ is that this makes sense even whenever X_{reg} is not connected, whereas we only defined twisted coefficients whenever X_{reg} is connected (one could extend the definition of our algebraic construction of twisted coefficients for intersection homology, but we opt to avoid the notational headache). We also note that when X_{reg} is connected, we have the equality

$$I^{\overline{p}}S_*(X;R^\tau) = I^{\overline{p}}\widetilde{S}_*^{\mathfrak{o}}(X;R^\tau).$$

We next show that $I^{\overline{p}}S_*(X;R^\tau) \cong I^{\overline{p}}S_*(X;o(X;R))$ where $o(X;R)$ is the R-orientation local coefficient system defined by $o(X;R) := R^\tau \times_{\mathbb{Z}_2} E(\mathfrak{o})$ which we recall is $R^\tau \times E(\mathfrak{o})$ modded out by the relations $(r,\widetilde{x}) \sim (r\cdot\tau, \tau\cdot\widetilde{x})$ for all $\widetilde{x} \in E(\mathfrak{o})$ and all $r \in R^\tau$.

Proposition 5.1.1. *For every stratified pseudomanifold X, $I^{\overline{p}}S_*(X;R^\tau) \cong I^{\overline{p}}S_*(X;o(X;R))$.*

Proof. The proof follows just as in Theorem 2.2.11. Define a map $\Phi : R \otimes_R \widehat{S}^{\mathfrak{o}}_*(X;R) \to \widehat{S}_*(X;o(X;R))$ by $\Phi(r \otimes (\widetilde{\sigma},\sigma)) = [(r,\widetilde{\sigma})]\sigma$ where $[(r,\widetilde{\sigma})] : \sigma^{-1}(X_{reg}) \to \mathfrak{o}(X;R)$ is the map defined by $[(r,\widetilde{\sigma})](x) = [(r,\widetilde{\sigma}(x))]$ for all $x \in \sigma^{-1}(X_{reg})$. Just as in Theorem 2.2.11 we have that $\left(R \otimes_R \widehat{S}^{\mathfrak{o}}_*(X;R)\right)/\ker(\Phi) \cong R^\tau \otimes_{\mathbb{Z}_2} \widehat{S}^{\mathfrak{o}}_*(X;R)$. So we have an injective map $\widehat{\Phi} : R^\tau \otimes_{\mathbb{Z}_2} \widehat{S}^{\mathfrak{o}}_*(X;R) \to \widehat{S}_*(X;o(X;R))$ and similarly we have an injective map $\widehat{\Phi} : R^\tau \otimes_{\mathbb{Z}_2} \widehat{S}^{\mathfrak{o}}_*(X;R) \to \widehat{S}_*(X;o(X;R))$. We then have that the boundary map commutes with these isomorphisms and the image of a $\overline{p}$-allowable chain is $\overline{p}$-allowable. Hence, just as in Theorem 2.2.11 we have $I^{\overline{p}}S_*(X;R^\tau) \cong I^{\overline{p}}S_*(X;o(X;R))$. $\square$

Remark 5.1.2. For an inclusion $i : U \hookrightarrow X$ of an open set and a local coefficient system $p : \mathcal{E} \to X$, one has that the local coefficient systems $i^*\mathcal{E} \to \mathcal{E}$ and $p^{-1}(U) \to U$ are equivalent. Applying this to $o(X;R) \to X$ we have that $i^*o(X;R) = R^\tau \times_{\mathbb{Z}_2} E(i^*\mathfrak{o})$. However, $i^*\mathfrak{o} = \mathfrak{o}_U$ where $\mathfrak{o}_U$ is the orientation cover of U_{reg}. Thus, $i^*o(X;R) \cong o(U;R)$. Combining this with the previous proposition we therefore have the commutative diagram

$$\begin{tikzcd}
I^{\overline{p}}S_*(U;R^\tau) \arrow[r, phantom, "\cong"] \arrow[d] & I^{\overline{p}}S_*(U;o(U;R)) \arrow[d] \\
I^{\overline{p}}S_*(X;R^\tau) \arrow[r, phantom, "\cong"] & I^{\overline{p}}S_*(X;o(X;R))
\end{tikzcd}$$

The next corollary follows by the previous proposition, the above remark, and the same arguments use to prove Proposition 2.3.3, Proposition 2.3.4, Corollary 2.4.5, and Corollary 2.4.3. Notice without the previous proposition this would not follow immediately from the results of Section 2. This is because we only proved versions of Proposition 2.3.3 and Proposition 2.3.4 for twisted coefficients defined using connected regular covers.

We will also consider relative homology $I^{\overline{p}}H_*(X,U;R^\tau)$. This is defined as

$$H_*\left(I^{\overline{p}}S_*(X;R^\tau)/I^{\overline{p}}S_*(U;R^\tau)\right).$$

However, we need to be careful because twisted coefficients is defined using tensor products so we need to verify that the map $I^{\overline{p}}S_*(U;R^\tau) \to I^{\overline{p}}S_*(X;R^\tau)$ is an injection. However, ${}^{\overline{p}}S_*^{\mathfrak{o}_U}(U;R)$ and ${}^{\overline{p}}S_*^{\mathfrak{o}_X}(X;R)$ are free $R[\mathbb{Z}_2]$-modules. Thus, the exact sequence $0 \to {}^{\overline{p}}S_*^{\mathfrak{o}_U}(U;R) \to {}^{\overline{p}}S_*^{\mathfrak{o}_X}(X;R)$ (recall $\mathfrak{o}_U = i^*\mathfrak{o}_X$) remains exact after applying the functor $R^\tau \otimes_{R[\mathbb{Z}_2]}$. Thus,

$$0 \to R^\tau \otimes_{R[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o}_U}(U;R) \to R^\tau \otimes_{R[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o}_X}(X;R)$$

is exact. Therefore, we have the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & R^{\tau} \otimes_{R[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o}_U}(U;R) \arrow[r] & {}^{\overline{p}}S_*^{\mathfrak{o}_X}(X;R) \\
0 \arrow[r] & I^{\overline{p}}S_*(U;R^{\tau}) \arrow[r] \arrow[u] & I^{\overline{p}}S_*(X;R^{\tau}) \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

where the vertical arrows are exact because $I^{\overline{p}}S_*(U;R^{\tau})$ and $I^{\overline{p}}S_*(X;R^{\tau})$ are by definition submodules of $R^{\tau} \otimes_{R[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o}_U}(U;R)$ and ${}^{\overline{p}}S_*^{\mathfrak{o}_X}(X;R)$; respectively. Therefore, commutativity of the diagram and exactness of the top row and the left column imply the bottom row must also be exact.

Thus, our definition of $I^{\overline{p}}H_*(X,U;R^{\tau})$ is justified and what's more we have the short exact sequence

$$0 \to I^{\overline{p}}S_*(U;R^{\tau}) \to I^{\overline{p}}S_*(X,U;R^{\tau}) \to I^{\overline{p}}S_*(X,U;R^{\tau}) \to 0$$

which induces a long exact sequence on homology. This is part 5. of the corollary below.

Corollary 5.1.3. *Let X be any $(n-1)$-dimensional stratified pseudomanifold, let U, V be open subsets of X, and let $C \subset U$ be a closed subspace.*

1. *Cone Formula:*

$$I^{\overline{p}}H_j(cX;R^\tau) \cong \begin{cases} 0 & \text{if } j \geq n-1-\overline{p}(\{v\}) \\ I^{\overline{p}}H_j(X;R^\tau) & \text{if } j < n-1-\overline{p}(\{v\}) \end{cases}$$

   where v is the cone vertex and the isomorphism in the second case is induced by the inclusion $X \hookrightarrow cX$.

2. *The inclusion $X \times \{0\} \hookrightarrow X \times \mathbb{R}^m$ induces an isomorphism*

$$I^{\overline{p}}H_*(X \times \mathbb{R}^m;R^\tau) \cong I^{\overline{p}}H_*(X;R^\tau)$$

3. *Excision holds. That is,*

$$I^{\overline{p}}H_*(U, U-C;R^\tau) \cong I^{\overline{p}}H_*(X, U;R^\tau)$$

4. *There is a Mayer-Vietoris long exact sequence.*

$$\longrightarrow I^{\overline{p}}H_j(U\cap V;R^\tau) \to I^{\overline{p}}H_j(U;R^\tau) \oplus I^{\overline{p}}H_j(V;R^\tau) \to I^{\overline{p}}H_j(U\cup V;R^\tau) \to I^{\overline{p}}H_{j-1}(U\cap V;R^\tau) \longrightarrow$$

5. *There is a long exact sequence of the pair (X, U)*

$$\longrightarrow I^{\overline{p}}H_j(U;R^\tau) \to I^{\overline{p}}H_j(X;R^\tau) \to I^{\overline{p}}H_j(X, U;R^\tau) \to I^{\overline{p}}H_{j-1}(U;R^\tau) \longrightarrow$$

We end the subsection with a lemma we will need for defining the twisted cap product.

Lemma 5.1.4. *Let X be a stratified pseudomanifold and let F be a field with char$(F) \neq 2$. Let $U \subset V \subset X$ be open subsets. Then the map $F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\circ}_*(U,V;F) \to I^{\overline{p}}S_*(U,V;F^\tau)$ is a quasi-isomorphism.*

Proof. We begin with the case $V = \emptyset$. We will apply Theorem 2.4.7 to the functors defined on open subsets $W \subset U$ by $\mathbf{F}_*(W) = H_*(F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\circ}_*(W;F))$ and $\mathbf{G}_*(W) = I^{\overline{p}}H_*(W;F^\tau)$. Notice that for $x \in I^{\overline{p}}S^{\circ}_*(W;F)$, $1 \otimes x$ is a $\overline{p}$-intersection chain according to the definition of $I^{\overline{p}}S_*(W;F^\tau)$. This gives a natural transformation $\mathbf{F}_* \to \mathbf{G}_*$.

By Maschke's theorem (Theorem 6.2.3) we have that $F[\mathbb{Z}_2]$ is semi-simple since char$(F) \neq 2$ by assumption. Therefore, by (18, Theorem 4.2.2) every module over $F[\mathbb{Z}_2]$ is projective; and therefore flat. Thus, we may apply the universal coefficient theorem (18, Theorem 3.6.1) so that we have the natural short exact sequence

$$0 \to F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}H^{\circ}_*(W;F) \to H_*\left(F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\circ}_*(W;F)\right) \to \mathrm{Tor}_1^{F[\mathbb{Z}_2]}\left(I^{\overline{p}}H^{\circ}_*(W;F), F^\tau\right) \to 0.$$

However, once again applying Maschke's theorem and (18, Theorem 4.2.2) we have that $I^{\overline{p}}H^{\circ}_*(W;F)$ is a projective $F[\mathbb{Z}_2]$-module, hence, a flat $F[\mathbb{Z}_2]$-module. Thus, $\mathrm{Tor}_1^{F[\mathbb{Z}_2]}(I^{\overline{p}}H^{\circ}_*(W;F), F^\tau) = 0$. So from naturality and exactness of the short exact sequence above we have a natural isomorphism $F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}H^{\circ}_*(W;F) \cong H_*\left(F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\circ}_*(W;F)\right)$.

These natural isomorphisms and tensoring the long exact sequence in Corollary 2.4.5 over F^τ (which remains exact by Theorem 6.2.3 and (18, Theorem 4.2.2)) induce Mayer-Vietoris long exact sequences for the functor $\mathbf{F}_*$. By Corollary 5.1.3 there is a Mayer-Vietoris long exact sequence for the functor $\mathbf{G}_*$. The natural transformation $\mathbf{F}_* \to \mathbf{G}_*$ then induces a diagram of Mayer-Vietoris sequences. The only non-trivial spot to check the commutativity of this diagram is at the boundary position

$$\begin{tikzcd}
F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}H^{\mathfrak{o}}_j(W_1 \cup W_2; F) \arrow[r, "1 \otimes \partial_*"] \arrow[d] & F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}H^{\mathfrak{o}}_{j-1}(W_1 \cap W_2; F) \arrow[d] \\
I^{\overline{p}}H_j(W_1 \cup W_2; F^\tau) \arrow[r, "\partial_*"] & I^{\overline{p}}H_{j-1}(W_1 \cap W_2; F^\tau)
\end{tikzcd}$$

where $W_i \subset U$ are open for $i = 1, 2$. By bilinearity of tensor products every element of $F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}H_j(W_1 \cup W_2; F)$ may be written as a sum of elements of the form $1 \otimes x$ where $x \in I^{\overline{p}}H^{\mathfrak{o}}_j(W_1 \cup W_2; F)$. Therefore, it suffices to show the diagram commutes for elements of the form $1 \otimes x$. Going across vertically, $1 \otimes x$ maps to $1 \otimes \partial_* x$. Recall from the "zig-zag" construction of Mayer-Vietoris long exact sequences that $\partial_* x$ is defined to be ∂x_1 where x is homologous to $x_1 + x_2$ for $x_i \in I^{\overline{p}}S^{\mathfrak{o}}_j(W_i; F)$. We then map down to $1 \otimes \partial x_1$, but considered as an element of $I^{\overline{p}}H_{j-1}(W_1 \cap W_2; F^\tau)$. Going around the diagram the other way, we first consider $1 \otimes x$ as an element of $I^{\overline{p}}H_j(W_1 \cup W_2; F^\tau)$ and then map to $\partial_*(1 \otimes x)$. However, because x is homologous to $x_1 + x_2$, we have that $1 \otimes x$ is homologous to $1 \otimes (x_1 + x_2) = 1 \otimes x_1 + 1 \otimes x_2$. Thus, by the zig-zag argument for Mayer-Vietoris sequences we may represent $\partial_*(1 \otimes x)$ by $\partial(1 \otimes x_1) = 1 \otimes \partial x_1$. So we see the boundary position commutes for the diagram of Mayer-Vietoris sequences induced by the transformation $\mathbf{F}_* \to \mathbf{G}_*$.

Now assume $W \subset U_{reg}$ is homeomorphic to Euclidean space. Then $I^{\overline{p}}S^{\mathfrak{o}}_*(W;F) = S^{\mathfrak{o}}_*(W;F)$ so that $\mathbf{F}_*(W) \to \mathbf{G}_*(W)$ is the identity map which of course is an isomorphism. If $W = \emptyset$, $\mathbf{F}_*(W) = 0 = \mathbf{G}_*(W)$ so $\mathbf{F}_*(W) \to \mathbf{G}_*(W)$ is obviously an isomorphism.

Next, assume $\{W_\alpha\}_{\alpha \in J|}$ is a collection of open subsets of U totally ordered by inclusion and such that $F_*(W_\alpha) \to G_*(W_\alpha)$ is an isomorphism for each $\alpha \in J$. We must show that $\mathbf{F}_*(\cup_{\alpha\in J} W_\alpha) \to \mathbf{G}_*(\cup_{\alpha\in J} W_\alpha)$ is an isomorphism. We have

$$\begin{aligned} F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\mathfrak{o}}_*(\cup_{\alpha\in J} W_\alpha; F) &\cong F^\tau \otimes_{F[\mathbb{Z}_2]} \lim_{\alpha\in J} I^{\overline{p}}S^{\mathfrak{o}}_*(W_\alpha; F) \\ &\cong \lim_{\alpha\in J} F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\mathfrak{o}}_*(W_\alpha; F) \\ &\cong_{q.i.} \lim_{\alpha\in J} I^{\overline{p}}S_*(W_\alpha; F^\tau) \\ &\cong I^{\overline{p}}S_*(\cup_{\alpha\in J} W_\alpha; F^\tau) \end{aligned}$$

where the first and last isomorphisms are because chains have compact support and because $\{W_\alpha\}$ is totally ordered by inclusion, the second is because direct limits preserve tensor products, and third isomorphism is because the direct limit of quasi-isomorphisms is a quasi-isomorphism. Hence, the entire composition above is a quasi-isomorphism which is what we wanted to show.

Next, assume $W \subset U$ is open and stratum-preserving homeomorphic to $cL \times \mathbb{R}^i$ where L is a $(k-1)$-dimensional compact stratified pseudomanifold and that $\mathbf{F}_*((cL - \{v\}) \times \mathbb{R}^i) \to \mathbf{G}_*((cL - \{v\}) \times \mathbb{R}^i)$ is an isomorphism where v is the cone vertex of cL. We must show that $\mathbf{F}_*(cL \times \mathbb{R}^i) \to \mathbf{G}_*(cL \times \mathbb{R}^i)$ is also an isomorphism. For $j \geq k - 1 - \overline{p}(\{v\})$ we have by Corollary 5.1.3 $\mathbf{G}_j(cL \times \mathbb{R}^i) = 0$.

Combining Corollary 2.4.5, Proposition 2.3.4, and our above work, we have $\mathbf{F}_j(cL \times \mathbb{R}^i) \cong F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} H^{\mathfrak{o}}_j(cL \times \mathbb{R}^i; F) = 0$. Hence, $\mathbf{F}_j(cL \times \mathbb{R}^i) \to \mathbf{G}_j(cL \times \mathbb{R}^i)$ is trivially an isomorphism whenever $j \geq k - 1 - \overline{p}(\{v\})$. Next, consider the case $j < k - 1 - \overline{p}(\{v\})$. We have the commutative diagram

$$\begin{tikzcd}
F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} H^{\mathfrak{o}}_*((cL - \{v\}) \times \mathbb{R}^i; F) \arrow[r] \arrow[d] & I^{\overline{p}} H_*((cL - \{v\}) \times \mathbb{R}^i; F^\tau) \arrow[d] \\
F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} H^{\mathfrak{o}}_*(cL \times \mathbb{R}^i; F) \arrow[r] & I^{\overline{p}} H_*(cL \times \mathbb{R}^i; F^\tau)
\end{tikzcd}$$

where the vertical maps are induced by inclusion. The left vertical map is an isomorphism in this dimension range by Corollary 2.4.5, Proposition 2.3.4, and because the inclusion map is equivariant over the action by τ. The right vertical map is an isomorphism in this dimension range by Corollary 5.1.3. The top map is an isomorphism by assumption so that commutativity of the diagram gives the bottom map must also be an isomorphism which is what we wanted to show. Therefore, by Theorem 2.4.7 we have that $\mathbf{F}_*(U) \to \mathbf{G}_*(U)$ is an isomorphism.

For the case V is possibly nonempty we have the commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} S^{\mathfrak{o}}_*(V; F) \arrow[r] \arrow[d] & F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} S^{\mathfrak{o}}_*(U; F) \arrow[r] \arrow[d] & F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}} S^{\mathfrak{o}}_*(U, V; F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{\overline{p}} S_*(V; F^\tau) \arrow[r] & I^{\overline{p}} S_*(U; F^\tau) \arrow[r] & I^{\overline{p}} S_*(U, V; F^\tau) \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by Theorem 6.2.3 and (18, Theorem 4.2.2). The commutative diagram of short exact sequences above induces a commutative diagram of long exact sequences on homology. By the previous case above and the five lemma, the lemma is then also true for the pair (U, V). $\square$

5.2 Branched orientation covers

Using the results of the last section we can show that every pseudomanifold has a branched covering space that is an orientable pseudomanifold. In general, not every pseudomanifold has an orientation covering space that evenly covers the entire pseudomanifold. For example, the orientation cover $SS^2 - \{n, s\} \to S\mathbb{R}P^2 - \{n, s\}$ cannot be extended to an even cover of the entire space $S\mathbb{R}P^2$ (where n and s are north and south poles). However, $SS^2 \to S\mathbb{R}P^2$ is a branched cover and SS^2 is an orientable pseudomanifold (in fact a non-trivially stratified manifold in this case).

The branched orientation cover

Let X be a connected normal stratified pseudomanifold and let $\mathfrak{o}$ denote the data associated to the orientation cover of X_{reg}. Assume X_{reg} is non-orientable. Then $E(\mathfrak{o}) \to X_{reg}$ is a connected 2-sheeted cover, and therefore, we may apply Theorem 4.3.3 to extend the orientation cover to a branched cover $\widehat{X} \to X$ with $\widehat{X}$ a connected normal stratified pseudomanifold. Because $E(\mathfrak{o})$ is orientable we have that $\widehat{X}$ is an orientable pseudomanifold. Moreover, the non-trivial orientation-reversing deck transformation involution $E(\mathfrak{o}) \to E(\mathfrak{o})$ extends uniquely to an involution $\tau : \widehat{X} \to \widehat{X}$ by Proposition 4.1.9.

If X_{reg} is orientable, then $E(\mathfrak{o}) \to X_{reg}$ is equivalent to the cover $X_{reg} \coprod X_{reg} \to X_{reg}$ which we may clearly extend to $\widehat{X} \to X$ with $\widehat{X} = X \coprod X$. We still clearly have an orientation-reversing involution $\widehat{X} \to \widehat{X}$ given by mapping one disjoint copy of X to the other.

Finally, we also note that the map $p : \widehat{X} \to X$ is a proper map. To see this note in our proof of Theorem 4.3.3 we actually also showed a finitely branched cover $Y \to Z$ of normal pseudomanifolds Y and Z is an open map. Thus, $p : \widehat{X} \to X$ is an open map. To see it is proper, let $K \subset X$ be compact. Let U_α, $\alpha \in J$ be an open cover of $p^{-1}(K)$. For each $x \in K$, choose a single lift $\widehat{x} \in p^{-1}(K)$ of x Then, for each $\widehat{x} \in p^{-1}(K)$, choose an open set $U_{\alpha_{\widehat{x}}} \ni \widehat{x}$. Set $V_{\widehat{x}} = U_{\alpha_{\widehat{x}}}$. Define similarly $V_{\tau\widehat{x}}$. Notice that the $V_{\widehat{x}}$ together with the $V_{\tau\widehat{x}}$ cover $p^{-1}(K)$. Then, $p(V_{\widehat{x}})$ is an open set and covers K. However, K is compact so there is a finite subcover, say, $p(V_{\widehat{x}_1}), \ldots, p(V_{\widehat{x}_k})$. Similarly, $p(V_{\tau\widehat{x}})$ is an open cover of K and so there is a finite subcover say $p(V_{\tau\widehat{y}_1}), \ldots, p(V_{\tau\widehat{y}_{k'}})$. But then, $V_{\widehat{x}_1}, \ldots, V_{\widehat{x}_k}, V_{\tau\widehat{y}_1}, \ldots, V_{\tau\widehat{y}_{k'}}$ is a finite subcover of $p^{-1}(K)$.

We summarize the above remarks as a proposition below.

Proposition 5.2.1. *Let X be a connected normal stratified pseudomanifold. Then there is a unique branched cover $\widehat{X} \to X$ extending the orientation cover of X_{reg} with $\widehat{X}$ an orientable normal stratified pseudomanifold. Moreover, the map $\widehat{X} \to X$ is a proper map and there is an orientation-reversing branched deck transformation involution $\tau : \widehat{X} \to \widehat{X}$.*

Remark 5.2.2. The space $\widehat{X}$ has a tautological orientation coming from the construction of orientation covers. Recall that for a manifold M^n, the orientation cover $\widehat{M}$ may be constructed to be the set of ordered pairs (x, o_x) where $x \in M$ and $o_x \in H_n(M, M - \{x\}; \mathbb{Z})$ is a generator. Let $B \subset M$ be an open set homeomorphic to an open ball and let $o_B \in$

$H_n(M, M-B;\mathbb{Z})$ represent a generator. Recall $\widehat{M}$ is given the topology generated by the basis $U(o_B)$ where $U(o_B)$ is the set of all (x, o_x) such that $x \in B$ and $o_B \mapsto o_x$ under the natural map $H_n(M, M-B;\mathbb{Z}) \to H_n(M; M-\{x\};\mathbb{Z})$. Then each (x, o_x) has a canonical local orientation given by the element $\widetilde{o}_{(x,o_x)} \in H_n(\widehat{M}; \widehat{M} - \{(x, o_x)\};\mathbb{Z})$ corresponding to (x, o_x) under the isomorphisms $H_n(\widehat{M}, \widehat{M} - \{(x, o_x)\};\mathbb{Z}) \cong H_n(U(o_B), U(o_B) - \{(x, o_x)\};\mathbb{Z}) \cong H_n(B, B-\{x\};\mathbb{Z})$.

We will call this the *tautological orientation* or *canonical orientation* of an orientation cover. For more on orientation covers of manifolds we refer the reader to (14, Section 3.3).

We extend orientation branched covers in an obvious way to non-connected normal pseudomanifolds in the following definition.

Definition 5.2.3. If X is a normal stratified pseudomanifold, but not necessarily connected, we may write $X = \coprod_i X_i$ where each X_i is a normal and connected stratified pseudomanifold. Then using the above proposition we have a unique branched cover $\widehat{X_i} \to X_i$ and an orientation-reversing deck transformation involution $\tau_i : \widehat{X_i} \to \widehat{X_i}$. Let $\widehat{X} = \coprod_i \widehat{X_i}$. We define the branched orientation cover of X to be $\widehat{X} = \coprod_i \widehat{X_i} \to \coprod_i X_i = X$ and we have an orientation reversing branched deck transformation $\tau : \widehat{X} \to \widehat{X}$ given by $\tau(x) = \tau_i(x)$ for the unique i such that $x \in \widehat{X_i}$.

The following proposition mirrors the construction of (14, Example 3.H.3) and will allow us to show the existence and uniqueness of twisted fundamental classes.

Proposition 5.2.4. *Let R be a commutative ring with unity and also assume $\frac{1}{2} \in R$. Let X be a stratified normal pseudomanifold with branched orientation cover $\widehat{X} \to X$ and let $U \subset X$ be open. Let $\overline{p} \leq \overline{t}$ be a perversity on X. Then there exists a long exact sequence*

$$\longrightarrow I^{\overline{p}}H_j(X, U; R^{\tau}) \longrightarrow I^{\overline{p}}H_j(\widehat{X}, \widehat{U}; R) \longrightarrow I^{\overline{p}}H_j(X, U; R) \longrightarrow I^{\overline{p}}H_{j-1}(X, U; R^{\tau}) \longrightarrow$$

Proof. We first consider the case that X is connected and normal. We first show we have an exact sequence of R-modules

$$0 \longrightarrow S_*^{\overline{p}}(U; R^{\tau}) \xrightarrow{\phi} S^{\overline{p}}(\widehat{U}; R) \xrightarrow{p_*} S_*^{\overline{p}}(U; R)$$

where $p : \widehat{X} \to X$ is the branched covering map and p_* is the map induced by p.

The map ϕ in the sequence above is defined to be $\phi(1 \otimes (\widetilde{\gamma}, \gamma)) = \widehat{\gamma}_{\widetilde{\gamma}} - \tau\widehat{\gamma}_{\widetilde{\gamma}}$, where $\gamma : \Delta^k \to U$, $\widetilde{\gamma} : \gamma^{-1}(U_{reg}) \to E(\mathfrak{o})$, and $\widehat{\gamma}_{\widetilde{\gamma}} : \Delta^k \to \widehat{U}$ is the extension of $\widetilde{\gamma}$. We then extend ϕ linearly. To verify the map ϕ is well-defined one easily checks that $1 \otimes (\widetilde{\gamma}, \gamma) = -1 \otimes \tau(\widetilde{\gamma}, \gamma)$ map to the same element under ϕ.

To show the sequence above is exact we begin by proving the map ϕ is injective. For each $\overline{p}$-allowable simplex $\sigma : \Delta^k \to U$ choose a single lift $\widetilde{\sigma} : \sigma^{-1}(U_{reg}) \to E(\mathfrak{o})$. Then the elements $(\widetilde{\sigma}, \sigma)$ generate a basis for $S_k^{\overline{p}}(U; R^{\tau})$ as a free R-module.

Let $\widehat{\sigma}$ denote the unique extension of $\widetilde{\sigma} : \Delta^k \to E(\mathfrak{o})$ to a map $\widehat{\sigma} : \Delta^k \to \widehat{U}$ guaranteed to exist by Proposition 4.1.9. Then the simplices $\widehat{\sigma}$ generate a basis for $S_k^{\overline{p}}(\widehat{U}; R)$ as a free $R[\mathbb{Z}_2]$-module.

Now assume $x \in S_*^{\overline{p}}(U; R^\tau)$ and that $\phi(x) = 0$. We may write x uniquely as $x = \sum_i r_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i))$. Since $\phi(x) = 0$ this means that $\sum_i r_i(\widehat{\sigma}_i - \tau\widehat{\sigma}_i) = 0$. Hence, $\sum_i r_i\widehat{\sigma}_i - \sum_i r_i\tau\widehat{\sigma}_i = 0$. By linear independence we therefore have that $r_i = 0$ for all i so that $x = 0$. Thus, ϕ is injective.

Next, we show that $\mathrm{im}(\phi) = \ker(p_*)$. Suppose $x \in S^{\overline{p}}(U; R^\tau)$ and write $x = \sum_i n_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i))$. Then $p_*\phi(x) = p_*\left(\sum_i n_i(\widehat{\sigma}_i - \tau\widehat{\sigma}_i)\right) = \sum_i n_i(p\widehat{\sigma}_i - p\tau\widehat{\sigma}_i) = \sum_i n_i(\sigma_i - \sigma_i) = 0$. Thus, $\mathrm{im}(\phi) \subset \ker(p_*)$. Conversely, assume $x \in \ker(p_*)$. Write $x = \sum_i n_i\widehat{\sigma}_i + \sum_i m_i\tau\widehat{\sigma}_i$. Then, $0 = p_*x = \sum_i(n_i + m_i)\sigma_i$. By linear independence, this means that $m_i = -n_i$. Hence, $x = \sum_i n_i(\widehat{\sigma}_i - \tau\widehat{\sigma}_i)$. Then, $\phi(\sum_i n_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i)) = x$. Thus, $\mathrm{im}(\phi) = \ker(p_*)$.

The same argument above shows that we also have an exact sequence of R-modules

$$0 \longrightarrow \widehat{S}_*(U; R^\tau) \xrightarrow{\phi} \widehat{S}_*(\widehat{U}; R) \xrightarrow{p_*} \widehat{S}_*(U; R).$$

Next, we show the exact sequences above restrict to an exact sequence of chain complexes

$$0 \longrightarrow I^{\overline{p}}S_*(U; R^\tau) \xrightarrow{\phi} I^{\overline{p}}S_*(\widehat{U}; R) \xrightarrow{p_*} I^{\overline{p}}S_*(U; R) \longrightarrow 0.$$

We need to verify that each map is well-defined with respect to $\overline{p}$-intersection chains. To this end, it suffices to show each map is a chain map. The map p_* is obviously a chain map so we focus our attention on ϕ. Let $x \in I^{\overline{p}}S_*(U; R^\tau)$ so that $x, \partial x \in S_*^{\overline{p}}(U; R^\tau)$. Notice that because $\phi(x) \in S_*^{\overline{p}}(\widehat{U}; R)$ we have that $\partial\phi(x) \in \widehat{S}_*(\widehat{U}; R)$. Write $x = \sum_i r_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i))$. Then,

$$\begin{aligned}
\partial\phi(x) &= \partial\left(\sum_i r_i(\widehat{\sigma_i} - \tau\widehat{\sigma_i})\right) \\
&= \sum_i \sum_j (-1)^j r_i(\widehat{\sigma_i} \circ \partial_j - \tau\widehat{\sigma_i} \circ \partial_j) \\
&= \sum_i \sum_j (-1)^j r_i \phi(1 \otimes (\widetilde{\sigma_i} \circ \partial_j, \sigma_i \circ \partial_j)) \\
&= \phi\left(\sum_i \sum_j (-1)^j r_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i))\right) \\
&= \phi(\partial x).
\end{aligned}$$

Hence, because ∂x is $\overline{p}$-allowable we have that $\phi(\partial x)$ is $\overline{p}$-allowable so that $\partial\phi(x)$ is $\overline{p}$-allowable. Hence, $\phi(x)$ is a $\overline{p}$-intersection chain. Thus, we have simultaneously shown that ϕ is well-defined and a chain map.

Now, because we are restricting the exact sequences above, we clearly still have that ϕ is injective and that $\mathrm{im}(\phi) \subset \ker(p_*)$ upon restriction to $\overline{p}$-intersection chains. We next verify that $\ker(p_*) \subset \mathrm{im}(\phi)$ upon restriction to $\overline{p}$-intersection chains. So assume that $x \in I^{\overline{p}}S_*(\widehat{U}; R)$ and $p_*x = 0$. Then there exists $y \in S_*^{\overline{p}}(U; R^\tau)$ such that $\phi(y) = x$. We will show that $\partial y \in S_*^{\overline{p}}(U; R^\tau)$.

Now $p_*(\partial x) = \partial p_* x = 0$ so that there exists $z \in S_*^{\overline{p}}(U; R^\tau)$ such that $\phi(z) = \partial x$. However, we have that $\phi(\partial y) = \partial\phi(y) = \partial x$. Hence, because ϕ is injective we have that $\partial y = z$ so that $\partial y \in S_*^{\overline{p}}(U; R^\tau)$.

To complete the proof that the sequence of $\overline{p}$-intersection chain complexes is exact, we show that $I^{\overline{p}}S_*(\widehat{U}; R) \xrightarrow{p_*} I^{\overline{p}}S_*(U; R)$ is surjective. Consider the map $\Psi : \widehat{S}_*(U; R) \to \widehat{S}_*(\widehat{U}; R)$ given by $\Psi(\sigma) = \frac{1}{2}(\widehat{\sigma} + \tau\widehat{\sigma})$ and extended linearly. Notice Ψ restricts to $\Psi : S_*^{\overline{p}}(U; R) \to S_*^{\overline{p}}(\widehat{U}; R)$. Let $x \in I^{\overline{p}}S_*(U; R)$ and write $x = \sum_i r_i\sigma_i$. Then,

$$
\begin{aligned}
\partial\Psi(x) &= \partial\left(\sum_i \frac{r_i}{2}(\widehat{\sigma_i} + \tau\widehat{\sigma_i})\right) \\
&= \sum_i\sum_j (-1)^j\frac{r_i}{2}(\widehat{\sigma}_i\partial_j + \tau\widehat{\sigma}_i\partial_j) \\
&= \sum_i\sum_j (-1)^j\frac{r_i}{2}(\widehat{\sigma_i\partial_j} + \tau\widehat{\sigma_i\partial_j}) \\
&= \sum_i\sum_j (-1)^j r_i\Psi(\sigma_i\partial_j) \\
&= \Psi\left(\sum_i\sum_j (-1)^j r_i\sigma_i\partial_j\right) \\
&= \Psi(\partial x).
\end{aligned}
$$

where the third line follows from the observation that if $\widehat{\sigma_i}$ and $\tau\widehat{\sigma_i}$ are the two lifts of σ_i, then $\widehat{\sigma}_i\partial_j$ and $\tau\widehat{\sigma}_i\partial_j$ are the two lifts of $\sigma_i\partial_j$. Thus, because ∂x is $\overline{p}$-allowable we have that $\partial\Psi(x)$ is $\overline{p}$-allowable since Ψ preserves $\overline{p}$-allowability. Hence, we have that $\Psi : I^{\overline{p}}S_*(U; R) \to I^{\overline{p}}S_*(\widehat{U}; R)$ is a chain map and $p_*\Psi = \mathrm{id}$ so that $p_* : I^{\overline{p}}S_*(\widehat{U}; R) \to I^{\overline{p}}S_*(U; R)$ is surjective as desired.

Thus, we have shown that the sequence

$$0 \longrightarrow I^{\overline{p}}S_*(U;R^\tau) \xrightarrow{\phi} I^{\overline{p}}S_*(\widehat{U};R) \xrightarrow{p_*} I^{\overline{p}}S_*(U;R) \longrightarrow 0.$$

is exact. Finally, to prove the theorem for the pair (X,U), we have short exact sequence below which is shown to be exact in the lemma proceeding this proposition.

$$0 \longrightarrow I^{\overline{p}}S_*(X,U;R^\tau) \xrightarrow{\phi} I^{\overline{p}}S_*(\widehat{X},\widehat{U};R) \xrightarrow{p_*} I^{\overline{p}}S_*(X,U:R) \longrightarrow 0$$

Thus, the short exact sequence above induces the desired long exact sequence for the pair (X,U). The case X is normal, but not necessarily connected follows by breaking up X into its connected components and observing that a direct sum of exact sequences is exact.

□

Lemma 5.2.5. *Under the assumptions of Proposition 5.2.4, the commutative diagram below has exact rows and exact columns.*

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I^{\overline{p}}S_*(X,U;R^\tau) \arrow[u] \arrow[r, "\phi"] & I^{\overline{p}}S_*(\widehat{X},\widehat{U},R) \arrow[u] \arrow[r, "p_*"] & I^{\overline{p}}S_*(X,U;R) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I^{\overline{p}}S_*(X;R^\tau) \arrow[u, "q_i"] \arrow[r, "\phi"] & I^{\overline{p}}S_*(\widehat{X}R) \arrow[u, "q_2"] \arrow[r, "p_*"] & I^{\overline{p}}S_*(X;R) \arrow[u, "q_3"] \arrow[r] & 0 \\
0 \arrow[r] & I^{\overline{p}}S_*(U;R^\tau) \arrow[u, "i_1"] \arrow[r, "\phi"] & I^{\overline{p}}S_*(\widehat{U},R) \arrow[u, "i_2"] \arrow[r, "p_*"] & I^{\overline{p}}S_*(U;R) \arrow[u, "i_3"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

where the q_j are quotient maps and each i_j is induced by topological inclusion.

Proof. The bottom two rows are exact from the case already proven in Proposition 5.2.4. The rightmost column is exact by ordinary intersection homology, the middle column is exact by Proposition 2.1.5, and the leftmost column is exact by Corollary 5.1.3.

It remains to show the top row is exact. We first show ϕ is exact. Assume $x \in I^{\overline{p}}S_*(X,U;R^\tau)$ and $\phi(x) = 0$. Then by exactness of the leftmost column there exists $y \in I^{\overline{p}}S_*(X;R^\tau)$ such that $q_1(y) = x$. By commutativity of the diagram we have that $q_2(\phi(y)) = \phi(x) = 0$. Thus, by exactness of the middle row there exists $z \in I^{\overline{p}}S_*(\widehat{U};R)$

such that $i_2(z) = \phi(y)$. By commutativity of the diagram we have that $i_3 p_* z = p_* \phi(y) = 0$. Hence, by exactness of the bottom row there exists a unique $u \in I^{\overline{p}}S_*(U; R^\tau)$ such that $\phi(u) = z$. By commutativity of the diagram, we have that $\phi(i_1(u)) = i_2\phi(u) = i_2(z) = i_2(z) = \phi(y)$. However, by exactness of the second row, this means that $i_1(u) = y$. Thus, $x = q_1(y) = q_1(i_1(u)) = 0$ by exactness of the leftmost column. Thus, the topmost ϕ is injective.

Next, we show in the top row that $\mathrm{im}(\phi) = \ker(p_*)$. To see $\mathrm{im}(\phi) \subset \ker(p_*)$, let $x \in I^{\overline{p}}S_*(X, U; R^\tau)$. We will show $p_*\phi(x) = 0$. By exactness of the left column, there exists $y \in I^{\overline{p}}S_*(X; R^\tau)$ such that $q_1(y) = x$. By commutativity of the diagram we then have that $p_*\phi(x) = q_3 p_* \phi(y) = q_3(0) = 0$ where we have used exactness of the middle row. Next, we will show that $\ker(p_*) \subset \mathrm{im}(\phi)$. So assume that $x \in I^{\overline{p}}S_*(\widehat{X}, \widehat{U}; R)$ and that $p_*(x) = 0$. Then by exactness of the middle column there exists $y \in I^{\overline{p}}S_*(\widehat{X}; R)$ such that $q_2(y) = x$. By commutativity of the diagram we then have that $q_3 p_*(y) = p_*(x) = 0$. Thus, by exactness of the rightmost column, there exists $z \in I^{\overline{p}}(U; R)$ such that $i_3(z) = p_*(y)$. By exactness of the bottom row there exists $w \in I^{\overline{p}}S_*(\widehat{U}; R)$ such that $p_*(w) = z$. By commutativity of the diagram we then have that $p_* i_2(w) = i_3 p_*(w) = i_3(z) = p_*(y)$. In particular, we have that $p_*(y - i_2(w)) = 0$. Thus, by exactness of the middle row, we have that there exists $v \in I^{\overline{p}}S_*(X; R^\tau)$ such that $\phi(v) = y - i_2(w)$. Let $p = q_1(v)$. Then by commutativity of the diagram, we have that $\phi(p) = q_2\phi(v) = q_2(y - i_2(w)) = q_2(y) - q_2 i_2(w) = x - 0 = x$ where we have used the definition of y and that the middle column is exact. Thus, we have shown $\ker(p_*) \subset \mathrm{im}(\phi)$.

Finally, we show that the map p_* in the top row is surjective. Let $z \in I^{\bar{p}}S_*(X, U; R)$. Then by exactness of the rightmost column there exists $x \in I^{\bar{p}}S_*(X; R)$ such that $q_3(x) = z$. By exactness of the middle row there exists $y \in I^{\bar{p}}S_*(\widehat{X}; R)$ such that $p_*(y) = x$. Let $u = q_2(y)$. Then by commutativity of the diagram we have that $p_*(u) = q_3 p_*(y) = q_3(x) = z$. So the map p_* in the top row is surjective. Hence, we have shown the top row is a short exact sequence. $\square$

5.3 Twisted fundamental classes for normal stratified pseudomanifolds

Using the long exact sequence in the previous proposition we can now show the existence of twisted fundamental classes along with other properties of twisted intersection homology for normal stratified pseudomanifolds.

Theorem 5.3.1. *Let X be a normal stratified n-dimensional pseudomanifold with orientation branched cover $p : \widehat{X} \to X$. Let $\overline{p} \leq \overline{t}$ be a perversity on X. Let R be a commutative ring with unity and assume $\frac{1}{2} \in R$. For a subspace $A \subset X$, let $\widehat{A} = p^{-1}(A)$ and let $K \subset X$ be compact.*

1. *$I^{\overline{p}}H_i(X, X - K; R^\tau) = 0$ for $i > n$.*

2. *There exists a unique $\Gamma_K \in I^{\overline{p}}H_n(X, X - K; R^\tau)$ such that $\Gamma_K \mapsto \Gamma_{\widehat{K}}$ in the exact sequence of Proposition 5.2.4, where $\Gamma_{\widehat{K}}$ is the fundamental class over $\widehat{K}$ (see (11, Definition 5.9) for the definition of fundamental classes over a compact set in the orientable case) with $\widehat{X}$ given the tautological orientation (Remark 5.2.2).*

3. *If $L \subset K$ is compact, then Γ_K maps to Γ_L under the map $I^{\overline{p}}H_n(X, X - K; R^\tau) \to I^{\overline{p}}H_n(X, X - L; R^\tau)$.*

Proof. Now $\widehat{X}$ is orientable by Proposition 5.2.1. So by (11, Remark 5.1.3, Proposition 5.1.4.) we have that $I^{\overline{p}}H_i(\widehat{X}, \widehat{X} - \widehat{K}; R) = 0$ for $i > n$ since $\widehat{K}$ is compact from the observation that p is a proper map (Proposition 5.2.1). We also have that the map $p_* : I^{\overline{p}}H_i(\widehat{X}, \widehat{X} - \widehat{K}; R) \to I^{\overline{p}}H_i(X, X - K; R)$ is surjective. This may be seen from the proof of Proposition 5.2.4 where we constructed the chain map $\Psi : I^{\overline{p}}S_*(X; R) \to I^{\overline{p}}S_*(\widehat{X}; R)$ which is natural for open sets $U \subset X$ and satisfies $p_*\Psi = \text{id}$. Thus, Ψ induces a map $\overline{\Psi} : I^{\overline{p}}S_*(X, X - K; R) \to I^{\overline{p}}S_*(\widehat{X}, \widehat{X} - \widehat{K}; R)$ given by $\overline{\Psi}([x]) = [\Psi(x)]$ where where the brackets $[\ \cdot\ ]$ denote the appropriate equivalence classes. This is well defined chain map since Ψ is a chain map over both X and $X - K$. We also have that $p_*\overline{\Psi}([x]) = p_*[\Psi(x)] = [p_*\Psi(x)] = [x]$. Altogether, this shows that $\overline{\Psi}$ induces a map $\overline{\Psi} : I^{\overline{p}}H_*(X, X - K; R) \to I^{\overline{p}}H_*(\widehat{X}, \widehat{X} - \widehat{K}; R)$ such that $p_*\overline{\Psi} = \text{id}$ as maps on intersection homology. Hence, $p_* : I^{\overline{p}}H_*(\widehat{X}, \widehat{X} - \widehat{K}; R) \to I^{\overline{p}}H_*(X, X -$

$K; R)$ is surjective. In particular, for $i > n$ this means that $I^{\overline{p}}H_i(X, X - K; R) = 0$ since $I^{\overline{p}}H_i(\widehat{X}, \widehat{X} - \widehat{K}; R) = 0$ for $i > n$ (11, Theorem 5.11). Now take $i > n$ and consider the portion of the exact sequence from Proposition 5.2.2. below

$$I^{\overline{p}}H_{i+1}(X, X - K; R) \longrightarrow I^{\overline{p}}H_i(X, X - K; R^{\tau}) \longrightarrow I^{\overline{p}}H_i(\widehat{X}, \widehat{X} - \widehat{K}; R).$$

By our above results we have that $I^{\overline{p}}H_{i+1}(X, X - K; R) = 0$ and $I^{\overline{p}}H_i(\widehat{X}, \widehat{X} - \widehat{K}; R) = 0$. Hence, by exactness $I^{\overline{p}}H_i(X, X - K; R^{\tau}) = 0$. This proves 1.

Next, to prove part 2. consider the portion of the exact sequence from Proposition 5.2.5. below

$$0 \longrightarrow I^{\overline{p}}H_n(X, X - K; R^{\tau}) \longrightarrow I^{\overline{p}}H_n(\widehat{X}, \widehat{X} - \widehat{K}; R) \xrightarrow{p_*} I^{\overline{p}}H_n(X, X - K; R)$$

where the 0 comes from the equality $I^{\overline{p}}H_{n+1}(X, X - K; R) = 0$. Now, $p\tau = p$ since τ is a deck transformation. Hence, $p_*(\tau(\Gamma_{\widehat{K}})) = p_*(\Gamma_{\widehat{K}})$. On the other hand, τ is an orientation reversing involution. Thus, $\tau(\Gamma_{\widehat{K}}) = -\Gamma_{\widehat{K}}$ so that $p_*(\Gamma_{\widehat{K}}) = p_*(\tau(\Gamma_{\widehat{K}})) = -p_*(\Gamma_{\widehat{K}})$. Hence, $2p_*(\Gamma_{\widehat{K}}) = 0$ and by multiplying both sides by $\frac{1}{2}$ we have that $p_*(\Gamma_{\widehat{K}}) = 0$. So we see that $\Gamma_{\widehat{K}} \in \ker(p_*)$. By exactness of the sequence above, there exists $\Gamma_K \in I^{\overline{p}}H_n(X, X - K; R^{\tau})$ mapping to $\Gamma_{\widehat{K}}$. Moreover, Γ_K is unique since the map $I^{\overline{p}}H_n(X, X - K; R^{\tau}) \to I^{\overline{p}}H_n(\widehat{X}, \widehat{X} - \widehat{K}; R)$ is injective as can be seen from the exact sequence above.

Finally, to prove part 3. observe we have the commutative diagram below

$$\begin{tikzcd}
I^{\overline{p}}H_n(X, X-K; R^{\tau}) \arrow[r] \arrow[d] & I^{\overline{p}}H_n(\widehat{X}, \widehat{X}-\widehat{K}; R) \arrow[d] \\
I^{\overline{p}}H_n(X, X-L; R^{\tau}) \arrow[r] & I^{\overline{p}}H_n(\widehat{X}, \widehat{X}-\widehat{L}; R)
\end{tikzcd}$$

where the horizontal maps are the ones coming from Proposition 5.2.5. and the vertical maps are induced by subspace inclusion. Because the diagram commutes and $\Gamma_{\widehat{K}}$ maps down vertically to $\Gamma_{\widehat{L}}$ (11, Remark 5.10) we have that the image of Γ_K in $I^{\overline{p}}H_n(X, X-L; R^{\tau})$ will map across horizontally to $\Gamma_{\widehat{L}}$. Thus, by uniqueness of part 2. this means that the image of Γ_K in $I^{\overline{p}}H_n(X, X-L; R^{\tau})$ must be Γ_L.

□

Definition 5.3.2. We call $\Gamma_K \in I^{\overline{p}}H_n(X, X-K; R^{\tau})$ from Theorem 5.3.1 the *twisted fundamental class over* K. In the special case $R = F$ is a field and $\mathrm{char}(F) = 2$, then $I^{\overline{p}}S_*(X; X-K; F^{\tau}) = I^{\overline{p}}S_*(X, X-K; F)$ and we will define the twisted fundamental class Γ_K to be the fundamental class guaranteed to exist by (11, Theorem 5.8).

Theorem 5.3.3. *Let R be a commutative ring with unity and $\frac{1}{2} \in R$. Let X be a compact n-dimensional stratified normal pseudomanifold.*

1. *$I^{\overline{p}}H_i(X; R^{\tau}) = 0$ for $i > n$.*

2. *The natural map*

$$\bigoplus_{Z} I^{\overline{p}}H_n(\overline{Z}; R^{\tau}) \to I^{\overline{p}}H_n(X; R^{\tau})$$

   is an isomorphism, where the sum is taken over the regular strata of X.

3. *If Z is a regular stratum of X, then $I^{\overline{p}}H_n(\overline{Z}; R^{\tau})$ is the free R-module generated by the fundamental class of $\overline{Z}$.*

Proof. Part 1. follows immediately from Theorem 5.3.1 by taking $K = X$.

To prove 2, notice that because X is normal we have that $X = \coprod_Z \overline{Z}$ where the disjoint union is over the regular strata of X. Thus, 2. follows our arguments at the end of the proof of Proposition 5.2.5.

Finally, we prove 3. Because X is normal, we have that if Z is a regular stratum of X, then $\overline{Z}$ is a connected normal pseudomanifold as an easy induction on depth shows. Thus, we assume without loss of generality that X is connected.

Consider the exact sequence below from our proof of part 2. of Theorem 5.3.1 with $K = X$.

$$0 \longrightarrow I^{\overline{p}}H_n(X; R^{\tau}) \longrightarrow I^{\overline{p}}H_n(\widehat{X}; R) \longrightarrow I^{\overline{p}}H_n(X; R) \longrightarrow 0$$

First assume X is non-orientable so that $\widehat{X}$ is a connected orientable normal pseudomanifold. Then $I^{\overline{p}}H_n(\widehat{X};R) \cong R$ and is generated by the fundamental class $\Gamma_{\widehat{X}}$ (11, Theorem 5.11). However, the map

$$I^{\overline{p}}H_n(X;R^{\tau}) \to I^{\overline{p}}H_n(\widehat{X};R)$$

is injective with Γ_X mapping to $\Gamma_{\widehat{X}}$. Since $\Gamma_{\widehat{X}}$ is a generator, this means the map is also surjective, hence, an isomorphism. This proves 3. in the case X is non-orientable. So assume now that X is orientable so that $\widehat{X} = X \coprod X$ and $I^{\overline{p}}H_n(X;R) \cong R$ (11, Theorem 5.11). So the exact sequence above has the form

$$0 \longrightarrow I^{\overline{p}}H_n(X;R^{\tau}) \longrightarrow R \oplus R \xrightarrow{p_*} R \longrightarrow 0$$

with $p_*(r_1 \oplus r_2) = r_1 + r_2$. One easily sees that $\ker(p_*)$ is the ideal generated by $1 \oplus -1$ which is isomorphic to R. Thus, by exactness of the sequence we have that $I^{\overline{p}}H_n(X;R^{\tau}) \cong R$ and is generated by Γ_X.

$\square$

5.4 Twisted fundamental classes for general pseudomanifolds

Next, we prove the results of the previous subsection hold for general pseudomanifolds, that is, not necessarily normal as we assumed in the previous section. To do so we must first define twisted fundamental classes over compact sets for general pseudomanifolds. Our definition follows the same spirit of (11, Definition 5.7) in that we make our definition via the normalization of the given pseudomanifold. First we prove a proposition which justifies our definition of twisted fundamental classes over compact sets for pseudomanifolds that are not necessarily connected.

Proposition 5.4.1. *Let R be a commutative ring with unity and $\frac{1}{2} \in R$. Let X be an n-dimensional pseudomanifold and let $K \subset X$ be compact. Assume $\mathbf{n} : X^N \to X$ is the normalization of X. Then $\mathbf{n}_* : I^{\bar{p}}H_n(X^N; X^N - \mathbf{n}^{-1}(K); R^\tau) \to I^{\bar{p}}H_n(X, X - K; R^\tau)$ is an isomorphism. Moreover, $\mathbf{n}$ is a proper map so that $\mathbf{n}^{-1}(K)$ is compact and $\Gamma_{\mathbf{n}^{-1}(K)}$ exists.*

Proof. Note that the normalization of $X - K$ is given by the restriction of $X^N \to X$ to $X - K$ and that $\mathbf{n}^{-1}(K)$ because $\mathbf{n}$ is a proper map (see (16, Proposition 2.5, Theorem 2.6)). Then, the fact that $\mathbf{n}_* : I^{\bar{p}}H_n(X^N; X^N - \mathbf{n}^{-1}(K); R^\tau) \to I^{\bar{p}}H_n(X, X - K; R^\tau)$ is an isomorphism follows by the obvious adaptation of Proposition 5.1.1 to the relative case and because normalizations preserve intersection homology with local coefficients. Moreover, by our remark above we have that $\mathbf{n}^{-1}(K)$ is compact which means that $\Gamma_{\mathbf{n}^{-1}(K)}$ exists by Theorem 5.3.1. □

Definition 5.4.2. Let X be an n-dimensional pseudomanifold and let $K \subset X$ be compact. Assume $\mathbf{n} : X^N \to X$ is the normalization of X. By the previous proposition, $\Gamma_{\mathbf{n}^{-1}(K)}$ exists. We define $\Gamma_K := \mathbf{n}_*\Gamma_{\mathbf{n}^{-1}(K)}$.

Remark 5.4.3. Our definition of twisted fundamental classes is consistent when X is normal since in this case $\mathbf{n}$ is the identity map.

Theorem 5.4.4. *Let X be a stratified n-dimensional pseudomanifold. Let R be a commutative ring with unity and assume $\frac{1}{2} \in R$. Let $K \subset X$ be a compact subspace.*

1. $I^{\bar{p}}H_i(X, X - K; R^\tau) = 0$ *for* $i > n$.

2. *If $L \subset K$ is compact, then Γ_K maps to Γ_L under the map* $I^{\bar{p}}H_n(X, X - K; R^\tau) \to I^{\bar{p}}H_n(X, X - L; R^\tau)$.

Proof. Let $\mathbf{n} : X^N \to X$ be the normalization of X. Part 1. follows by Theorem 5.3.1 and the isomorphism $I^{\bar{p}}H_n(X^N, X^N - \mathbf{n}^{-1}(K); R^\tau) \cong I^{\bar{p}}H_n(X, X - K; R^\tau)$.

Part 2. follows by Theorem 5.3.1 and the commutative diagram below.

$$\begin{tikzcd}
I^{\bar{p}}H_n(X^N, X^N - \mathbf{n}^{-1}(K); R^\tau) \arrow[r, "\mathbf{n}_*", "\cong"'] \arrow[d] & I^{\bar{p}}H_n(X, X - K; R^\tau) \arrow[d] \\
I^{\bar{p}}H_n(X^N, X^N - \mathbf{n}^{-1}(L); R^\tau) \arrow[r, "\mathbf{n}_*", "\cong"'] & I^{\bar{p}}H_n(X, X - L; R^\tau)
\end{tikzcd}$$

$\square$

Theorem 5.4.5. *Let X be a compact n-dimensional stratified pseudomanifold.*

1. $I^{\overline{p}}H_i(X;R^\tau) = 0$ *for* $i > n$.

2. *The natural map*

$$\bigoplus_Z I^{\overline{p}}H_n(\overline{Z};R^\tau) \to I^{\overline{p}}H_n(X;R^\tau)$$

   is an isomorphism, where the sum is taken over the regular strata of X.

3. *If Z is a regular stratum of X, then $I^{\overline{p}}H_n(\overline{Z};R^\tau)$ is the free R-module generated by the fundamental class of $\overline{Z}$.*

Proof. Part 1. follows by Theorem 5.4.4 upon taking $K = X$. To see 2. let $\mathbf{n} : X^N \to X$ be the normalization of X. Let Z be a regular stratum of X and let $\mathrm{cl}(\mathbf{n}^{-1}(Z))$ denote the closure of $\mathbf{n}^{-1}(Z)$ in X^N. Then $\mathbf{n}$ restricts to a normalization $\mathbf{n}|_{\mathrm{cl}(\mathbf{n}^{-1}(Z))} : \mathrm{cl}(\mathbf{n}^{-1}(Z)) \to \overline{Z}$ of the stratified pseudomanifold $\overline{Z}$. Moreover, $X^N = \coprod_Z \mathrm{cl}(\mathbf{n}^{-1}(Z))$. Thus, we have the commutative diagram below where the horizontal maps are the obvious maps.

$$\begin{tikzcd}
\bigoplus_Z I^{\overline{p}}H_n(\mathrm{cl}(\mathbf{n}^{-1}(Z));R^\tau) \arrow[r, "\cong"] \arrow[d, "\mathbf{n}_*"', "\cong"] & I^{\overline{p}}H_n(X^N;R^\tau) \arrow[d, "\mathbf{n}_*"', "\cong"] \\
\bigoplus_Z I^{\overline{p}}H_n(\overline{Z};R^\tau) \arrow[r] & I^{\overline{p}}H_n(X;R^\tau)
\end{tikzcd}$$

The top horizontal map is clearly an isomorphism since it is just the connected component direct sum decomposition, and the vertical maps are isomorphisms by invariance of intersection homology with twisted coefficients under normalization. Hence, the bottom horizontal map is an isomorphism by commutativity of the diagram.

Finally, to prove 3. notice that if X_{reg} is connected, then X^N is a connected normal stratified pseudomanifold so that $R \cong I^{\overline{p}}H_n(X^N; R^\tau) \xrightarrow{\cong} I^{\overline{p}}H_n(X; R^\tau)$ where the arrow is the normalization map. Since Γ_X is by definition $\mathbf{n}_*\Gamma_{X^N}$ this case follows by the above composition of isomorphisms and Theorem 5.3.3. $\square$

5.5 Cross products of fundamental classes

Let X be a stratified pseudomanifold and M be an oriented manifold with compact subsets $K_1 \subset X$ and $K_2 \subset M$. Our goal this subsection is to show $\Gamma_{K_1} \times \Gamma_{K_2} = \Gamma_{K_1 \times K_2}$. However, we have not defined a cross product for twisted coefficients as Theorem 3.4.1 only applied to untwisted intersection homology for coverings of the regular stratum. To begin we show the cross product we defined in Section 3 may be extended to coefficients twisted by the orientation character.

Notice that an orientation on M implies that the orientation cover of $X \times M$ may be identified with $\mathfrak{o}_X \times \mathrm{id}_M$ where $\mathfrak{o}_X$ is the orientation cover of X. We give ${}^{\overline{p}}S^{\mathfrak{o}}_*(X; F) \otimes_F S_*(M; F)$ the structure of a left $F[\mathbb{Z}_2]$-module by defining $\tau(x \otimes y) = \tau x \otimes y$ for $x \in {}^{\overline{p}}S^{\mathfrak{o}}_*(X; F) \otimes_F S_*(M; F)$ and extending linearly.

Thus, we have the equalities below

$$S_*^{\overline{p}}(X; F^\tau) \otimes_F S_*(M; F) = \left(F^\tau \otimes_{F[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o}}(X; F)\right) \otimes_F S_*(M; F)$$
$$= F^\tau \otimes_{F[\mathbb{Z}_2]} \left({}^{\overline{p}}S_*^{\mathfrak{o}}(X; F) \otimes_F S_*(M; F)\right).$$

where the first equality is by definition while the second follows by our above definition.

The cross product induces a map $\times : S_*^{\mathfrak{o}_X}(X; F) \otimes_F S_*(M; F) \to S_*^{\mathfrak{o}_X \times \mathrm{id}_M}(X \times M; F)$. Moreover, if $x \in S_*^{\mathfrak{o}_X}(X; F)$ and $y \in S_*(M; F)$ we have $\tau(x \times y) = (\tau x) \times y$ as may be verified directly from the definition of the cross product. Thus, the cross product induces a map of left $F[\mathbb{Z}_2]$-modules. Hence, applying the functor $F^\tau \otimes_{F[\mathbb{Z}_2]}$, the cross product induces a map

$$F^\tau \otimes_{F[\mathbb{Z}_2]} \left({}^{\overline{p}}S_*^{\mathfrak{o}}(X; F) \otimes_F S_*(M; F)\right) \to F^\tau \otimes_{F[\mathbb{Z}_2]} {}^{\overline{p}}S_*^{\mathfrak{o} \times \mathrm{id}}(X \times M; F)$$

The same arguments used in Section 3 show that upon restricting the cross product we have a map $\times : I^{\overline{p}}S_*(X; F^\tau) \otimes_F S_*(M; F) \to I^{\overline{p}}S_*(X \times M; F^\tau)$. More generally, if $U \subset X$ and $V \subset M$ are open we have a map $\times : I^{\overline{p}}S_*(X, U; F^\tau) \otimes_F S_*(M, V; F) \to I^{\overline{p}}S_*(X \times M, (X \times V) \cup (U \times M); F^\tau)$.

The next proposition says that in the case char$(F) \neq 2$, this cross product map is a quasi-isomorphism.

Proposition 5.5.1. *Let F be a field with char$(F) \neq 2$. Let X be a stratified pseudomanifold with perversity $\overline{p} \leq \overline{t}$ and let $U \subset X$ be open. Let M be an oriented manifold and $V \subset M$ be open. Then the cross product induces a quasi-isomorphism*

$$I^{\overline{p}}S_*(X, U; F^\tau) \otimes_F S_*(M, V; F) \to I^{\overline{p}}S_*(X \times M, (X \times V) \cup (U \times M); F^\tau)$$

Proof. Consider the commutative diagram below.

$$\begin{tikzcd}
F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\mathfrak{o}}_*(X, U; F) \otimes_F S_*(M, V; F) \arrow[r] \arrow[d, "1 \otimes \times"'] & I^{\overline{p}}S_*(X, U; F^\tau) \otimes_F S_*(M, V; F) \arrow[d, "\times"'] \\
F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\overline{p}}S^{\mathfrak{o}\times \mathrm{id}}_*(X \times M, (X \times V) \cup (U \times M); F) \arrow[r] & I^{\overline{p}}S_*(X \times M, (X \times V) \cup (U \times M); F^\tau)
\end{tikzcd}$$

The diagram commutes because the right cross product is by definition a restriction of the left cross product map $1 \otimes \times$ (and using that $F[\mathbb{Z}_2]$ is semi-simple whenever char$(F) \neq 2$). The top horizontal map is quasi-isomorphism by Lemma 5.1.4 and the algebraic Künneth theorem, the bottom horizontal map is a quasi-isomorphism by Lemma 5.1.4.

Finally, if we can show the left vertical map is a quasi-isomorphism we will be done. To see that the left vertical map is a quasi-isomorphism we have that the map $\times : I^{\bar{p}}S_*^{\mathfrak{o}}(X,U;F)\otimes_F S_*(M,V;F)$ is a quasi-isomorphism by Theorem 3.4.2. What's more, the cross product is a map of $F[\mathbb{Z}_2]$-modules as we saw from our comments preceding this proposition. But notice that $F[\mathbb{Z}_2]$ is semi-simple by Theorem 6.2.3 so that every module over $F[\mathbb{Z}_2]$ is flat. Thus, by the universal coefficient theorem (18, Theorem 3.6.1) we have the commutative diagram whose rows are exact

$$\begin{tikzcd}
0 \arrow[r] & F^\tau \otimes_{F[\mathbb{Z}_2]} H_*\left(I^{\bar{p}}S_*^{\mathfrak{o}}(X,U;F)\otimes_F S_*(M,V;F)\right) \arrow[r] \arrow[d, "1\otimes\times"'] & H_*\left(F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\bar{p}}S_*^{\mathfrak{o}}(X,U;F)\otimes_F S_*(M,V;F)\right) \arrow[r] \arrow[d, "1\otimes\times"'] & 0 \\
0 \arrow[r] & F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\bar{p}}H_*^{\mathfrak{o}\times\mathrm{id}}(X\times M,(X,U)\times(M,V);F) \arrow[r] & H_*\left(F^\tau \otimes_{F[\mathbb{Z}_2]} I^{\bar{p}}S_*^{\mathfrak{o}\times\mathrm{id}}(X\times M,(X,U)\times(M,V);F)\right) \arrow[r] & 0
\end{tikzcd}$$

and where the righthand $0's$ come from the fact once again that $F[\mathbb{Z}_2]$ is semi-simple so that torsion groups vanish. The left vertical map is an isomorphism because the functor $F^\tau\otimes_{F[\mathbb{Z}_2]}$ preserves isomorphisms of $F[\mathbb{Z}_2]$-modules. Thus, the right vertical map is a isomorphism as was to be shown. $\square$

Proposition 5.5.2. *Let X be a stratified pseudomanifold and let $K_1 \subset X$ be a compact subset. Let M be an oriented manifold with compact subset $K_2 \subset M$. Then $\Gamma_{K_1} \times \Gamma_{K_2} = \Gamma_{K_1 \times K_2}$.*

Proof. If $\mathrm{char}(F) = 2$, then $I^{\bar{p}}S_*(X; F^\tau) = I^{\bar{p}}S_*(X; F)$ and the proposition follows by (11, Proposition 5.18).

For the case $\mathrm{char}(F) \neq 2$, we first assume X is normal. Consider the diagram

$$\begin{tikzcd}
I^{\bar{p}}S_*(X, X-K_1; F^\tau) \otimes_F S_*(M, M-K_2; F) \arrow[r, "\phi \otimes 1"] \arrow[d, "\times"'] & I^{\bar{p}}S_*(\widehat{X}, \widehat{X}-\widehat{K}; F) \otimes_F S_*(M, M-K_2; F) \arrow[d, "\times"'] \\
I^{\bar{p}}S_*(X \times M, (X, X-K_1) \times (M, M-K_2); F^\tau) \arrow[r, "\phi"] & I^{\bar{p}}S_*(\widehat{X} \times M, (\widehat{X}, \widehat{X}-\widehat{K_1}) \times (M, M-K_2); F)
\end{tikzcd}$$

where each map ϕ is the map defined in Proposition 5.2.4. We first verify the diagram commutes. Starting in the top right we verify the diagram commutes for elements of the form $x \otimes y$ which suffices since these elements are generators. So let $x \in I^{\bar{p}}S_*(X, X-K_1; F^\tau)$ and let $y \in S_*(M, M-K_2; F)$. Write $x = \sum_i f_i(1 \otimes (\widetilde{\sigma}_i, \sigma_i))$ and $y = \sum_j g_j \gamma_j$ where $f_i, g_j \in F$ and $\sigma_i : \Delta^* \to X$, $\widetilde{\sigma}_i : \sigma_i^{-1}(X_{reg}) \to E(\mathfrak{o})$ is a lift of σ_i, and $\gamma_j : \Delta^* \to M$. Let $\widehat{\sigma}_{i\widetilde{\sigma}_i} : \Delta^k \to \widehat{X}$ denote the unique extension of $\widetilde{\sigma}_i$. We recall that by the definition of ϕ in Proposition 5.2.4 that $\phi(x) = \sum_i (f_i \widehat{\sigma}_{i\widetilde{\sigma}_i} + f_i \tau \widehat{\sigma}_{i\widetilde{\sigma}_i})$.

Going around the diagram clockwise, we have that $x \otimes y$ maps to

$$\phi(x) \times y = \left(\sum_i (f_i \widehat{\sigma}_{i\widetilde{\sigma}_i} + f_i \tau \widehat{\sigma}_{i\widetilde{\sigma}_i}) \right) \times \left(\sum_j g_j \gamma_j \right)$$
$$= \sum_{i,j} (f_i g_j \widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j + f_i g_j \tau \widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j)$$

where the last equality follows from bilinearity of the cross product. On the other hand, going around the diagram counter-clock wise, we have that $x \otimes y$ maps to

$$\phi(x \times y) = \phi \left(\sum_{i,j} (f_i g_j (\widetilde{\sigma}_i, \sigma_i) \times \gamma_j) \right)$$

However, notice that $\widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j$ is the extension of $\widetilde{\sigma}_i \times \gamma_j$. Thus, by definition of ϕ we have

$$\phi \left(\sum_{i,j} (f_i g_j (\widetilde{\sigma}_i, \sigma_i) \times \gamma_j) \right) = \sum_{i,j} (f_i g_j \widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j + f_i g_j \tau (\widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j))$$
$$= \sum_{i,j} (f_i g_j \widehat{\sigma}_{i\widetilde{\sigma}_i} \times \gamma_j + f_i g_j (\tau \widehat{\sigma}_{i\widetilde{\sigma}_i}) \times \gamma_j)$$

so that $\phi(x) \otimes y = \phi(x \times y)$ as was to be shown.

Because $\Gamma_{\widehat{K_1}} \times \Gamma_{K_2} = \Gamma_{\widehat{K_1 \times K_2}}$ (11, Proposition 5.18) commutativity of this diagram and uniqueness of Theorem 5.3.1 proves the proposition in the case X is normal.

If X is not normal, we first note that if $\mathbf{n} : X^N \to X$ is the normalization of X, then $\mathbf{n} \times \mathrm{id} : X^N \times M \to X \times M$ is the normalization of $X \times M$. Let $K_1^N = \mathbf{n}^{-1}(K_1)$. Then we have

$$\begin{aligned}
\Gamma_{K_1} \times \Gamma_{K_2} &= \mathbf{n}_*(\Gamma_{K_1^N}) \times \Gamma_{K_2} \\
&= (\mathbf{n}_* \times \mathrm{id})(\Gamma_{K_1^N} \times \Gamma_{K_2}) \\
&= (\mathbf{n}_* \times \mathrm{id})(\Gamma_{K_1^N \times K_2}) \\
&= \Gamma_{K_1 \times K_2}
\end{aligned}$$

where the second to last equality follows by the previous case and the last equality follows by the definition (Definition 5.4.2) of fundamental classes for possibly non-normal pseudo-manifolds. □

6 Technical preliminaries

In this section we prove some technical preliminaries we will need in order to define the algebraic diagonal map for the next section. For a covering space ν we will let π denote the group of deck transformations.

6.1 Subdivisions and chain homotopy equivalences over the group ring of deck transformations

If $\mathcal{U}$ is a collection of open sets in X, then we let $I^{\overline{p}}_{\mathcal{U}}S^{\nu}_{*}(X;R)$ be the subcomplex of $\overline{p}$-intersection chains which have base support in some open set that is an element of $\mathcal{U}$ (see (8, Section 2.5)).

Theorem 6.1.1. *Let R be a commutative ring with unity. Let $\mathcal{U}$ be a locally finite open cover for a stratified pseudomanifold X with X_{reg} connected and with perversity $\overline{p} \leq \overline{t}$. Let ν be any connected covering for X_{reg}.*

(i) There exists a subdivision map $T : I^{\overline{p}}S^{\nu}_{*}(X;R) \to I^{\overline{p}}_{\mathcal{U}}S^{\nu}_{*}(X;R)$ equivariant over $R[\pi]$ which is an inverse to $I^{\overline{p}}_{\mathcal{U}}S^{\nu}_{*}(X;R) \hookrightarrow I^{\overline{p}}S^{\nu}_{*}(X;R)$ up to chain homotopy equivalence.

(ii) For open $U \subset X$, there exists a $R[\pi]$-equivariant map $T_U : I^{\overline{p}}S^{\nu}_{*}(X;R) \to I^{\overline{p}}S^{\nu}_{*}(U;R)$ which is a splitting for the inclusion.

Proof. The first claim of the theorem follows by (8, Proposition 2.9) and applying Theorem 2.2.11. The second claim of the theorem follows by applying the modification given in (5, Lemma 7.6) to our situation. Instead of reproducing these results in their entirety, we sketch out the main idea while filling in only some details.

We will use the notation $S_*^{\mathcal{U}}(X)$ to denote the subchain-complex of $S_*(X)$ generated by singular simplices which have support in some $U \in \mathcal{U}$. Let $\Phi : I^{\bar{p}}S_*^{\nu}(X;R) \to I^{\bar{p}}S_*(X;\mathcal{E})$ be the isomorphism of chain complexes given by Corollary 2.2.14. Also, let $\Phi_{\mathcal{U}}$ be the restriction of Φ to $I_{\mathcal{U}}^{\bar{p}}S_*^{\nu}(X;R)$ which gives an isomorphism of chain complexes onto $I_{\mathcal{U}}^{\bar{p}}S_*(X;\mathcal{E})$. In (8, Proposition 2.9) Friedman shows we have have a subdivision map $\widehat{T} : I^{\bar{p}}S_*(X;\mathcal{E}) \to I_{\mathcal{U}}^{\bar{p}}S_*(X;\mathcal{E})$. Thus, we let $T = \Phi_{\mathcal{U}}^{-1}\widehat{T}\Phi$. Also let $\hat{\iota} : I_{\mathcal{U}}^{\bar{p}}S_*(X;\mathcal{E}) \hookrightarrow I^{\bar{p}}S_*(X;\mathcal{E})$ and $\iota : I_{\mathcal{U}}^{\bar{p}}\widetilde{S}_*^{\nu}(X;R) \hookrightarrow I^{\bar{p}}S_*^{\nu}(X;R)$. Then, T and ι are inverses up to chain homotopy equivalence because $\Phi_{\mathcal{U}}$ and Φ are isomorphisms of chain complexes, hence, preserve chain homotopies and because the following diagram commutes.

$$\begin{tikzcd}
I_{\mathcal{U}}^{\bar{p}}S_*^{\nu}(X;R) \arrow[r, "\iota"] \arrow[d, "\Phi_{\mathcal{U}}"'] & I^{\bar{p}}S_*^{\nu}(X;R) \arrow[d, "\Phi"'] \\
I_{\mathcal{U}}^{\bar{p}}S_*(X;\mathcal{E}) \arrow[r, "\widehat{\iota}"] & I^{\bar{p}}S_*(X;\mathcal{E}).
\end{tikzcd}$$

The subdivision map T is equivariant over $R[\pi]$ by the construction of $\hat{T}$. More specifically, in (8) the map $\hat{T}$ is constructed from a subdivision map $T' : S_*(X) \to S_*^{\mathcal{U}}(X)$ (which must satisfy certain properties) in the following way. If $\xi \in I^{\bar{p}}S_*(X;\mathcal{E})$ and $\xi = \sum_i e^i\sigma_i$ with $\sigma_i : \Delta^n \to X$ and e^i a lift of $\sigma_i^{-1}(X_{reg})$ and $T'(\sigma_i) = \sum_j \sigma_i \circ \tau_{ij}$ where the $\tau_{ij} : \Delta^n \hookrightarrow \Delta^n$ give a subdivision of Δ^n, then $\hat{T}(\xi) = \sum_{i,j}(e^i \circ \tau_{ij})\sigma_i\tau_{ij}$.

Because $T = \Phi_{\mathcal{U}}^{-1}\hat{T}\Phi$, we have that if $\xi \in I^{\overline{p}}S_*^{\nu}(X;R)$ with $\xi = \sum_i r_i(\widetilde{\sigma_i}, \sigma_i)$ and $r_i \in R$, then $T(\xi) = \sum_{i,j} r_i(\widetilde{\sigma_i} \circ \tau_{ij}, \sigma_i \circ \tau_{ij})$. Therefore, if $\alpha \in \pi$ we have

$$\begin{aligned} T(\alpha \cdot \xi) &= T\left(\sum_i r_i(\alpha \cdot \widetilde{\sigma_i}, \sigma_i)\right) \\ &= \sum_{ij} r_i(\alpha \cdot \widetilde{\sigma_i} \circ \tau_{ij}, \sigma_i \circ \tau_{ij}) \\ &= \alpha \cdot \left(\sum_{ij} r_i(\widetilde{\sigma_i} \circ \tau_{ij}, \sigma_i \circ \tau_{ij})\right) \\ &= \alpha \cdot T(\xi). \end{aligned}$$

Thus, T is equivariant over $R[\pi]$ as claimed.

Finally to prove 2., if $\mathcal{U} = \{U, X\}$, we wish to show that we can construct $T_U : I^{\overline{p}}S_*^{\nu}(X;R) \to I^{\overline{p}}S_*^{\nu}(U;R)$ which is a splitting for the inclusion map $\iota : I^{\overline{p}}S_*^{\nu}(U;R) \hookrightarrow I^{\overline{p}}S_*^{\nu}(X;R)$. However, in (5, Lemma 7.6), the author shows that for an ordered covering of the form $\{U, X\}$ with ordering given by $U < X$, we can alter $T' : S_*(X) \to S_*^{\mathcal{U}}(X)$ to be such that $T'(\sigma) = \sigma$ whenever $|\sigma| \subset U$ and in such a way that the properties listed in (8, Proposition 2.9) are still satisfied. Thus, from the proof of (8, Proposition 2.9) the induced map $\hat{T}$ will still be a subdivision map for $\overline{p}$-intersection chains. If $i : U \hookrightarrow X$ is the inclusion map, then from the construction of $\hat{T}$, we can write $\hat{T} = \hat{T}_U + \hat{T}_X$ where $\hat{T}_U : I^{\overline{p}}S_*(X;E) \to I^{\overline{p}}S_*(U;i^*E)$ and $\hat{T}_X : I^{\overline{p}}S_*(X;E) \to I^{\overline{p}}S_*(X;E)$ are subdivision maps.

Because we ordered $U < X$ we have as in (5, Lemma 7.6) that for $\xi \in I^{\bar{p}}S_*(U; i^*E)$, $\hat{T}(\xi) = \hat{T}_U(\xi) = \xi$. Thus, $\hat{T}_U$ gives a splitting for the inclusion $\hat{\imath} : I^{\bar{p}}S_*(U; i^*E) \hookrightarrow I^{\bar{p}}S_*(X; E)$. That is, $\hat{T}\hat{\imath} = \mathrm{id}$. But if we set $T_U = \Phi_{\mathcal{U}}^{-1}\hat{T}_U\Phi$ as we did before, then we have

$$\begin{aligned} \Phi_{\mathcal{U}}^{-1}\hat{T}_U\hat{\imath}\Phi_{\mathcal{U}} &= \Phi_{\mathcal{U}}^{-1}\mathrm{id}\Phi_{\mathcal{U}} \\ \Phi_{\mathcal{U}}^{-1}\hat{T}_U\Phi\iota &= \mathrm{id} && \text{because } \hat{\imath}\Phi_{\mathcal{U}} = \Phi\iota. \end{aligned}$$

Thus, the computation above gives us $T_U\iota = \mathrm{id}$ so that T_U is a splitting as desired. The map T_U is equivariant over $R[\pi]$ just as the map T was above from the construction of $\hat{T}_U$ because it only depends on the base support of an extended simplex.

□

Because for a right $R[\pi]$-module A, the functor $A \otimes_{R[\pi]}$ preserves chain homotopy equivalences over $R[\pi]$, the previous theorem implies the following corollary.

Corollary 6.1.2. *The map $A \otimes_{R[\pi]} I^{\bar{p}}_{\mathcal{U}}S^{\nu}_*(X; R) \to A \otimes_{R[\pi]} I^{\bar{p}}S^{\nu}_*(X; R)$ induced by the inclusion $I^{\bar{p}}_{\mathcal{U}}\widetilde{S}^{\nu}_*(X; R) \to I^{\bar{p}}S^{\nu}_*(X; R)$ is a chain homotopy equivalence.*

Let $\mathcal{U}$ be an open covering for a stratified pseudomanifold X with perversity $\bar{p}$. Let $\mathcal{C}$ be the category of finite intersections of sets in $\mathcal{U}$ with inclusion maps as morphisms. Let $i_W : W \hookrightarrow X$ be the inclusion of an open set W. In (10, Proposition 6.3) the authors prove there is an isomorphism

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}}S_*(V, V \cap W; R) \to I^{\bar{p}}_{\mathcal{U}}S_*(X, W; R).$$

However, the proof does not rely on R being a constant system of local coefficients. In fact, the same result holds if we replace R with a local coefficient system of R-modules, say $\mathcal{E}$, defined on X_{reg}. That is, the canonical map gives us an isomorphism

$$\varinjlim_{V \in \mathcal{C}} I^{\overline{p}} S_*(V, V \cap W; i_V^* \mathcal{E}) \to I^{\overline{p}}_{\mathcal{U}} S_*(X, W; \mathcal{E}).$$

This allows us to prove the following proposition.

Proposition 6.1.3. *Let X be a stratified pseudomanifold with perversity $\overline{p} \leq \bar{t}$. Let ν be any covering for X_{reg}. The canonical map induces an isomorphism*

$$\varinjlim_{V \in \mathcal{C}} I^{\overline{p}} S^{\nu}_*(V, V \cap W; R) \to I^{\overline{p}}_{\mathcal{U}} S^{\nu}_*(X, W; R).$$

Proof. Let $\mathcal{E}_\nu$ be the system of local coefficients associated to the cover ν such that for each open V we have natural isomorphisms $I^{\overline{p}} S^{\nu}_*(V; R) \cong I^{\overline{p}} S_*(V; i_V^* \mathcal{E}_\nu)$. Thus, we also have $I^{\overline{p}} S^{\nu}_*(V, V \cap W; R) \cong I^{\overline{p}} S_*(V, V \cap W; i_V^* \mathcal{E}_\nu)$. Thus, we have the commutative diagram below.

$$\begin{tikzcd}
\varinjlim_{V \in \mathcal{C}} I^{\overline{p}} S^{\nu}_*(V, V \cap W; R) \arrow[r] \arrow[d] & I^{\overline{p}}_{\mathcal{U}} S^{\nu}_*(X, W; R) \arrow[d] \\
\varinjlim_{V \in \mathcal{C}} I^{\overline{p}} S^{\nu}_*(V, V \cap W; \mathcal{E}_\nu) \arrow[r] & I^{\overline{p}}_{\mathcal{U}} S_*(X, W; \mathcal{E}_\nu).
\end{tikzcd}$$

The bottom horizontal map is an isomorphism by the comments preceding the proposition and the two vertical maps are isomorphisms since direct limits preserve isomorphisms. By commutativity of the diagram this implies the top horizontal map is an isomorphism. $\square$

Combining Theorem 6.1.1 with Proposition 6.1.3 gives us the following corollary

Corollary 6.1.4. *Let R be a commutative ring with unity. Let X be a stratified pseudomanifold with X_{reg} connected and assume ν is a connected cover. The canonical map*

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}} S_*^{\nu}(V, V \cap W; R) \to I^{\bar{p}} S_*^{\nu}(X, W; R)$$

is a chain homotopy equivalence over $R[\pi]$.

6.2 Intersection homology computations for finitely branched covers

The next definition describes the coverings of X_{reg} to which our universal duality theorem will apply. We also note that whenever X is normal and connected, the completion of the associated pre-branched cover to a locally finite unbranched cover will be a finitely branched cover.

Definition 6.2.1. Let X be a stratified normal pseudomanifold. If ν is the data associated to a covering space of X_{reg} with the property that for every $x \in X$ there is a connected open set $U \ni x$ such that the connected components of $i_U^* \nu$ are finitely fibered coverings of U_{reg} (which is connected since X is normal), then we call ν a locally finite unbranched cover of X.

If X is a stratified pseudomanifold with X_{reg} connected and $\mathbf{n} : X^N \to X$ is the normalization of X. Then ν is a cover of $(X^N)_{reg} = X_{reg}$ and we will say that ν is locally finite unbranched cover of X if it is a locally finite unbranched cover of X^N [7].

Remark 6.2.2. The definition above is evidently equivalent to the condition that the completion of the unbranched cover ν be a finitely branched covering.

Our definition of locally finite unbranched covers will allow us to apply useful algebraic theorems. For a finite group G and a field F with $\mathrm{char}(F) \nmid |G|$, the group algebra $F[G]$ has nice properties. In particular, it is a semisimple ring. A semisimple ring is a ring R such that every module over R is projective (18, Theorem 4.2.2). We state the theorem, but we will not prove it. A proof may be found in (15, Chapter XVIII, Theorem 1.2).

Theorem 6.2.3 (Maschke)**.** *Let G be a finite group and let F be a field whose characteristic does not divide the order of G. Then the group ring $F[G]$ is semi-simple.*

Let $U \subset X$ be a connected open subset of a normal stratified pseudomanifold X such that the connected components of $\nu|_U$ are finite coverings where ν is a regular cover (we take a regular cover to also mean a connected cover) of X_{reg}. Fix a connected component ν' of ν. Let $\pi' \subset \pi$ denote the subgroup of the group of deck transformations of ν which map ν' to itself. More explicitly, $\pi' = \{g \in \pi : g(E(\nu')) = E(\nu')\}$. Then ν' is a regular covering of U and π' is isomorphic to the deck transformation group of ν'. To see the latter statement let $G(\nu')$ denote the group of deck transformations of ν' and consider the map $\pi' \to G(\nu')$ given by $g \mapsto g|_{E(\nu')}$. This map is injective since deck transformations of connected covers

[7] We could have made a similar definition of finitely unbranched cover for non-normal pseudomanifolds as we did in the normal case, but since U_{reg} may be non-connected whenever U is connected there is difficulty in defining a branching index which we need in Definition 6.2.4.

are determined by where a single point maps. Moreover, it is surjective because if $h \in G(\nu')$ and $x_0 \in E(\nu')$ then by regularity of ν there exists $g \in \pi$ such that $g(x_0) = h(x_0)$ and since $h(x_0) \in E(\nu')$ and $E(\nu')$ is connected we must have that $g(E(\nu')) = E(\nu')$. Thus, $g \in \pi'$ so we have shown $\pi' \cong G(\nu')$. The fact that ν' is regular now follows because ν is regular.

The next definition provides the fields to which the main theorem of the present paper applies.

Definition 6.2.4. Let X be a normal connected stratified pseudomanifold and with ν a locally finite unbranched cover of X_{reg}. We call a field F a $\nu -$ *good field* if the characteristic of F does not divide $j(x)$ for all $x \in X$. We recall $j(x)$ is the branching index of the point x and is defined in this context by Proposition 4.3.4.

Remark 6.2.5. Although the condition that the characteristic of F not divide $j(x)$ for every $x \in X$ appears excessive, by Corollary 4.3.6 there will only be a finite list which the characteristic of F cannot divide whenever X is compact. We also note that this condition is always satisfied if $\mathrm{char}(F) = 0$.

Remark 6.2.6. Note that if $x \in X$ and $U \ni x$ is chosen so that a fiber of ν' has $j(x) < \infty$ elements, then because $|\pi'|$ is in bijective correspondence with a fiber of ν' (it's regular), we have that a ν-good field F will be such that $\mathrm{char}(F)$ does not divide $|\pi'|$. Hence, $F[\pi']$ will be semi-simple by Theorem 6.2.3.

Lemma 6.2.7. *Let U, ν', π' be as above and let $V \subset W \subset U$ be open. Assume that π' is finite and that char(F) does not divide $|\pi'|$. Let A be any right $F[\pi]$-module. Then A also restricts to a $F[\pi']$-module structure and the inclusion map $\iota : \nu' \hookrightarrow \nu$ induces F-vector space isomorphisms of chain complexes*

$$I^{\overline{p}}\widetilde{S}_*^{\nu'}(W, V; A) \xrightarrow{1\otimes\iota} I^{\overline{p}}\widetilde{S}_*^{\nu}(W, V; A)$$

and

$$A \otimes_{F[\pi']} I^{\overline{p}}S_*^{\nu'}(W, V; F) \xrightarrow{1\otimes\iota} A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(W, V; F).$$

Proof. We first prove the second isomorphism and first consider the case $V = \emptyset$. For this case, we first show that $A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F) \to A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F)$ is an isomorphism of F-vector spaces.

For each singular simplex $\sigma : \Delta^k \to X$ with $\mathrm{int}(\Delta^k) \subset \sigma^{-1}(W_{reg}$ choose a single lift $\widetilde{\sigma} : \sigma^{-1}(X_{reg}) \to E(\nu')$. These lifts provide a basis for $\widehat{S}_*^{\nu'}(W; F)$ as an $F[\pi']$-module. What's more, because ν is regular they also provide a basis for $\widehat{S}_*^{\nu}(W; F)$ as an $F[\pi]$-module. In particular, this implies the map $A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F) \to A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F)$ is surjective.

Next, we show the map $A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F) \to A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F)$ is injective. To this end, define a map $\Phi : A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F) \to A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F)$ in the following way. We first define the map for elements of the form $a \otimes (\widetilde{\gamma}, \gamma)$, where $a \in A$ and $(\widetilde{\gamma}, \gamma)$ is an ν-extended simplex, and then we extend linearly. By regularity there is a unique $\alpha \in \pi$ such that $\alpha \cdot (\widetilde{\gamma}, \gamma) = (\widetilde{\sigma}, \sigma)$ where $(\widetilde{\sigma}, \sigma)$ is a ν'-extended simplex which is a generator of our chosen basis from above. Then define $\Phi(a \otimes (\widetilde{\gamma}, \gamma)) = (a \cdot \alpha) \otimes (\widetilde{\sigma}, \sigma)$. To see this map is well-defined, consider

$(a\cdot\beta^{-1})\otimes(\beta\cdot(\widetilde{\gamma},\gamma))$. Then $\alpha'=\beta\alpha\in\pi$ is the unique element such that $\beta\cdot(\widetilde{\gamma},\gamma)=\alpha'\cdot(\widetilde{\sigma},\sigma)$. Thus, by definition of Φ we have $\Phi\left((a\cdot\beta^{-1})\otimes(\beta\cdot(\widetilde{\gamma},\gamma)\right)=((a\cdot\beta^{-1})\cdot(\beta\cdot\alpha))\otimes(\widetilde{\sigma},\sigma)$. However, $(a\cdot\beta^{-1})\cdot(\beta\cdot\alpha)=a\cdot(\beta^{-1}\beta\alpha)=a\cdot\alpha$. Thus, we see that Φ is well-defined.

Let us show that $\Phi(1\otimes\iota)=\mathrm{id}$. By bilinearity it suffices to show this identity for elements of the form $a\otimes(\widetilde{\gamma},\gamma)$ where $a\in A$ and $(\widetilde{\gamma},\gamma)$ is an ν'-extended simplex. By regularity of ν' there exists $\alpha\in\pi'$ such that $(\widetilde{\gamma},\gamma)=\alpha\cdot(\widetilde{\sigma},\sigma)$ where $(\widetilde{\sigma},\sigma)$ is an ν'-extended simplex which is a generator from our chosen basis above. Then we have that

$$\begin{aligned}
\Phi(1\otimes\iota)\left(a\otimes_{F[\pi']}(\widetilde{\gamma},\gamma)\right)&=\Phi(1\otimes\iota)\left(a\otimes_{F[\pi']}\alpha\cdot(\widetilde{\sigma},\sigma)\right)\\
&=\Phi(a\otimes_{F[\pi]}\alpha\cdot(\widetilde{\sigma},\sigma))\\
&=a\cdot\alpha\otimes_{F[\pi']}(\widetilde{\sigma},\sigma)\\
&=a\otimes_{F[\pi']}\alpha\cdot(\widetilde{\sigma},\sigma)\\
&=a\otimes_{F[\pi']}(\widetilde{\gamma},\gamma)
\end{aligned}$$

where the third equality follows by definition of Φ since $(\widetilde{\sigma},\sigma)$ is an ν'-extended simplex coming from the chosen basis above and the third equality follows because $\alpha\in\pi'$ and the tensor product is over $F[\pi']$. Thus, $1\otimes\iota$ is injective and so we have shown that $A\otimes_{F[\pi']}\widehat{S}_*^{\nu'}(W;F)\xrightarrow{1\otimes\iota}A\otimes_{F[\pi]}\widehat{S}_*^{\nu}(W;F)$ is an isomorphism of F-vector spaces.

Consider the following commutative diagram.

$$\begin{tikzcd}
A \otimes_{F[\pi']} I^{\overline{p}} S_*^{\nu'}(W;F) \arrow[r] \arrow[d] & A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu}(W;F) \arrow[d] \\
A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W;F) \arrow[r, phantom, "\cong"] & A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W;F)
\end{tikzcd}$$

The left vertical map is injective because $F[\pi']$ is semi-simple by Theorem 6.2.3. and therefore the functor $A \otimes_{F[\pi']}$ is exact. Thus, by a simple diagram chase we see that the top horizontal map must also be injective. It remains to show it is surjective. So let $\xi \in A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu}(W;F)$ be of the form $\xi = a \otimes_{F[\pi]} z$ where $z \in I^{\overline{p}} S_*^{\nu}(W;F)$ and $a \in A$. To prove surjectivity it suffices to show our map is surjective onto generating elements of this form. Now by Proposition 2.1.6 we can write $z = \sum_i z_i$ where z_i is a $\overline{p}$-intersection chain and is in a connected component of $\nu|_U$.

But by regularity there exists $g_i \in \pi$ such that $g_i \cdot z_i \in I^{\bar{p}}S_*^{\nu'}(W;F)$. Let $a_i = a \cdot g_i^{-1}$. Then,

$$\begin{aligned}
\xi &= a \otimes_{F[\pi]} z \\
&= a \otimes_{F[\pi]} (\sum_i z_i) \\
&= \sum_i a \otimes_{F[\pi]} z_i \\
&= \sum_i a \otimes_{F[\pi]} g_i^{-1} g_i \cdot z_i \\
&= \sum_i a \cdot g_i^{-1} \otimes_{F[\pi]} g_i \cdot z_i \\
&= \sum_i a_i \otimes_{F[\pi]} g_i \cdot z_i
\end{aligned}$$

Thus, the above computation shows $\left(\sum_i a_i \otimes_{F[\pi']} g_i \cdot z_i\right) \in A \otimes_{F[\pi']} I^{\bar{p}}S_*^{\nu'}(W;F)$ maps to ξ. So the horizontal map in the diagram above is surjective, and therefore, an isomorphism as was to be shown. Moreover, $A \otimes_{F[\pi']} I^{\bar{p}}S_*^{\nu'}(W;F) \to A \otimes_{F[\pi]} I^{\bar{p}}S_*^{\nu}(W;F)$ is clearly a chain map so that the map is an isomorphism of chain complexes.

Now consider the case V is possibly non-empty. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{F[\pi']} I^{\bar{p}}S_*^{\nu'}(V;F) \arrow[r] \arrow[d] & A \otimes_{F[\pi']} I^{\bar{p}}S_*^{\nu'}(W;F) \arrow[r] \arrow[d] & A \otimes_{F[\pi']} I^{\bar{p}}S_*^{\nu'}(W,V;F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \otimes_{F[\pi]} I^{\bar{p}}S_*^{\nu}(V;F) \arrow[r] & A \otimes_{F[\pi]} I^{\bar{p}}S_*^{\nu}(W;F) \arrow[r] & A \otimes_{F[\pi]} I^{\bar{p}}S_*^{\nu}(W,V;F) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact because $F[\pi']$ is semi-simple while the bottom row is exact because the functor $A\otimes_{F[\pi]}$ is right exact and the left two vertical maps are isomorphisms by our argument above. Thus, a simple diagram chase completes the proof to show the right vertical map is also an isomorphism.

Finally, we show the first isomorphism. For this part of the proof F may be any field. From our argument above we showed that $A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F) \cong A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F)$. The same proof shows also that $A \otimes_{F[\pi']} {}^{\overline{p}}S_*^{\nu'}(W; F) \cong A \otimes_{F[\pi]} {}^{\overline{p}}S_*^{\nu}(W; F)$. Moreover, we have the commutative diagram

$$\begin{tikzcd}
A \otimes_{F[\pi']} \widehat{S}_*^{\nu'}(W; F) \arrow[r, "1 \otimes \iota"] & A \otimes_{F[\pi]} \widehat{S}_*^{\nu}(W; F) \\
A \otimes_{F[\pi']} {}^{\overline{p}}S_*^{\nu'}(W; F) \arrow[r, "1 \otimes \iota"] \arrow[u, "\partial"] & A \otimes_{F[\pi]} {}^{\overline{p}}S_*^{\nu}(W; F). \arrow[u, "\partial"']
\end{tikzcd}$$

Thus, by Lemma 2.2.10 we have that $I^{\overline{p}}\widetilde{S}_*^{\nu'}(W; A) \cong I^{\overline{p}}\widetilde{S}_*^{\nu}(W; A)$,and therefore, we also have the relative version. □

Corollary 6.2.8. *Let X be a stratified pseudomanifold with X_{reg} connected and let $V \subset U \subset X$ be open sets. Let ν be a locally finite unbranched regular cover of X_{reg}. If F is a ν-good field, then $I^{\overline{p}}S_*^{\nu}(U, V; F)$ is chain homotopy equivalent to a chain complex of non-negatively graded flat $F[\pi]$-modules.*

Proof. We first consider the case X is normal. Let $x \in X$ and $W \subset X$ be an connected open subset of x such that the connected components of $\nu|_W$ are finite coverings with fibers having $j(x)$ elements. Fix a connected component ν' of $\nu|_W$ and denote the deck transformation group of ν' by π'. We first show that $I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F)$ is a flat $F[\pi]$-module. For an exact sequence of $F[\pi]$-modules, say $0 \to A \to B$, this follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{F[\pi']} I^{\overline{p}}S^{\nu'}_{*}(W, W \cap V; F) \arrow[r] \arrow[d] & B \otimes_{F[\pi']} I^{\overline{p}}S^{\nu'}_{*}(W, W \cap V; F) \arrow[d] \\
0 \arrow[r] & A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F) \arrow[r] & B \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F)
\end{tikzcd}$$

The top row is exact because $F[\pi']$ is semi-simple. The two vertical maps are isomorphisms by Lemma 6.2.7 so the bottom row must also be exact. Hence, $I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F)$ is a flat $F[\pi]$-module.

Hence, we may choose an open covering $\mathcal{U}$ of X so that for each $W \in \mathcal{U}$, $I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F)$ is a flat $F[\pi]$-module. By Corollary 6.1.4 then,

$$\varinjlim_{V \in \mathcal{C}} I^{\overline{p}}S^{\nu}_{*}(W, W \cap V; F) \to I^{\overline{p}}S^{\nu}_{*}(U, U \cap V; F)$$

is a chain homotopy equivalence over $F[\pi]$. Because a direct limit of flat modules is flat (18, Corollary 2.6.17), we have that $I^{\overline{p}}S^{\nu}_{*}(U, U \cap V; F)$ is chain homotopy equivalent to a complex of flat $F[\pi]$-modules.

For the case X is not normal let $X^N \to X$ be the normalization of X. The isomorphism $I^{\overline{p}}S^{\nu}_{*}(U^N, U^N \cap V^N; F) \cong I^{\overline{p}}S^{\nu}_{*}(U, U \cap V; F)$ is clearly equivariant over $F[\pi]$. However, by our previous case $I^{\overline{p}}S^{\nu}_{*}(U^N; U^N \cap V^N; F)$ is chain homotopy equivalent to a chain complex of flat $F[\pi]$-modules. Hence, $I^{\overline{p}}S^{\nu}_{*}(U, U \cap V; F)$ is also chain homotopy equivalent to a chain complex of flat $F[\pi]$-modules. $\square$

Lemma 6.2.9. *Let X be a stratified pseudomanifold with X_{reg} connected and let ν be a locally finite unbranched regular cover and F be a ν-good field. Suppose $L \subset X$ is a connected $(k-1)$-dimensional link such the connected components of $\nu|_L$ are finite coverings. Assume A is any right $F[\pi]$-module. Then we have a cone formula*

$$H_i(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(cL;F)) \cong \begin{cases} 0 & \text{if } i \geq k-1-\overline{p}(\{v\}) \\ H_i(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(L;F)) & \text{if } i < k-1-\overline{p}(\{v\}) \end{cases}$$

where v is the cone vertex of cL.

Proof. Let ν' be a connected component of $\nu|_L$ with deck transformation group π'. By Remark 4.3.5 and because F is a ν-good field we have that $F[\pi']$ is semi-simple.

From standard covering space theory we have that a connected component of $\nu|_{cL}$ is isomorphic to $c\nu'$. Because $F[\pi']$ is semi-simple (so every module is flat over $F[\pi']$) we have the following commutative diagram of short exact sequences by (18, Theorem 3.6.1)

$$\begin{tikzcd} 0 \arrow[r] & A \otimes_{F[\pi']} I^{\overline{p}}H_i^{c\nu'}(cL;F) \arrow[r] & H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{c\nu'}(cL;F)) \arrow[r] & \mathrm{Tor}_1^{F[\pi']}(I^{\overline{p}}H_{i-1}^{c\nu'}(cL;F), A) \arrow[r] & 0 \\ 0 \arrow[r] & A \otimes_{F[\pi']} I^{\overline{p}}H_i^{\nu'}(L;F) \arrow[r] \arrow[u, "\mathrm{id}\otimes\iota_*"] & H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{\nu'}(L;F)) \arrow[r] \arrow[u, "(\mathrm{id}\otimes\iota)_*"] & \mathrm{Tor}_1^{F[\pi']}(I^{\overline{p}}H_{i-1}^{\nu'}(L;F), A) \arrow[r] \arrow[u, "\mathrm{Tor}_1^{F[\pi']}(\iota_*, \mathrm{id})"] & 0 \end{tikzcd}$$

However, because $F[\pi']$ is semi-simple the torsion groups vanish so that $A \otimes_{F[\pi']} I^{\overline{p}}H_i^{c\nu'}(cL;F) \to H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{c\nu'}(cL;F))$ and $A \otimes_{F[\pi']} I^{\overline{p}}H_i^{\nu'}(L;F) \to H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{\nu'}(L;F))$ are isomorphisms.

Now by the cone formula we have for $i \geq k-1-\overline{p}(\{v\})$, that $I^{\overline{p}}H_i^{c\nu'}(cL;F) = 0$. Thus, $H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{c\nu'}(cL;F)) \cong A \otimes_{F[\pi']} I^{\overline{p}}H_i^{c\nu'}(cL;F) = 0$ and so by Lemma 6.2.7 we also have $H_i(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(cL;F)) = 0$ for $i \geq k-1-\overline{p}(\{v\})$.

For $i < k-1-\overline{p}(\{v\})$ we have that the inclusion map $\iota : L \hookrightarrow cL$ induces an isomorphism of $F[\pi']$ modules $\iota_* : I^{\overline{p}}H_i^{\nu'}(L;F) \to I^{\overline{p}}H_i^{c\nu'}(cL;F)$. Thus, for this dimension range the left vertical map in the diagram above is an isomorphism. Hence, the middle map in the diagram above is an isomorphism as by above it is a composition of isomorphisms. We then have the commutative diagram

$$\begin{tikzcd}
H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{c\nu'}(L;F)) \arrow[r] \arrow[d] & H_i(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{c\nu'}(cL;F)) \arrow[d] \\
H_i(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(L;F)) \arrow[r] & H_i(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(cL;F))
\end{tikzcd}$$

The vertical maps are induced by the inclusions $\nu' \hookrightarrow \nu$ and $c\nu' \hookrightarrow c\nu$ and the horizontal maps are induced by the inclusion $L \hookrightarrow cL$. The two vertical maps are isomorphisms by Lemma 6.2.7. while the top horizontal map is an isomorphism in this dimension range by above. Thus, the bottom horizontal map is also an isomorphism in this dimension range by commutativity of the diagram. $\square$

Lemma 6.2.10. *Let X be a stratified pseudomanifold with X_{reg} connected and with ν a covering of X_{reg}. Let A be a right $F[\pi]$-module. Then we have the isomorphism*

$$H_*(A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu \times id_{\mathbb{R}^i}}(X \times \mathbb{R}^i; F)) \cong H_*(A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu}(X; F)).$$

Proof. This is immediate from the observations that $X \times \mathbb{R}^i$ is stratified homotopy equivalent to X via straight line homotopy. Moreover, the lift of this homotopy is evidently equivariant over the deck transformation action $\alpha(\widetilde{x}, r) = (\alpha \cdot \widetilde{x}, r)$ for $\widetilde{x} \in E(\nu)$, $r \in \mathbb{R}^i$, and $\alpha \in \pi$. Thus, there is a chain homotopy equivalence of $I^{\overline{p}} S_*^{\nu}(X; F)$ and $I^{\overline{p}} S_*^{\nu \times \mathrm{id}_{\mathbb{R}^i}}(X \times \mathbb{R}^i; F)$ as $F[\pi]$-modules. This implies $\mathrm{id}_A \otimes i_* : H_*(A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu}(X; F)) \to H_*(A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu \times \mathrm{id}_{\mathbb{R}^i}}(X \times \mathbb{R}^i; F))$ is an isomorphism where $i : X \times \{0\} \hookrightarrow X \times \mathbb{R}^i$ is the inclusion map.

$\square$

Before proceeding, we recall for the reader that $I^{\overline{p}} \widetilde{S}_*^{\nu}(X; A)$ refers to a twisted intersection chain complex, that is, it is the complex of $\overline{p}$-intersection chains of $A \otimes_{F[\pi]} S_*^{\nu}(X; A)$.

The next proposition should be thought of as the analogue of (10, Proposition 6.1.3). Loosely speaking, this proposition says one is allowed to "pull out" the $I^{\overline{p}}$ symbol from the chain complex $A \otimes_{F[\pi]} I^{\overline{p}} S_*^{\nu}(U, V; F)$ and the result will have isomorphic homology. Notice that this will by no means be an isomorphism of chain complexes. The chain complex $I^{\overline{p}} \widetilde{S}_*^{\nu}(U, V; A)$ consists of $\overline{p}$-intersection chains of $A \otimes_{F[\pi]} S_*^{\nu}(U, V; F)$ which may increase $\overline{p}$-allowability of boundaries because of increased amounts of cancellations due to twisted coefficients. Nonetheless, we have the proposition below.

Proposition 6.2.11. *Let X be a stratified pseudomanifold with X_{reg} connected. Let $V \subset U \subset X$ be open. Let ν be a locally finite unbranched regular cover of X_{reg} and let π denote the deck transformation group. Let F be a ν-good field and A be a right $F[\pi]$-module. Then the natural map*

$$A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U, V; F) \to I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U, V; A)$$

induces a quasi-isomorphsm

Proof. We first consider the case X is normal and $V = \emptyset$.

We proceed by induction on the depth of X. First consider the case $\text{depth}(X) = 0$ so X is a manifold. Let $i_U : U \hookrightarrow X$ be the inclusion map. In this case we have equality

$$\begin{aligned} A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U; F) &= A \otimes_{F[\pi]} S_{*}(E(i_U^{*}\nu); F) \\ &= I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U; A). \end{aligned}$$

Now assume the statement of the theorem holds for all regular coverings of spaces with depth less than N and $N > 0$. To finish the theorem we will apply Theorem 2.4.7 to the functor $\mathbf{F}_*$ defined on open subsets of U by $\mathbf{F}_*(W) = H_*(A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(W; F))$ and the functor $\mathbf{G}_*(W) = I^{\overline{p}}\widetilde{H}^{\nu}_{*}(W; A)$. The natural transformation $\mathbf{F}_* \to \mathbf{G}_*$ is the obvious map.

First assume that $W \subset U_{reg}$ is homeomorphic to euclidean space. Then $\text{depth}(W) = 0$ so by the base case above we again have equality.

Next, assume that $\{U_\alpha\}$ is a totally ordered increasing sequence of open subsets such that each $\mathbf{F}_*(U_\alpha) \to \mathbf{G}_*(U_\alpha)$ is an isomorphism. However, because chains have compact support and $\{U_\alpha\}$ is a totally ordered increasing sequence, we have the commutative diagram below whose vertical maps are isomorphisms.

$$
\begin{tikzcd}
\mathbf{F}_*(\cup_\alpha U_\alpha) \arrow[r] & \mathbf{G}_*(\cup_\alpha U_\alpha) \\
\lim\limits_{\alpha} \mathbf{F}_*(U_\alpha) \arrow[r] \arrow[u] & \lim\limits_{\alpha} \mathbf{G}_*(U_\alpha) \arrow[u]
\end{tikzcd}
$$

The bottom horizontal map is then an isomorphism since a direct limit of isomorphisms is an isomorphism. Thus, the top horizontal map is also an isomorphism.

Next, we need to show that if $\mathbb{R}^i \times cL^{k-1}$ is stratified homeomorphic to an open subset of U, and the theorem holds for $\mathbb{R}^i \times (cL - \{v\})$, then it also holds for $\mathbb{R}^i \times cL$. Since X is currently assumed to be normal notice we have that L is a connected normal stratified pseudomanifold and therefore has a single regular stratum (5, Lemma 2.69). Thus, applying Lemma 6.2.9 and Lemma 6.2.10 we have that whenever $j \geq k - 1 - \bar{p}(\{v\})$ (where v is the cone vertex)

$$
\begin{aligned}
H_j(A \otimes_{F[\pi]} I^{\bar{p}} S_*^{\nu}(\mathbb{R}^i \times cL; F)) &\cong H_j(A \otimes_{F[\pi]} I^{\bar{p}} S_*^{\nu}(cL; F)) \\
&\cong 0.
\end{aligned}
$$

On the other hand, by Proposition 2.3.1 we have that $\mathbf{G}_j(\mathbb{R}^i \times cL) = 0$ whenever $j \geq k - 1 - \overline{p}(\{v\})$.

If $j < k - 1 - \overline{p}(\{v\})$, then we have the commutative diagram below (the outer vertical maps are the same map) where ν' denotes a connected component of $\nu|_{\mathbb{R}^i \times cL}$ and so is a connected regular cover of $(\mathbb{R}^i \times cL)_{reg} = \mathbb{R}^i \times L_{reg} \times (0,1)$.

$$\begin{tikzcd}
I^{\overline{p}}\widetilde{H}_j^{\nu'}(\mathbb{R}^i \times (cL - \{v\}); A) \arrow[d] & H_j(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{\nu'}(\mathbb{R}^i \times (cL - \{v\}); F)) \arrow[l] \arrow[r] \arrow[d] & H_j(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(\mathbb{R}^i \times (cL - \{v\}); F)) \arrow[r] \arrow[d] & I^{\overline{p}}\widetilde{H}_j^{\nu}(\mathbb{R}^i \times (cL - \{v\}); A) \arrow[d] & I^{\overline{p}}\widetilde{H}_j^{\nu'}(\mathbb{R}^i \times (cL - \{v\}); A) \arrow[l] \arrow[d] \\
I^{\overline{p}}\widetilde{H}_j^{\nu'}(\mathbb{R}^i \times cL; A) & H_j(A \otimes_{F[\pi']} I^{\overline{p}}S_*^{\nu'}(\mathbb{R}^i \times cL; F)) \arrow[l] \arrow[r] & H_j(A \otimes_{F[\pi]} I^{\overline{p}}S_*^{\nu}(\mathbb{R}^i \times cL; F)) \arrow[r] & I^{\overline{p}}\widetilde{H}_j^{\nu}(\mathbb{R}^i \times cL; A) & I^{\overline{p}}\widetilde{H}_j^{\nu'}(\mathbb{R}^i \times cL; A) \arrow[l]
\end{tikzcd}$$

The vertical maps are isomorphisms by Lemma 6.2.9, Lemma 6.2.10, Proposition 2.3.1, and Proposition 2.3.2. The top and bottom horizontal maps in the square to the right of the leftmost square are isomorphisms by Lemma 6.2.7. The top leftmost horizontal map is an isomorphism by the inductive hypothesis since ν' is a connected cover and $\operatorname{depth}(\mathbb{R}^i \times (cL - \{v\})) < \operatorname{depth}(X)$. So by commutativity of the diagram the bottom left horizontal map is also an isomorphism. The two rightmost horizontal maps are isomorphisms by Lemma 6.2.7. Thus, by commutativity of the diagram, top and bottom horizontal maps in the square to the left of the rightmost square will also be isomorphisms. In particular the bottom horizontal map is an isomorphism so that $\mathbf{F}_*(\mathbb{R}^i \times cL) \to \mathbf{G}_*(\mathbb{R}^i \times cL)$ is an isomorphism as was to be shown.

Finally, for open sets $U_1, U_2 \subset U$ consider the short-exact sequence

$$0 \to I^{\overline{p}}S_*^{\nu}(U_1 \cap U_2; F) \to I^{\overline{p}}S_*^{\nu}(U_1; F) \oplus I^{\overline{p}}S_*^{\nu}(U_2; F) \to I^{\overline{p}}S_*^{\nu}(U_1; F) + I^{\overline{p}}S_*^{\nu}(U_2; F) \to 0.$$

The sequence will remain exact upon tensoring with A over $F[\pi]$ because the map $I^{\overline{p}}S^{\nu}_{*}(U_1 \cap U_2; F) \to I^{\overline{p}}S^{\nu}_{*}(U_1; F) \oplus I^{\overline{p}}S^{\nu}_{*}(U_2; F)$ splits over $F[\pi]$ by Theorem 6.1.1. So we have an exact sequence

$$0 \to A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_1 \cap U_2; F) \to \left(A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_1; F)\right) \oplus \left(A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_2; F)\right) \to A \otimes_{F[\pi]} \left(I^{\overline{p}}S^{\nu}_{*}(U_1; F) + I^{\overline{p}}S^{\nu}_{*}(U_2; F)\right) \to 0.$$

Moreover, this short exact sequence fits into the commutative diagram of short exact sequences below.

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_1 \cap U_2; F) \arrow[r] \arrow[d] & \left(A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_1; F)\right) \oplus \left(A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_2; F)\right) \arrow[r] \arrow[d] & A \otimes_{F[\pi]} \left(I^{\overline{p}}S^{\nu}_{*}(U_1; F) + I^{\overline{p}}S^{\nu}_{*}(U_2; F)\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_1 \cap U_2; A) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_1; A) \oplus I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_2; A) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_1; A) + I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_2; A) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are the obvious ones inducing the natural transformation $\mathbf{F}_* \to \mathbf{G}_*$. Thus, the commutative diagram above induces a commutative diagram of long exact sequences on homology. To finish the proof that we have a commutative diagram of Mayer-Vietoris long exact sequences we just point out that the diagram below obviously commutes

$$\begin{tikzcd}
A \otimes_{F[\pi]} \left(I^{\overline{p}}S^{\nu}_{*}(U_1; F) + I^{\overline{p}}S^{\nu}_{*}(U_2; F)\right) \arrow[r] \arrow[d] & I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_1; A) + I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_2; A) \arrow[d] \\
A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_{*}(U_1 \cup U_2; F) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_{*}(U_1 \cup U_2; A)
\end{tikzcd}$$

where the horizontal maps are the obvious ones inducing the natural transformation $\mathbf{F}_* \to \mathbf{G}_*$, the left vertical map is a quasi-isomorphism by Corollary 6.1.2, and the right vertical map is a quasi-isomorphism by combining Proposition 2.4.1 and Corollary 2.2.14.

Hence, $\mathbf{F}_*(U) \to \mathbf{G}_*(U)$ is an isomorphism by Theorem 2.4.7.

Next, we consider the case that $U \cap V$ is possibly nonempty. Because the map $I^{\overline{p}}S^{\nu}_*(U \cap V; F) \hookrightarrow I^{\overline{p}}S^{\nu}_*(U; F)$ splits over $F[\pi]$ by Theorem 6.1.1 we have that the top row in the diagram below is exact.

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U \cap V; F) \arrow[r] \arrow[d] & A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U; F) \arrow[r] \arrow[d] & A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U, U \cap V; F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_*(U \cap V; A) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_*(U; A) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_*(U, U \cap V; A) \arrow[r] & 0
\end{tikzcd}$$

Hence, the diagram above induces a commutative diagram of long exact sequences on homology so that the case $U \cap V$ is possibly non-empty follows by the five lemma.

Finally, for the general case that X is possibly not normal let $\mathbf{n} : X^N \to X$ be the normalization of X. Then we have the commutative diagram

$$\begin{tikzcd}
A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U^N, U^N \cap V^N; F) \arrow[r] \arrow[d] & I^{\overline{p}}\widetilde{S}^{\nu}_*(U^N, U^N \cap V^N; A) \arrow[d] \\
A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U, U \cap V; F) \arrow[r] & I^{\overline{p}}\widetilde{S}^{\nu}_*(U, U \cap V; A)
\end{tikzcd}$$

where the vertical maps are induced by $\mathbf{n} : X^N \to X$. The left vertical map is a quasi-isomorphism by Proposition 2.1.8 and because $\mathbf{n}$ preserves the action by π. The right vertical map is a quasi-isomorphism by combining Corollary 2.2.14 and the fact that normalization preserves intersection homology with local coefficients. Thus, since the top horizontal map is a quasi-isomorphism by the case already proven, commutativity of the diagram implies the bottom horizontal map is also a quasi-isomorphism.

□

For the next proposition we let $F[\pi]$ act via the diagonal action.

Proposition 6.2.12. *Let X be a stratified pseudomanifold with X_{reg} connected and with ν a locally finite unbranched cover of X_{reg} with deck transformation group π and let F be a ν-good field. Let $U \subset X$ be open and let V and W be open subsets of U. Let A be a right $F[\pi]$-module. The cross product induces a quasi-isomorphism*

$$A \otimes_{F[\pi]} \big(I^{\bar{p}}S^{\nu}_{*}(U, V; F) \otimes_F I^{\bar{q}}S^{\nu}_{*}(U, W; F)\big) \to A \otimes_{F[\pi]} \big(I^{Q}S^{\nu\times\nu}_{*}(U \times U, (U \times W) \cup (V \times U); F)\big) .$$

Proof. Unmarked tensor products are assumed to be over F. Denote $I^{\bar{p}}S^{\nu}_{*}(U, V; F) \otimes_F I^{\bar{q}}S^{\nu}_{*}(U, W; F)$ by C and $I^{Q}S^{\nu\times\nu}_{*}(U \times U, (U \times W) \cup (V \times U); F)$ by D. By Theorem 3.4.2 the cross product induces a quasi-isomorphism $\times : C \to D$. Moreover, via the diagonal action of π, this is a quasi-isomorphism as $F[\pi]$-modules as the formula $\alpha \cdot (\xi \times \xi') = (\alpha \cdot \xi) \times (\alpha \cdot \xi')$ is readily verified from the definition of the cross product. By Corollary 6.2.8 we have there exists a chain complex, which we denote by D', which is a flat $F[\pi \times \pi] \cong F[\pi] \otimes F[\pi]$-module and chain homotopy D. By Corollary 6.2.8 there exists a chain complexes K and L which are flat $F[\pi]$-modules and are chain homotopy equivalent to $I^{\bar{p}}S^{\nu}_{*}(U, V; F)$ and $I^{\bar{p}}S_{*}(U, W; F)$; respectively. Thus, $K \otimes L$ is chain homotopy equivalent to C. Let us verify that $K \otimes L$ is a flat $F[\pi] \otimes F[\pi]$-module.

Suppose $0 \to A \to B$ is an exact sequence of $F[\pi] \otimes F[\pi]$-modules. Give A the structure of a right $F[\pi]$-module by defining $x \cdot \alpha := x \cdot (\alpha \otimes 1)$ for all $x \in A$ and $\alpha \in \pi$. Similarly give B the analogous right $F[\pi]$-module structure. Thus,

$$0 \to A \otimes_{F[\pi]} K \to B \otimes_{F[\pi]} K$$

is exact since K is a flat $F[\pi]$-module. Now give $A \otimes_{F[\pi]} K$ the structure of a right $F[\pi]$-module by definition $(x \otimes y) \cdot \alpha := (x \cdot (1 \otimes \alpha)) \otimes y$ for all $x \in A$, $y \in K$, and $\alpha \in \pi$. The equality $(\alpha \otimes 1)(1 \otimes \beta) = (\alpha \otimes \beta) = (1 \otimes \beta)(\alpha \otimes 1)$ shows this is well-defined. We similarly give $B \otimes_{F[\pi]} K$ the structure of a right $F[\pi]$-module. We then have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \left(A \otimes_{F[\pi]} K\right) \otimes_{F[\pi]} L \arrow[r] \arrow[d, "\cong"'] & \left(B \otimes_{F[\pi]} K\right) \otimes_{F[\pi]} L \arrow[d, "\cong"'] \\
0 \arrow[r] & A \otimes_{F[\pi] \otimes F[\pi]} (K \otimes L) \arrow[r] & B \otimes_{F[\pi] \otimes F[\pi]} (K \otimes L)
\end{tikzcd}$$

where the vertical maps are the obvious maps. The top row is exact because L is a flat $F[\pi]$-module. Hence, the bottom row must also be exact. Thus, $K \otimes L$ is a flat $F[\pi] \otimes F[\pi]$-module. Let C' denote $K \otimes L$.

We next show that C' and D' are flat $F[\pi]$-modules under the diagonal action of π. Let $0 \to A \to B$ be an exact sequence of $F[\pi]$-modules. Now $F[\pi] \otimes F[\pi]$ is free over $F[\pi]$ under the diagonal action (with basis given by elements of the form $\alpha \otimes 1$ where $\alpha \in \pi$), hence it is a flat $F[\pi]$-module. Thus,

$$0 \to A \otimes_{F[\pi]} F[\pi] \otimes F[\pi] \to B \otimes_{F[\pi]} F[\pi] \otimes F[\pi]$$

is exact. By above C' is a flat $F[\pi] \otimes F[\pi]$-module so that we have

$$0 \to \big(A \otimes_{F[\pi]} F[\pi] \otimes F[\pi]\big) \otimes_{F[\pi]\otimes F[\pi]} C' \to \big(B \otimes_{F[\pi]} F[\pi] \otimes F[\pi]\big) \otimes_{F[\pi]\otimes F[\pi]} C'$$

is exact. However, we have natural isomorphisms $\big(A \otimes_{F[\pi]} F[\pi] \otimes F[\pi]\big) \otimes_{F[\pi]\otimes F[\pi]} C' \cong A \otimes_{F[\pi]} C'$ and $\big(B \otimes_{F[\pi]} F[\pi] \otimes F[\pi]\big) \otimes_{F[\pi]\otimes F[\pi]} C' \cong B \otimes_{F[\pi]} C'$. Thus,

$$0 \to A \otimes_{F[\pi]} C' \to B \otimes_{F[\pi]} C'$$

is exact so that C' is a flat $F[\pi]$-module. The same argument shows D' is a flat $F[\pi]$-module.

Next, we have the composition

$$C' \to C \xrightarrow{\times} D \to D'$$

with each map a quasi-isomorphism of $F[\pi]$-modules. Hence, because $C' \to D'$ is a quasi-isomorphism of flat $F[\pi]$ modules we have by (18, Theorem 5.6.4) that

$$A \otimes_{F[\pi]} C' \to A \otimes_{F[\pi]} D'$$

is a quasi-isomorphism.

Finally, consider the composition

$$A \otimes_{F[\pi]} C' \to A \otimes_{F[\pi]} C \to A \otimes_{F[\pi]} D \to A \otimes_{F[\pi]} D'$$

The maps $A \otimes_{F[\pi]} C' \to A \otimes_{F[\pi]} C$ and $A \otimes_{F[\pi]} D \to A \otimes_{F[\pi]} D'$ are quasi-isomorphism because the functor $\otimes_{F[\pi]}$ preserves chain homotopy equivalences over $F[\pi]$. Hence, the entire composition above is a quasi-isomorphism, so we conclude that $A \otimes_{F[\pi]} C \to A \otimes_{F[\pi]} D$ must also be a quasi-isomorphism which is what we wanted to show.

□

6.3 Universal cohomology and oriented coverings

In this subsection we develop the version of cohomology we will need to generalize universal Poincaré duality for ordinary (co)homology. We continue to assume that X is a stratified pseudomanifold with X_{reg} connected and that ν is a locally finite unbranched regular cover.

Definition 6.3.1. We define the *universal intersection cochain complex* by

$$I_{\overline{p}}\overline{S}^*_{\nu}(X;F) := \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S^{\nu}_*(X;F), F[\pi])$$

and denote the homology of this chain complex by $I_{\overline{p}}\overline{H}^*_{\nu}(X;F)$.

Remark 6.3.2. We will use a different sign convention from the standard one for the coboundary map: For a cochain α, we define $\delta\alpha(x) = -(-1)^{|\alpha|}\alpha(\partial x)$. This is the Koszul sign convention (for a detailed discussion see (3, Section VI.10)).

We aim to give a version of the cone-formula for cohomology, but first we prove a lemma.

Lemma 6.3.3. *Let X be a stratified pseudomanifold with X_{reg} connected and let ν be a regular cover of X_{reg}. Let $V \subset X$ be open and let $W \subset U$ be open sets with U connected. Let ν' be a connected component of $i_U^*\nu$ with deck transformation group π'. Then the restriction map gives an isomorphism*

$$Hom_{F[\pi]}(I^{\overline{p}}S^{\nu}_*(W, W\cap V;F), F[\pi]) \xrightarrow{\iota^*} Hom_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(W, W\cap V;F), F[\pi]).$$

Proof. To see ι^* is injective suppose $\iota^* f = 0$. Let $x \in I^{\overline{p}}S_*^{\nu}(W, W \cap V; F)$, then x can be written as a sum of intersection chains each having relative extended support (Definition 2.1.2) in a connected component of ν. We assume a chain representative for x has relative extended support in a single connected component, and then our argument can be extended linearly for general intersection chains. With this assumption, regularity of π guarantees there exists $g \in \pi$ such that $g \cdot x \in I^{\overline{p}}S_*^{\nu'}(W, W \cap V; F)$. Thus, $0 = (\iota^* f)(g \cdot x) = f(g \cdot x) = g \cdot f(x)$ so that $g \cdot f(x) = 0$. Multiplying both sides by $\overline{g}$ (the inverse of g in π) we see that $f(x) = 0$. Hence, f must be the zero function.

To see the restriction map is surjective, let $f \in \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{\nu'}(W, W \cap V; F), F[\pi])$. To define $\widetilde{f}$, let $x \in I^{\overline{p}}S_*^{\nu}(W, W \cap V; F)$. Once again we assume x is in a single connected component of ν and then we will be able to extend linearly. By regularity there exists $g \in \pi$ and $x' \in I^{\overline{p}}S_*^{\nu'}(W, W \cap V; F)$ such that $g \cdot x' = x$. Define $\widetilde{f}(x) = g \cdot f(x')$. Once we verify $\widetilde{f}$ is well-defined we will be done since $\iota^* \widetilde{f} = f$. So suppose we choose another $h \in \pi$ and $y' \in I^{\overline{p}}S_*^{\nu'}(W, W \cap V; F)$ such that $h \cdot y' = x$. Then, $g \cdot x' = h \cdot y'$. So $\overline{h}g \cdot x' = y'$ which means the automorphism $\overline{h}g$ must send ν' to ν' and therefore we have $\overline{h}g \in \pi'$. Thus, $f(y') = f(\overline{h}g \cdot x') = \overline{h}g \cdot f(x)$ so that $h \cdot f(y') = g \cdot f(x')$. Hence, our construction is well-defined. We need only show that $\widetilde{f} \in \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(W, W \cap V; F), F[\pi])$.

Let $\alpha \in \pi$ and let $x \in I^{\overline{p}}S_*^{\nu}(W, W \cap V; F)$ and write $x = \sum_i x_i$ with x_i in a single connected component of ν. Then as above we can find $z_i \in I^{\overline{p}}S_*^{\nu'}(W, W \cap V; F)$ and $g_i \in \pi$ such that $g_i \cdot z_i = x_i$. Therefore, $\alpha \cdot x = \sum_i \alpha g_i z_i$ so that by definition of $\widetilde{f}$ we have

$$\begin{aligned} \widetilde{f}(\alpha x) &= \sum_i \alpha g_i f(z_i) \\ &= \alpha \sum_i g_i f(z_i) \\ &= \alpha \widetilde{f}(x). \end{aligned}$$

Thus, $\widetilde{f} \in \operatorname{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(W, W \cap V; F), F[\pi])$ and we see that ι^* is an isomorphism.

□

Proposition 6.3.4. *Let X be a normal connected stratified pseudomanifold. Let ν be a locally finite unbranched regular cover of X_{reg} and let F be a ν-good field. Also, let L be a $(n-1)$-dimensional link of X. Then,*

$$I_{\overline{p}}\overline{H}^i_{\nu}(cL; F) \cong \begin{cases} 0, & \text{if } i \geq n-1-\overline{p}(\{v\}), \\ I_{\overline{p}}\overline{H}^i_{\nu}(L; F), & \text{if } i < n-1-\overline{p}(\{v\}) \end{cases}$$

Proof. Let ν' denote a connected component of $\nu|_L$ with deck transformation group π'. Recall we also have that a connected component of $\nu|_{cL}$ is isomorphic to $c\nu'$. By Lemma 6.3.3 we have that $\operatorname{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(cL; F), F[\pi]) \cong \operatorname{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{c\nu'}(cL; F), F[\pi])$ and $\operatorname{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(L; F), F[\pi]) \cong \operatorname{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{\nu'}(L; F), F[\pi])$.

However, π' is finite which means $F[\pi']$ is semi-simple by Theorem 6.2.3. Hence, combining (18, Theorem 3.6.1) and (18, Theorem 4.2.2) we have the isomorphisms

$$H^*(\mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{c\nu'}(cL;F), F[\pi])) \cong \mathrm{Hom}_{F[\pi']}(H_*(I^{\overline{p}}S_*^{c\nu'}(cL;F)), F[\pi])$$

and

$$H^*(\mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{\nu'}(L;F), F[\pi])) \cong \mathrm{Hom}_{F[\pi']}(H_*(I^{\overline{p}}S_*^{\nu'}(L;F)), F[\pi]).$$

Combining this with our above work we therefore have that

$$H^*(\mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{c\nu}(cL;F), F[\pi])) \cong \mathrm{Hom}_{F[\pi']}(H_*(I^{\overline{p}}S_*^{c\nu'}(cL;F)), F[\pi])$$

and

$$H^*(\mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(L;F), F[\pi'])) \cong \mathrm{Hom}_{F[\pi']}(H_*(I^{\overline{p}}S_*^{\nu'}(L;F)), F[\pi]).$$

Thus, for $i \geq n-1-\overline{p}(\{v\})$ we have $H^i(\mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{c\nu}(cL;F), F[\pi])) = 0$ because $I^{\overline{p}}H_i^{c\nu'}(cL;F) = 0$ by Proposition 2.3.3. For $i < n-1-\overline{p}(\{v\})$, we have the commutative diagram

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^i_{\nu}(cL;F) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}H_i^{c\nu'}(cL;F), F[\pi]) \arrow[d, "\cong"] \\
I_{\overline{p}}\overline{H}^i_{\nu}(L;F) \arrow[r, "\cong"] & \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}H_i^{\nu'}(L;F), F[\pi])
\end{tikzcd}$$

where the vertical maps are induced by the inclusion $L \hookrightarrow cL$. We know the horizontal maps are isomorphisms by our work above. The right vertical map is an isomorphism because in this dimension range $I^{\overline{p}}H_i^{c\nu'}(cL;F) \cong I^{\overline{p}}H_i^{\nu'}(L;F)$ by Proposition 2.3.3 and so the pull back is also an isomorphism since the isomorphism is equivariant over $F[\pi']$. Thus, by commutativity of the diagram the left vertical map is also an isomorphism in this dimension range as was to be shown.

$\square$

There is also a relative version of the cone formula.

Proposition 6.3.5. *Let X be a connected normal stratified pseudomanifold. Let ν be a locally finite unbranched regular cover of X_{reg} and let F be a ν-good field. Also, let L be an $(n-1)$-dimensional link of X. Then,*

$$I_{\overline{p}}\overline{H}^{i}_{\nu}(cL, L; F) \cong \begin{cases} I_{\overline{p}}\overline{H}^{i-1}_{\nu}(L; F), & \text{if } i \geq n - \overline{p}(\{v\}), \\ 0, & \text{if } i < n - \overline{p}(\{v\}) \end{cases}$$

Proof. Let ν' be a connected component of $\nu|_L$ with deck transformation group π'. Consider the diagram below

$$0 \longrightarrow I^{\overline{p}}S^{\nu'}_{*}(L; F) \longrightarrow I^{\overline{p}}S^{c\nu'}_{*}(cL; F) \longrightarrow I^{\overline{p}}S^{c\nu'}_{*}(cL, L; F) \longrightarrow 0$$

Note that the functor $\mathrm{Hom}_{F[\pi']}(-, F[\pi])$ is exact because modules over the semi-simple ring $F[\pi']$ are injective by (18, Theorem 4.2.2). Thus, applying the functor $\mathrm{Hom}_{F[\pi']}(-, F[\pi])$ to the short exact sequence above we obtain the short exact sequence

$$0 \longleftarrow \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_{*}(L; F), F[\pi]) \longleftarrow \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{c\nu'}_{*}(cL; F), F[\pi]) \longleftarrow \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{c\nu'}_{*}(cL, L; F), F[\pi]) \longleftarrow 0$$

which fits into the commutative diagram

$$\begin{tikzcd}
0 & \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{\nu'}(L;F),F[\pi]) \arrow[l] & \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{c\nu'}(cL;F),F[\pi]) \arrow[l] & \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S_*^{c\nu'}(cL,L;F),F[\pi]) \arrow[l] & 0 \arrow[l] \\
0 & \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{\nu}(L;F),F[\pi]) \arrow[l] \arrow[u] & \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{c\nu}(cL;F),F[\pi]) \arrow[l] \arrow[u] & \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S_*^{c\nu}(cL,L;F),F[\pi]) \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

with the left two vertical arrows isomorphisms by Lemma 6.3.3. Hence, the bottom row must also be exact which induces a long exact sequence of cohomology. Consider the case that $i \geq n-\overline{p}(\{v\})$. Then by the cone formula in Proposition 6.3.4 we have that $I_{\overline{p}}\overline{H}_{\nu}^{i}(cL;F)=0$ and $I_{\overline{p}}\overline{H}_{\nu}^{i-1}(cL;F)=0$ in this dimension range. Thus, from the long exact sequence on homology we have the exact sequence

$$0 \to I_{\overline{p}}\overline{H}_{\nu}^{i-1}(L;F) \xrightarrow{\delta^*} I_{\overline{p}}\overline{H}_{\nu}^{i}(cL,L;F) \to 0$$

which gives us the desired isomorphism for $i \geq n-\overline{p}(\{v\})$. For the case $i = n-1-\overline{p}(\{v\})$, we have by Proposition 6.3.4 that $I_{\overline{p}}\overline{H}_{\nu}^{n-2-\overline{p}(\{v\})}(cL;F) \xrightarrow{i^*} I_{\overline{p}}\overline{H}_{\nu}^{n-2-\overline{p}(\{v\})}(L;F)$ is an isomorphism and also that $I_{\overline{p}}\overline{H}_{\nu}^{n-1-\overline{p}(\{v\})}(cL;F)=0$. So from the long exact sequence on cohomology we have the exact sequence

$$I_{\overline{p}}\overline{H}^{n-2-\overline{p}(\{v\})}(cL;F) \xrightarrow{\cong} I_{\overline{p}}\overline{H}^{n-2-\overline{p}(\{v\})}(L;F) \xrightarrow{0} I_{\overline{p}}\overline{H}^{n-1-\overline{p}(\{v\})}(cL,L;F) \to 0.$$

Hence, by exactness $I_{\overline{p}}\overline{H}^{n-1-\overline{p}(\{v\})}(cL,L;F)=0$. Finally, for $i < n-1-\overline{p}(\{v\})$ the cone formula in Proposition 6.3.4 gives us that $I_{\overline{p}}\overline{H}^i_\nu(cL;F) \xrightarrow{i^*} I_{\overline{p}}\overline{H}^i_\nu(L;F)$ and $I_{\overline{p}}\overline{H}^{i-1}_\nu(cL;F) \xrightarrow{i^*} I_{\overline{p}}\overline{H}^{i-1}_\nu(L;F)$ are isomorphisms in this dimension range. Thus, from the long exact sequence on cohomology we have the exact sequence

$$I_{\overline{p}}\overline{H}^{i-1}_\nu(cL;F) \xrightarrow{\cong} I_{\overline{p}}\overline{H}^{i-1}_\nu(L;F) \xrightarrow{0} I_{\overline{p}}\overline{H}^i_\nu(cL,L;F) \xrightarrow{0} I_{\overline{p}}\overline{H}^i_\nu(cL;F) \xrightarrow{\cong} I_{\overline{p}}\overline{H}^i_\nu(L;F).$$

So by exactness we have $I_{\overline{p}}\overline{H}^i_\nu(cL,L;F) = 0$ for this dimension range. Thus, we have shown the desired cone formula.

□

Next, we define oriented coverings for which our duality theorem will apply. A treatment of manifolds may be found in (17, Section 4.5).

Definition 6.3.6. For a pseudomanifold X with X_{reg} connected, we will say ν is an *oriented covering* if ν is a connected regular cover and factors through the orientation cover $\mathfrak{o}$ of X_{reg}. That is, we have a commutative diagram

$$\begin{tikzcd} E(\nu) \arrow[rr] \arrow[dr] & & E(\mathfrak{o}) \arrow[dl] \\ & X_{reg} & \end{tikzcd}$$

where each map is a covering map (technically we mean in the generalized sense of covering map as we do not require the map $E(\nu) \to E(\mathfrak{o})$ to be surjective in the case $E(\mathfrak{o})$ is not connected). In this situation we have an induced map $\tau_\pi : \pi \to \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ where an orientation preserving deck transformation maps to 1 and an orientation reversing deck transformation maps to -1. We define the τ_π-*twisted* involution on $F[\pi]$ by

$$- : F[\pi] \to F[\pi] : a = \sum_{g\in\pi} n_g g \mapsto \overline{a} = \sum_{g\in\pi} n_g \tau_\pi(g) g^{-1}$$

The τ_π-twisted involution gives $I_{\overline{p}}\overline{S}^*_\nu(U,V;F)$ the structure of a left $F[\pi]$ module by defining $(\alpha \cdot f)(x) = f(x) \cdot \overline{\alpha}$ for $f \in I_{\overline{p}}\overline{S}^*_\nu(U,V;F)$, $\alpha \in \pi$ and $x \in I^{\overline{p}}S^\nu_*(U,V;F)$.

The next remark will be used in our definition of the twisted algebraic diagonal map.

Remark 6.3.7. Let ν be a locally finite unbranched orientable (thus connected) cover of X_{reg}, F be a ν-good field, and X_{reg} be non-orientable. Let $V \subset U$ be open, then we have that the map $E(\nu) \to E(\mathfrak{o})$ induces an isomorphism

$$F^{\tau_\pi} \otimes_{F[\pi]} {}^{\overline{p}}S^\nu_*(U;F) \to F^\tau \otimes_{F[\mathbb{Z}_2]} {}^{\overline{p}}S^{\mathfrak{o}}_*(U;F)$$

(see (14, Example 3H.2) for the ordinary homology case, since $\overline{p} \leq \overline{t}$ the argument here is completely analogous).

Similarly, we have the isomorphism below and commutative diagram involving boundary maps

$$\begin{tikzcd} F^{\tau_\pi} \otimes_{F[\pi]} {}^{\overline{p}}S^{\nu}_*(U;F) \arrow[r, "\cong"] \arrow[d, "\partial"'] & F^{\tau} \otimes_{F[\mathbb{Z}_2]} {}^{\overline{p}}S^{\mathfrak{o}}_*(U;F) \arrow[d, "\partial"'] \\ F^{\tau_\pi} \otimes_{F[\pi]} \widehat{S}^{\nu}_*(U;F) \arrow[r, "\cong"] & F^{\tau} \otimes_{F[\mathbb{Z}_2]} \widehat{S}^{\mathfrak{o}}_*(U;F). \end{tikzcd}$$

Thus, by Lemma 2.2.10 this implies that $I^{\overline{p}}S^{\nu}_*(U;F^{\tau_\pi}) \to I^{\overline{p}}S_*(U;F^{\tau})$ is an isomorphism of chain complexes. Thus, we also have that $I^{\overline{p}}S^{\nu}_*(U,V;F^{\tau_\pi}) \to I^{\overline{p}}S_*(U,V;F^{\tau})$

Combining the above isomorphism with Proposition 6.2.11 we have the composition below is a quasi-isomorphism.

$$F^{\tau_\pi} \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U,V;F) \to I^{\overline{p}}S^{\nu}_*(U,V;F^{\tau_\pi}) \to I^{\overline{p}}S^{\nu}_*(U,V;F^{\tau}).$$

If we take X to be orientable, then the orientable cover ν of X_{reg} factors as $E(\nu) \to E(\mathfrak{o}) \to X_{reg}$. However, $E(\mathfrak{o})$ is the trivial 2-sheeted cover of X_{reg} so that $E(\nu)$ (which is connected) maps onto one of these sheets. This induces a preferred identification of $I^{\overline{p}}S_*(U,V;F^{\tau})$ with $I^{\overline{p}}S_*(U,V;F)$. Moreover, the induced map $\pi \to \mathbb{Z}_2$ is the trivial map so that F^{τ_π} has the trivial $F[\pi]$-module structure. So we have the commutative diagram

$$\begin{tikzcd}
F^{\tau_\pi} \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U,V;F) \arrow[r] \arrow[d, "\cong"'] & I^{\overline{p}}S_*(U,V;F^{\tau}) \arrow[d, "="'] \\
I^{\overline{p}}S^{\nu}_*(U,V;F^{\tau_\pi}) \arrow[r] & I^{\overline{p}}S_*(U,V;F).
\end{tikzcd}$$

The bottom horizontal map is an quasi-isomorphism by applying the same argument above and the left vertical map is a quasi-isomorphism by Proposition 6.2.11 so that the top horizontal map must also be a quasi-isomorphism.

So in either case, we have a quasi-isomorphism

$$F^{\tau_\pi} \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(U,V;F) \to I^{\overline{p}}S_*(U,V;F^{\tau}).$$

We have the following algebraic lemma we will need in the proof of our main theorem.

Lemma 6.3.8. *Let X be a pseudomanifold with X_{reg} connected and let ν be a regular covering of X_{reg}. Let $U \subset X$ be a connected open subset and let $V \subset U$ also be open. Let ν' be a connected component of $\nu|_U$ with deck transformation group π'. Let J denote a set consisting of a choice of a coset representatives from the set of left cosets π/π'. Then,*

1. *$F[\pi]$ is a free left $F[\pi']$-module with a basis $\overline{J} = \{\overline{\alpha} : \alpha \in J\}$ (recall $\overline{\alpha}$ is the twisted involution of α).*

2. *$I^{\overline{p}}S^{\nu}_*(U,V;F) \cong \bigoplus_{\alpha\in J} I^{\overline{p}}S^{\nu'}_*(U,V;F)$ as F-vector space chain complexes*

3. *$I_{\overline{p}}\overline{S}^*_{\nu}(U,V;F) \cong \bigoplus_{\alpha\in J} I_{\overline{p}}\overline{S}^*_{\nu'}(U,V;F)$ as F-vector space chain complexes.*

Proof. 1. To prove part 1 it suffices to show every $x \in F[\pi]$ can be written uniquely as $x = \sum_{\alpha \in J} h_\alpha \overline{\alpha}$ where $h_\alpha \in F[\pi']$. We first note that $\overline{J}$ consists of a choice of a single element from each right coset by standard group theory. Now let $x \in F[\pi]$ and write $x = \sum_{g \in \pi} f_g g$ where $f_g \in F$. For each $g \in \pi$ we have that $g = h_g \alpha_g^{-1}$ for some unique $\alpha_g \in J$ and unique $h_g \in \pi'$ so that $g = \tau_\pi(\alpha_g) h_g \overline{\alpha_g}$. Thus,

$$\begin{aligned} x &= \sum_{g \in \pi} f_g g \\ &= \sum_{\alpha \in J} \sum_{\substack{g: \\ \alpha_g = \alpha}} \tau_\pi(\alpha) f_g h_g \overline{\alpha} \\ &= \sum_{\alpha \in J} \left(\sum_{\substack{g: \\ \alpha_g = \alpha}} \tau_\pi(\alpha) f_g h_g \right) \overline{\alpha}. \end{aligned}$$

The second sum is by just by the proof of Lagrange's theorem from group theory and our computation above. However, $(\sum_{\substack{g: \\ \alpha_g = \alpha}} \tau_\pi(\alpha) f_g h_g) \in F[\pi']$ so x can be written as $x = \sum_{\alpha \in J} h_\alpha \overline{\alpha}$ for some $h_\alpha \in F[\pi']$. To complete the proof of part 1 we need to show the h_α are unique. Suppose $\sum_{\alpha \in J} h_\alpha \overline{\alpha} = \sum_{\alpha \in J} h'_\alpha \overline{\alpha}$ for $h_\alpha, h'_\alpha \in F[\pi']$. Then we have $\sum_{\alpha \in J} (h_\alpha - h'_\alpha)\overline{\alpha} = 0$. Writing $h_\alpha = \sum_{h \in \pi'} f_h h$ where $f_h \in F$ and similarly writing $h'_\alpha = \sum_{h \in \pi'} f'_h h$ we have that

$$\sum_{\alpha \in J} (h_\alpha - h'_\alpha)\overline{\alpha} = \sum_{\alpha \in J} \sum_{h \in \pi'} (f_h - f'_h) h \overline{\alpha}$$

which means that $\sum_{\alpha \in J} \sum_{h \in \pi'} (f_h - f'_h) h \overline{\alpha} = 0$. However, we have that $h\overline{\alpha} = \tau_\pi(\alpha) h \alpha^{-1}$ so by above we have that $\sum_{\alpha \in J} \sum_{h \in \pi'} \tau_\pi(\alpha)(f_h - f'_h) h \alpha^{-1} = 0$. However, because the sum is over unique coset representatives, each term $h\alpha^{-1}$ in this sum is a unique basis element of $F[\pi]$. Hence, we must have that $\tau_\pi(\alpha)(f_h - f_h') = 0$ which means $f_h = f_h'$ and this means also that $h_\alpha = h_\alpha'$.

2. To prove 2. we first note that if for $\alpha_1, \alpha_2 \in \pi$, $\alpha_1\pi' = \alpha_2\pi'$, then $\alpha_2^{-1}\alpha_1 \in \pi'$. Hence, $\alpha_2^{-1}\alpha_1(E(\nu')) = E(\nu')$ so that $\alpha_1(E(\nu')) = \alpha_2(E(\nu'))$ which means α_1 and α_2 send $E(\nu')$ to the same connected component of $E(\nu|_U)$. What's more, by regularity of ν, the action of π on connected components of $\nu|_U$ is transitive. Thus, the set of left cosets is in bijective correspondence with the connected components of $\nu|_U$. From this observation and Proposition 2.1.6 this proves 2.

3. We have the composition of isomorphisms:

$$\begin{aligned} I_{\overline{p}}\overline{S}^*_\nu(U,V;F) &= \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S^\nu_*(U,V;F), F[\pi]) \\ &\cong \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(U,V;F), F[\pi]) \\ &\cong \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(U,V;F), \bigoplus_{\alpha\in J} F[\pi']) \\ &\cong \bigoplus_{\alpha\in J} \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(U,V;F), F[\pi']). \end{aligned}$$

The first isomorphism is by Lemma 6.3.3, the second is by part 1. above, and the third is elementary.

□

7 Poincaré duality theorems

7.1 Universal algebraic diagonal map and cap product

Recall ((5, Definition 3.5)) the dual of a perversity $\overline{p}$ is denoted $D\overline{p}$ and defined by

$$D\overline{p} = \overline{t} - \overline{p}$$

We also recall that if perversities $\overline{p}, \overline{q}$, and $\overline{r}$ satisfy $D\overline{r} \geq D\overline{p} + D\overline{q}$, then the diagonal map induces a map on intersection chains (11, Proposition 4.2)

$$d : I^{\overline{r}}S_*(X, U \cup V; F) \to I^{Q_{\overline{p},\overline{q}}}S_*(X \times X, (U \times X) \cup (X \times V); F).$$

Similarly, in our setting we have a diagonal map

$$d : I^{\overline{r}}S_*^{\nu}(X, U \cup V; F) \to I^{Q_{\overline{p},\overline{q}}}S_*^{\nu \times \nu}(X \times X, (U \times X) \cup (X \times V); F).$$

Following the ideas of J. McClure and G. Friedman in (10) and (11), we will utilize the diagonal map above to define an algebraic diagonal map which enables us to extend the cap product to our setting.

Definition 7.1.1. Suppose that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let U, V be open subsets of a stratified pseudomanifold X with X_{reg} connected, ν a locally finite unbranched oriented regular cover, and F a ν-good field. We define the *twisted-algebraic diagonal map*

$$\widetilde{d} : I^{\bar{r}}H_*(X, U \cup V; F^\tau) \to H_*((I^{\bar{p}}S^\nu_*(X, U; F))^t \otimes_{F[\pi]} I^{\bar{q}}S^\nu_*(X, V; F))$$

to be the composition

$$\begin{aligned}
I^{\bar{r}}H_*(X, U \cup V; F^\tau) \cong & H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{\bar{r}}S^\nu_*(X, U \cup V; F)) \\
\xrightarrow{1 \otimes d} & H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\bar{p},\bar{q}}}S^{\nu\times\nu}_*(X \times X, (U \times X) \cup (X \times V); F)) \\
\cong & H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{\bar{p}}S^\nu_*(X, U; F) \otimes_F I^{\bar{q}}S^\nu_*(X, V; F)) \\
\cong & H_*((I^{\bar{p}}S^\nu_*(X, U; F))^t \otimes_{F[\pi]} I^{\bar{q}}S^\nu_*(X, V; F)).
\end{aligned}$$

The first isomorphism is from Remark 6.3.7, the second is by Proposition 6.2.12, and the third is elementary. Here $(I^{\bar{p}}S^\nu_*(X, U; F))^t$ is the right $F[\pi]$ module structure induced by the τ_π-twisted involution. More explicitly, for $x \in I^{\bar{p}}S^\nu_*(X, U; F)^t$ and $g \in \pi$, we define $x \cdot g = \bar{g} \cdot x$.

Using the twisted algebraic diagonal map we may now define the cap product.

Definition 7.1.2. Under the same assumptions of Definition 7.1.1, let $\alpha \in I^{\bar{q}}\overline{H}_{\nu}^{m}(X, U; F)$ and $x \in I^{\bar{r}}H_n(X, U \cup V; F^{\tau})$. We define $\alpha \cap x \in I^{\bar{p}}H^{\nu}_{m-n}(X, V; F)$ by

$$\alpha \cap x := (1 \otimes \alpha)(\widetilde{d}(x)).$$

That is, if we write $\widetilde{d}(x) = \sum_i y_i \otimes z_i$ where $y_i \in (I^{\bar{p}}S^{\nu}_{*}(X, U; F))^t$ and $z_i \in I^{\bar{q}}S^{\nu}_{*}(X, V; F)$, then

$$\alpha \cap x = \sum_i (-1)^{m \cdot |y_i|}\overline{\alpha(z_i)}y_i$$

where α evaluates to 0 for $|z_i| \neq m$.

Proposition 7.1.3. *Assume that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let $i : (X'; U', V') \hookrightarrow (X; U, V)$ be an inclusion of triads with each set open in X. Let $\alpha \in I_{\bar{q}}\overline{H}^*_\nu(X, V; F)$ and $x \in I^{\bar{r}}H^\nu_*(X', U' \cup V'; F^\tau)$. Then $\alpha \cap i_*x = i_*(i^*\alpha \cap x) \in I^{\bar{p}}H^\nu_*(X, U; F)$.*

Proof. We will use Q to denote $Q_{\bar{p},\bar{q}}$. First we note we have the commutative diagram

$$\begin{tikzcd}
I^{\bar{r}}S^\nu_*(X', U' \cup V'; F) \arrow[r, "i_*"] \arrow[d, "d"] & I^{\bar{r}}S^\nu_*(X, U \cup V; F) \arrow[d, "d"] \\
I^Q S^{\nu\times\nu}_*(X' \times X', (U' \times X') \cup (X' \times V'); F) \arrow[r, "i_* \times i_*"] & I^Q S^{\nu\times\nu}_*(X \times X, (U \times X) \cup (X \times V); F) \\
I^{\bar{p}}S^\nu_*(X', U'; F) \otimes I^{\bar{q}}S^\nu_*(X', V'; F) \arrow[r, "i_* \otimes i_*"] \arrow[u, "\times"] & I^{\bar{p}}S^\nu_*(X, U; F) \otimes I^{\bar{q}}S^\nu_*(X, V; F) \arrow[u, "\times"]
\end{tikzcd}$$

The top square commutes on the level of spaces and the bottom square commutes by naturality of the cross product. Commutativity of this diagram combined with the fact that the diagonal map preserves the diagonal action of the deck transformation group π implies that $\widetilde{d}(i_*x) = (i_* \otimes i_*)(\widetilde{d}(x))$. Moreover, we have the equality $(1 \otimes \alpha)(i_* \otimes i_*) = i_* \otimes \alpha i_* = i_* \otimes i^*\alpha$

where the last equality follows by definition of pullbacks. Combining all of this we have

$$\begin{aligned}\alpha \cap i_*x =&(1 \otimes \alpha)(\widetilde{d}(i_*(x)))\\ =&(1 \otimes \alpha)(i_* \otimes i_*)(\widetilde{d}(x))\\ =&(i_* \otimes i^*\alpha)(\widetilde{d}(x))\\ =&i_*\left((1 \otimes i^*\alpha)(\widetilde{d}(x))\right)\\ =&i_*(i^*\alpha \cap x)\end{aligned}$$

□

The above applies to inclusions of spaces, we also have a similar computation for inclusions of coverings.

Proposition 7.1.4. *Assume that* $D\overline{r} \geq D\overline{p} + D\overline{q}$*. Let* $U \subset X$ *be a connected open subset and let* ν' *be a connected component of* $i_U^*\nu$ *with deck transformation group* π'*. Let* $W, V \subset U$ *also be open. Let* $\alpha \in I_{\overline{q}}\overline{H}^*_{\nu'}(U, V; F)$ *and let* $\widetilde{\alpha} \in I_{\overline{q}}\overline{H}^*_{\nu}(U, V; F)$ *be the extension of* α *(Lemma 6.3.3). Let* $x \in I^{\overline{r}}H_*(U, W \cup V; F^\tau)$*. Let* $\iota : \nu' \hookrightarrow \nu$ *be the inclusion map. Then*

$$\iota_*(\alpha \cap x) = \widetilde{\alpha} \cap x$$

where the left hand cap product uses the algebraic diagonal map

$$I^{\overline{r}}H_*(U, W \cup V; F^\tau) \to H_*((I^{\overline{p}}S_*^{\nu'}(U, W; F))^t \otimes_{F[\pi']} I^{\overline{q}}S_*^{\nu'}(U, V; F))$$

and the right hand cap product uses the algebraic diagonal map

$$I^{\overline{r}}H_*(U, W \cup V; F^\tau) \to H_*((I^{\overline{p}}S_*^{\nu}(U, W; F))^t \otimes_{F[\pi]} I^{\overline{q}}S_*^{\nu}(U, V; F))$$

Proof. Denote the algebraic diagonal map

$$I^{\bar{r}}H_*(U, W \cup V; F^\tau) \to H_*((I^{\bar{p}}S^\nu_*(U, W; F))^t \otimes_{F[\pi]} I^{\bar{q}}S^\nu_*(U, V; F))$$

by $\widetilde{d}_\nu$ and similarly define $\widetilde{d}_{\nu'}$. We also denote $Q_{\bar{p},\bar{q}}$ by Q. We have the commutative diagram

$$\begin{tikzcd}
I^{\bar{r}}S^{\nu'}_*(U, W \cup V; F) \arrow[r, "\iota_*"] \arrow[d, "d"'] & I^{\bar{r}}S^{\nu}_*(U, W \cup V; F) \arrow[d, "d"] \\
I^{Q}S^{\nu' \times \nu'}_*(U \times U, (W \times U) \cup (U \times V); F) \arrow[r, "\iota_* \times \iota_*"] & I^{Q}S^{\nu \times \nu}_*(X \times X, (W \times X) \cup (X \times V); F) \\
I^{\bar{p}}S^{\nu'}_*(U, W; F) \otimes I^{\bar{q}}S^{\nu'}_*(U, V; F) \arrow[r, "\iota_* \otimes \iota_*"] \arrow[u, "\times"] & I^{\bar{p}}S^{\nu}_*(X, W; F) \otimes I^{\bar{q}}S^{\nu}_*(X, V; F) \arrow[u, "\times"']
\end{tikzcd}$$

The top square commutes on the level of spaces and the bottom square commutes by naturality of the cross product or direct computation. By commutativity of the diagram and because the diagonal map respects the diagonal action by $F[\pi]$ and $F[\pi']$ we have that $\widetilde{d}_\nu = (\iota_* \otimes \iota_*)\widetilde{d}_{\nu'}$. As in Proposition 7.1.3 we have $(1 \otimes \alpha)(\iota_* \otimes \iota_*) = \iota_* \otimes \alpha\iota_*$. We also note that because $\widetilde{\alpha}$ is the extension of α, we have that $\widetilde{\alpha}\iota_* = \alpha$.

Thus,

$$\begin{aligned}
\widetilde{\alpha} \cap x &= (1 \otimes \widetilde{\alpha})(\widetilde{d}_{\nu}(x)) \\
&= (1 \otimes \widetilde{\alpha})(\iota_* \otimes \iota_*)(\widetilde{d}_{\nu'}(x)) \\
&= (\iota_* \otimes \widetilde{\alpha}\iota_*)(\widetilde{d}_{\nu'}(x)) \\
&= (\iota_* \otimes \alpha)(\widetilde{d}_{\nu'}(x)) \\
&= \iota_* \Big((1 \otimes \alpha)(\widetilde{d}_{\nu'}(x))\Big) \\
&= \iota_*(\alpha \cap x).
\end{aligned}$$

$\square$

We also have a naturally result for cap products involving normalizations which we will need to carry over our main theorem from normal pseudomanifolds to pseudomanifolds which are not necessarily normal. The proof is exactly the same as Proposition 7.1.3 upon replacing i by $\mathbf{n}$.

Proposition 7.1.5. *Assume that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let $(X; U, V)$ be a triad with U, V open in X. Let $\mathbf{n} : X^N \to X$ be the normalization of X. Let $\alpha \in I_{\bar{q}}\overline{H}^*_{\nu}(X; V; F)$ and $x \in I^{\bar{r}}H^{\nu}_*(X^N, U^N \cup V^N; F^{\tau})$. Then $\alpha \cap \mathbf{n}_* x = \mathbf{n}_*(\mathbf{n}^*\alpha \cap x) \in I^{\bar{p}}H^{\nu}_*(X, U; F)$.*

Proposition 7.1.6. *Assume that $D\bar{r} \geq D\bar{p} + D\bar{q}$. Let $V \subset X$ be open. Let $\alpha \in I_{\bar{q}}\overline{H}^*_{\nu}(V; F)$ and that $x \in I^{\bar{r}}H^{\nu}_*(X, V; F^{\tau})$. Then $(\delta\alpha) \cap x = -(-1)^{|\alpha|} i_*(\alpha \cap \partial x)$ where δ and ∂ are the connecting homomorphisms.*

Proof.

$$
\begin{aligned}
(\delta\alpha) \cap x =& (1 \otimes \delta\alpha)(\widetilde{d}(x)) \\
=& -(-1)^{|\alpha|}(1 \otimes \alpha)(1 \otimes \partial)(\widetilde{d}(x)) \\
=& -(-1)^{|\alpha|}(1 \otimes \alpha)(i_* \otimes 1)(\widetilde{d}(\partial x)) \\
=& -(-1)^{|\alpha|} i_*(\alpha \cap \partial x)
\end{aligned}
$$

□

Remark 7.1.7. We we will also need a version of the cohomology cross product in the proof of our main theorem. Specifically, we let X be a psuedomanifold with X_{reg} connected, ν be a locally finite unbranched regular cover of X_{reg} with deck transformation group π, and F be a ν-good field. Let $U \subset X$ be a distinguished neighborhood so that the connected components of $\nu|_U$ are finitely fibered.

Let ν' be a connected component of $\nu|_U$ with deck transformation group π'. We also let K_1 be any closed ball in $\mathbb{R}^i$ and K_2 be any compact subset of U. Set $A = \mathbb{R}^i - K_1$ and $B = U - K_2$. We define the cohomology cross product in this special situation to be the composition of isomorphisms

$$\begin{aligned}
H^*(\mathbb{R}^i, A; F) \otimes_F I_{\overline{p}}\overline{H}^*_{\nu}(U, B; F) &\cong H^*(\mathbb{R}^i, A; F) \otimes_F H^*(\mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(U, B; F), F[\pi])) \\
&\cong \mathrm{Hom}_F(H_*(\mathbb{R}^i, A; F), F) \otimes_F \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}H^{\nu'}_*(U, B; F), F[\pi]) \\
&\cong \mathrm{Hom}_{F[\pi']}(H_*(\mathbb{R}^i, A; F) \otimes_F I^{\overline{p}}H^{\nu'}_*(U, B; F), F[\pi]) \\
&\cong \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}H^{\mathrm{id}_{\mathbb{R}^i}\times\nu'}_*(\mathbb{R}^i \times U, \mathbb{R}^i \times B \cup A \times X; F), F[\pi]) \\
&\cong H^*(\mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\mathrm{id}_{\mathbb{R}^i}\times\nu'}_*(\mathbb{R}^i \times U, \mathbb{R}^i \times B \cup A \times X; F), F[\pi])) \\
&\cong I_{\overline{p}}\overline{H}^*_{\mathrm{id}\times\nu}(\mathbb{R}^i \times U, \mathbb{R}^i \times B \cup A \times X; F)
\end{aligned}$$

The first and last isomorphisms are by Lemma 6.3.3 the second and second to last isomorphisms follow from the universal coefficients theorem (18, Theorem 3.6.5) and the fact that $F[\pi']$ is semi-simple, the third isomorphism (note this uses the Koszul sign convention) follows because $H_*(\mathbb{R}^i, A; F)$ is finitely generated (compare (11, Remark 4.20)). Finally, the fourth isomorphism follows from Theorem 3.4.2 and the compatibility of the cross product with the deck transformation actions. What's more, a simple verification shows that the above isomorphism is an isomorphism of left $F[\pi]$-modules.

More explicitly, let $\alpha \in H^*(\mathbb{R}^i, A; F)$, $\beta \in I_{\overline{p}}\overline{H}^*_{\nu}(U, B; F)$, $a \in S_*(\mathbb{R}^i, A; F)$, and $b \in I^{\overline{p}}S_*(X, B; F)$. Then, $(\alpha \times \beta)(a \times b) = (-1)^{|\beta|\cdot|a|}\alpha(a)\beta(b)$.

The following well known relationship between cross products and cap products for ordinary homology and cohomology extends to our setting.

Proposition 7.1.8. *Suppose that $D\overline{r} \geq D\overline{p} + D\overline{q}$. Let $K_1 \subset \mathbb{R}^i$ be a closed ball and let $K_2 \subset X$ be compact. Let $U \subset X$ be a distinguished neighborhood so that the connected components of $\nu|_U$ are finitely fibered (so that the cohomology cross product above is defined). Let $\alpha \in H^*(\mathbb{R}^i, \mathbb{R}^i - K_1; F)$, $\beta \in I_{\overline{r}}\overline{H}^*_{\nu}(U, U - K_2; F)$, $x \in H_*(\mathbb{R}^i, \mathbb{R}^i - K_1; F)$, $y \in I^{\overline{q}}H_*(U, U - K_2; F^{\tau})$. Then,*

$$(\alpha \times \beta) \cap (x \times y) = (-1)^{|\beta|\cdot|x|}(\alpha \cap x) \times (\beta \cap y)$$

as elements of $I^{\overline{p}}H_*^{id_{\mathbb{R}^i} \times \nu}(\mathbb{R}^i \times U; F)$

Proof. For convenience set $A = \mathbb{R}^i - K_1$, $B = X - K_2$, and $C = \mathbb{R}^i \times X - K_1 \times K_2$ and we will also use id to mean $\mathrm{id}_{\mathbb{R}^i}$. We have the following commutative diagram where field coefficients are assumed and the unmarked tensor products are over F.

$$\begin{tikzcd}
H_*(S_*(\mathbb{R}^i, A) \otimes I^{\overline{p}}S_*(X, B; F^{\tau})) \arrow[rr, "\widetilde{d} \otimes \widetilde{d}"] \arrow[dr, "d \otimes d"] \arrow[ddd, "\times"'] & & H_*(S_*(\mathbb{R}^i) \otimes S_*(\mathbb{R}^i, A)) \otimes H_*(I^{\overline{p}}S^{\nu}_*(X)^t \otimes_{F[\pi]} I^{\overline{q}}S^{\nu}_*(X, B)) \arrow[dl, "\times \otimes \times"'] \arrow[d, "\cong"'] \\
& H_*(S_*(\mathbb{R}^i \times \mathbb{R}^i, \mathbb{R}^i \times A) \otimes (F^{\tau} \otimes_{F[\pi]} I^{Q}S^{\nu \times \nu}_*(X \times X, X \times B))) \arrow[d] & H_*\left(\left(S_*(\mathbb{R}^i) \otimes I^{\overline{p}}S^{\nu}_*(X)\right)^t \otimes_{F[\pi]} \left(S_*(\mathbb{R}^i, A) \otimes I^{\overline{q}}S^{\nu}_*(X, B)\right)\right) \arrow[dd, "\times \otimes \times"'] \\
& I^{Q}H_*^{\mathrm{id} \times \nu \times \mathrm{id} \times \nu}(\mathbb{R}^i \times X \times \mathbb{R}^i \times X, \mathbb{R}^i \times X \times C; F^{\tau}) & \\
I^{\overline{r}}H_*^{\mathrm{id} \times \nu}(\mathbb{R}^i \times X, C; F^{\tau}) \arrow[ur, "\mathrm{d}"] \arrow[rr, "\widetilde{d}"] & & H_*\left(I^{\overline{p}}S_*^{\mathrm{id} \times \nu}(\mathbb{R}^i \times X)^t \otimes_{F[\pi]} I^{\overline{q}}S_*^{\mathrm{id} \times \nu}(\mathbb{R}^i \times X, C)\right) \arrow[ul, "\times"']
\end{tikzcd}$$

The unmarked middle vertical arrow is the composition of the cross product and the coordinate swap map $t : \mathbb{R}^i \times (\mathbb{R}^i \times X) \times X \to \mathbb{R}^i \times (X \times \mathbb{R}^i) \times X$.

The top and bottom portions of the diagram commutes by the definition of $\widetilde{d}$, the left side commutes by looking at the chain level shuffle-product and the observation that $d_{\mathbb{R}^i \times X} = t \circ (d_{\mathbb{R}^i} \times d_X)$ where t is the coordinate swap map above and d_X, $d_{\mathbb{R}^i}$, $d_{\mathbb{R}^i \times X}$ denote the appropriate diagonal maps, and finally the right side commutes by associativity (Theorem 3.5.2) and commutativity (Theorem 3.5.1) of the cross product (note the top right isomorphism uses the Koszul sign convention).

Now let's show why this diagram proves our desired identity. Let Φ denote the isomorphism in the upper right corner of the diagram. Write $\widetilde{d}(x) = \sum_i a_i \otimes b_i$ where $a_i \in S_*(\mathbb{R}^i)$, $b_i \in S_*(\mathbb{R}^i, A)$ and write $\widetilde{d}(y) = \sum_i c_j \otimes_{F[\pi]} d_j$ where $c_j \in \left(I^{\bar{p}}S_*^{\nu}(X)\right)^t$ and $d_j \in I^{\bar{q}}S_*^{\nu}(X, B)$,. Notice that $|x| = |a_i| + |b_i|$ and $|y| = |c_j| + |d_j|$ for each i, j. Then by commutativity of the outside of the diagram above we have that

$$
\begin{aligned}
(\alpha \times \beta) \cap (x \times y) &= (1 \otimes (\alpha \times \beta))(\widetilde{d}(x \times y)) \\
&= (1 \otimes (\alpha \times \beta)) \left((\times \otimes \times)\Phi(\widetilde{d}(x) \otimes \widetilde{d}(y))\right) \\
&= (1 \otimes (\alpha \times \beta))(\times \otimes_{F[\pi]} \times)\Phi \left(\left(\sum_i a_i \otimes b_i \right) \otimes \left(\sum_j c_j \otimes_{F[\pi]} d_j \right) \right) \\
&= (1 \otimes (\alpha \times \beta))(\times \otimes_{F[\pi]} \times)\Phi \left(\sum_{i,j} (a_i \otimes b_i) \otimes (c_j \otimes_{F[\pi]} d_j) \right) \\
&= (1 \otimes (\alpha \times \beta))(\times \otimes_{F[\pi]} \times) \left(\sum_{i,j} (-1)^{|b_i| \cdot |c_j|} (a_i \otimes c_j) \otimes_{F[\pi]} (b_i \otimes d_j) \right) \\
&= (1 \otimes (\alpha \times \beta)) \left(\sum_{i,j} (-1)^{|b_i| \cdot |c_j|} (a_i \times c_j) \otimes_{F[\pi]} (b_i \times d_j) \right) \\
&= \sum_{i,j} (-1)^{|b_i| \cdot |c_j|} (-1)^{|\alpha \times \beta| \cdot |a_i \otimes c_j|} \overline{(\alpha \times \beta)(b_i \times d_j)} a_i \times c_j.
\end{aligned}
$$

Recall from the cohomology cross product (Remark 7.1.7) that $(\alpha \times \beta)(b_i \times d_j) = (-1)^{|\beta| \cdot |b_i|} \alpha(b_i)\beta(d_j)$. What's more, we have that α and β evaluate to zero unless $|\alpha| = |b_i|$ and $|\beta| = |d_j|$and because $\alpha(b_i) \in F$, we have $\overline{\alpha(b_i)} = \alpha(b_i)$. Thus, we have

$$
\begin{aligned}
(\alpha \times \beta) \cap (x \times y) &= \sum_{i,j} (-1)^{|b_i|\cdot|c_j|}(-1)^{|\alpha\times\beta|\cdot|a_i\otimes c_j|}\overline{(\alpha \times \beta)(b_i \times d_j)}a_i \times c_j \\
&= \sum_{i,j} (-1)^{|b_i|\cdot|c_j|}(-1)^{(|\alpha|+|\beta|)\cdot(|a_i|+|c_j|)}\overline{(-1)^{|\beta|\cdot|b_i|}\alpha(b_i)\beta(d_j)}a_i \times c_j \\
&= \sum_{i,j} (-1)^{|b_i|\cdot|c_j|}(-1)^{(|\alpha|+|\beta|)\cdot(|a_i|+|c_j|)}(-1)^{|\beta|\cdot|b_i|}\overline{\beta(d_j)}\ \overline{\alpha(b_i)}a_i \times c_j \\
&= \sum_{i,j} (-1)^{|b_i|\cdot|c_j|}(-1)^{(|\alpha|+|\beta|)\cdot(|a_i|+|c_j|)}(-1)^{|\beta|\cdot|b_i|}\overline{\beta(d_j)}\ \alpha(b_i)a_i \times c_j \\
&= \sum_{i,j} (-1)^{|b_i|\cdot|c_j|}(-1)^{(|\alpha|+|\beta|)\cdot(|a_i|+|c_j|)}(-1)^{|\beta|\cdot|b_i|}a_i \times (\alpha(b_i)\overline{\beta}(d_j)c_j)
\end{aligned}
$$

where we have used that $\alpha(b_i) \in F$ and the action of π on $\mathrm{id}_{\mathbb{R}^i} \times \nu$.

Looking at the sign term above we see have -1 to the power $|b_i| \cdot |c_j| + |\alpha| \cdot |a_i| + |\alpha| \cdot |c_j| + |\beta| \cdot |a_i| + |\beta| \cdot |c_j| + |\beta| \cdot |b_i|$. However, using that $|\alpha| = |b_i|$ this means that

$$|b_i| \cdot |c_j| + |\alpha| \cdot |a_i| + |\alpha| \cdot |c_j| + |\beta| \cdot |a_i| + |\beta| \cdot |c_j| + |\beta| \cdot |b_i|$$

is the same as

$$|\alpha| \cdot |a_i| + 2|b_i| \cdot |c_j| + |\beta| \cdot |c_j| + |\beta| \cdot (|a_i| + |b_i|)$$

and then using that $|a_i| + |b_i| = |x|$ this is

$$|\alpha| \cdot |a_i| + 2|b_i| \cdot |c_j| + |\beta| \cdot |c_j| + |\beta| \cdot |x|.$$

Reducing modulo 2 we have

$$|\alpha| \cdot |a_i| + |\beta| \cdot |c_j| + |\beta| \cdot |x|.$$

Thus, we have shown

$$(\alpha \times \beta) \cap (x \times y) = \sum_{i,j} (-1)^{|\beta|\cdot|x| + |\alpha|\cdot|a_i| + |\beta|\cdot|c_j|} a_i \times (\alpha(b_i)\overline{\beta(d_j)}c_j).$$

Next, we consider the righthand side of our identity.

$$\begin{aligned}
(-1)^{|\beta|\cdot|x|}(\alpha \cap x) \times (\beta \cap y) &= (-1)^{|\beta|\cdot|x|} \left((1 \otimes \alpha)(\widetilde{d}(x))\right) \times \left((1 \otimes \beta)(\widetilde{d}(y))\right) \\
&= (-1)^{|\beta|\cdot|x|} \left((1 \otimes \alpha)(\sum_i a_i \otimes b_i)\right) \times \left((1 \otimes \beta)(\sum_j c_j \otimes d_j)\right) \\
&= (-1)^{|\beta|\cdot|x|} \left(\sum_i (-1)^{|\alpha|\cdot|a_i|}\overline{\alpha(b_i)}a_i\right) \times \left((-1)^{|\beta|\cdot|c_j|}\overline{\beta(d_j)}c_j\right) \\
&= \sum_{i,j} (-1)^{|\beta|\cdot|x| + |\alpha|\cdot|a_i| + |\beta|\cdot|c_j|} \left(\overline{\alpha(b_i)}a_i\right) \times \left(\overline{\beta(d_j)}c_j\right) \\
&= \sum_{i,j} (-1)^{|\beta|\cdot|x| + |\alpha|\cdot|a_i| + |\beta|\cdot|c_j|} (\alpha(b_i)a_i) \times (\overline{\beta(d_j)}c_j) \\
&= \sum_{i,j} (-1)^{|\beta|\cdot|x| + |\alpha|\cdot|a_i| + |\beta|\cdot|c_j|} a_i \times (\alpha(b_i)\overline{\beta(d_j)}c_j)
\end{aligned}$$

where we have used bilinearity of the cross product and again used that $\alpha(b_i) \in F$. Thus, we see this agrees with the lefthand side we computed above. $\square$

7.2 Universal Poincaré duality

Let $K \subset X$ be compact. Let $\Gamma_K \in I^{\bar{0}}H_n(X, X-K; F^\tau)$ be the twisted fundamental class over K defined in Definition 5.4.2 . We define $\mathscr{D}_K : I_{\bar{p}}\overline{H}^i_\nu(X, X-K; F) \to I^{\bar{p}}H^\nu_{n-i}(X; F)$ by

$$\mathscr{D}_K(\alpha) = (-1)^{in}\alpha \cap \Gamma_K.$$

We note this is consistent if we let K vary. More precisely, let $K \subset K'$ and let $j : (X, X-K') \to (X, X-K)$ be the inclusion map (so $j|_X : X \to X$ is the identity map) of pairs. By uniqueness of fundamental classes we must have that $j_*(\Gamma_{K'}) = \Gamma_K$. So we have the following computation

$$\begin{aligned}
\mathscr{D}_{K'}(j^*(\alpha)) =& (-1)^{|\alpha|n} j^*(\alpha) \cap \Gamma_{K'} \\
=& (-1)^{|\alpha|n} \alpha \cap j_*(\Gamma_{K'}) \\
=& (-1)^{|\alpha|n} \alpha \cap \Gamma_K \\
=& \mathscr{D}_K(\alpha).
\end{aligned}$$

We define universal cohomology with compact supports to be the limit over compact subsets $K \subset X$

$$I^c_{\bar{p}}\overline{H}^*_\nu(X; F) := \varinjlim I_{\bar{p}}\overline{H}^*_\nu(X, X-K; F).$$

The computation above implies we have a well-defined map $\mathscr{D} : I^c_{\overline{p}}\overline{H}^*_\nu(X;F) \to I^{\overline{p}}H^\nu_*(X;F)$ given by

$$\mathscr{D} := \varinjlim \mathscr{D}_K$$

We now arrive at our main theorem. Observe that the theorem always holds over $\mathbb{Q}$ and for compact pseudomanifolds will hold over any field as long as the characteristic of the field does not divide branching indices of the branched cover which will be a finite set (Corollary 4.3.6).

Theorem 7.2.1. *Let X be a stratified pseudomanifold with X_{reg} connected and with perversity $\overline{0} \leq \overline{p} \leq \overline{t}$ and dual perversity $\overline{q}$. Let ν be locally finite unbranched oriented regular connected cover of X_{reg} with deck transformation group π. Let F be a ν-good field. Then,*

$$\mathscr{D} : I^c_{\overline{p}}\overline{H}^*_\nu(X;F) \to I^{\overline{q}}H^\nu_*(X;F)$$

is an isomorphism of $F[\pi]$-modules.

Proof. We proceed by induction on depth(X). If depth$(X) = 0$, then X is a manifold and the theorem follows by standard Universal Poincaré duality for manifolds (17, Theorem 4.65). We will apply Theorem 2.4.7 to prove the theorem.

First assume X is normal. Let $\mathbf{F}_*$ be a functor defined on open sets of X by $\mathbf{F}_*(U) = I^c_{\overline{p}}\overline{H}^*(U;F)$ and define a functor $\mathbf{G}_*$ by $\mathbf{G}_*(U) = I^{\overline{q}}H^\nu_*(U;F)$. Consider the natural transformation $\mathbf{F}_* \to \mathbf{G}_*$ induced by $\mathscr{D}$.

Consider the case $U \subset X$ is open and homeomorphic to Euclidean space. Then U is evenly covered and $\mathbf{F}_*(U) \to \mathbf{G}_*(U)$ is an isomorphism of $F[\pi]$-moduels by (17, Theorem 4.65).

Next, assume that $\{U_\alpha\}$ is a totally ordered set of open subsets of X. Then because $\{U_\alpha\}$ is totally ordered and because the direct limit in the definition of cohomology with compact supports is over compact subsets, we have that the natural map $\varinjlim_\alpha \mathbf{F}_*(U_\alpha) \to \mathbf{F}_*(\cup_\alpha U_\alpha)$ is an isomorphism. What's more, we clearly have that $\varinjlim_\alpha \mathbf{G}_*(U_\alpha) \to \mathbf{G}_*(\cup_\alpha U_\alpha)$ is an isomorphism. Thus, the second condition of Theorem 2.4.7 is met.

Now assume the theorem holds for a link $L \subset X$ of dimension $k-1$. We will show that the theorem holds for $cL \subset X$. To prove it for cL we choose cofinal compact sets of the form $K = \bar{c}_r L$ where $\bar{c}_r L$ is the image of $[0, r] \times L$ in $cL = [0, 1) \times L/\sim$ and where $0 < r < 1$. For a moment fix r. We claim that

$$\mathscr{D}_K : I^{\bar{p}}\overline{H}^*_\nu(cL, cL - c_r L; F) \to I^{\bar{q}}H^\nu_*(cL; F)$$

is already an isomorphism. Let $b \in (r, 1)$ and let $j : L \to cL$ take x to (x, b). Because j gives a stratified homotopy equivalence $cL - c_r L \simeq L$ we have that j induces an isomorphism $I^{\bar{r}}H^\nu_*(cL, cL - c_r L; F) \cong I^{\bar{r}}H^\nu_*(cL, L; F)$ for any perversity $\bar{r}$ with $\bar{0} \leq \bar{r} \leq \bar{t}$. Let v denote the cone vertex.

For $i < k - \overline{p}(\{v\})$ the domain and range of $\mathscr{D}_{c_rL}$ are both zero by Proposition 6.3.4 and Proposition 2.3.3. So the isomorphism is trivial in this dimension range. So take $i \geq k - \overline{p}(\{v\})$. We have the following commutative diagram

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^{i}_{\nu}(cL, cL - c_rL; F) \arrow[d, "\mathscr{D}_{c_rL}"] & I_{\overline{p}}\overline{H}^{i-1}_{\nu}(cL - c_rL; F) \arrow[l, "\delta"'] \arrow[r, "j^*", "\cong"'] \arrow[d, "\cap j_*\Gamma_L"] & I_{\overline{p}}\overline{H}^{i-1}_{\nu}(L; F) \arrow[d, "\cap\Gamma_L"] \\
I^{\overline{q}}H^{\nu}_{k-i}(cL; F) & I^{\overline{q}}H^{\nu}_{k-i}(cL - c_rL; F) \arrow[l, "\text{inclusion}"'] & I^{\overline{q}}H^{\nu}_{k-i}(L; F) \arrow[l, "j_*"', "\cong"]
\end{tikzcd}$$

The right square commutes by Proposition 7.1.3 and the left square commutes up to sign by Proposition 7.1.6 and noting that $j_*\Gamma_L = \partial\Gamma_{c_rL}$. The boundary map δ is an isomorphism by Proposition 6.3.5 and the map labelled inclusion is an isomorphism from the cone formula Proposition 2.3.3 and the fact that $L \hookrightarrow cL - c_rL$ is a stratified homotopy equivalence. We need to show that the right vertical map is an isomorphism. Because X is normal we have that L is a connected normal compact pseudomanifold so that L_{reg} is connected. Fix a connected component of $\nu|_L$ and call this ν'. Notice that ν' is a locally finite unbranched oriented regular connected cover of L_{reg}. Also notice that by Proposition 4.3.4 we have that because F is a ν-good field, it is also ν'-good field. Let $\pi' \subset \pi$ be the subgroup of π of deck transformations of ν'. Let J be a set containing a single choice of element from each left coset in π/π'.

By Lemma 6.3.8 we have the isomorphisms of F-vector spaces $I_{\overline{p}}\overline{S}^{*}_{\nu}(L; F) \cong \bigoplus_{\alpha\in J} \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_{*}(L; F), F[\pi'])$ and $I^{\overline{q}}S^{\nu}_{*}(L; F) \cong \bigoplus_{\alpha\in J} I^{\overline{q}}S^{\nu'}_{*}(L; F)$. What's more, by taking the cap product with Γ_L we have a map $\cap\Gamma_L : I_{\overline{p}}\overline{H}^{*}_{\nu'}(L; F) \to (I^{\overline{q}}H^{\nu'}_{*}(L; F)^t)$.

By induction on depth, these maps are isomorphisms of $F[\pi]'$-modules. Therefore, to complete our argument we show that the following diagram commutes

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_{\nu}(L;F) \arrow[r, "\cong"] \arrow[d, "\cap\Gamma_L"] & \bigoplus_{\alpha\in J} I_{\overline{p}}\overline{H}^*_{\nu'}(L;F) \arrow[d, "\bigoplus\cap\Gamma_L"] \\
I^{\overline{q}}H^{\nu}_*(L;F) & \bigoplus_{\alpha\in J} I^{\overline{q}}H^{\nu'}_*(L;F). \arrow[l, "\cong"']
\end{tikzcd}$$

Taking the cap product with Γ_L from the left vertical arrow we use the algebraic diagonal map $I^{\overline{0}}H_*(L;F^{\tau}) \to H_*((I^{\overline{q}}S^{\nu}_*(L;F))^t \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(L;F))$ while the direct sum of cap products on the right use the algebraic diagonal map $I^{\overline{0}}H_*(L;F^{\tau}) \to H_*((I^{\overline{q}}S^{\nu'}_*(L;F))^t \otimes_{F[\pi']} I^{\overline{p}}S^{\nu'}_*(L;F))$. Let $f \in I_{\overline{p}}\overline{H}^*_{\nu}(L;F)$. By the top isomorphism we can write $f = \sum_{\alpha\in J}\alpha\widetilde{f_\alpha}$ where $\alpha \in J$ and $\widetilde{f_\alpha} \in \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S^{\nu}_*(L;F), F[\pi'])$ is the extension of $f_\alpha \in \mathrm{Hom}_{F[\pi']}(I^{\overline{p}}S^{\nu'}_*(L;F), F[\pi'])$. Going around the square right, down, and then left, f maps to $\sum_{\alpha\in J}\alpha\cdot\iota_*((f_\alpha\cap\Gamma_L))$. On the other hand, f maps down to $\sum_{\alpha\in J}\alpha\cdot(\widetilde{f_\alpha}\cap\Gamma_L)$ which is the same as $\sum_{\alpha\in J}\alpha\cdot\iota_*((f_\alpha\cap\Gamma_L))$ by Proposition 7.1.4. Thus, the diagram commutes so the left vertical map is an isomorphism of F-vector spaces (what's more $F[\pi]$-modules since the map is easily seen to be a map of $F[\pi]$-modules). Thus, in the previous diagram the leftmost vertical map is also an isomorphism of $F[\pi]$-modules.

Next, we show the theorem holds for open sets of the form $Y = \mathbb{R}^i \times cL$. Note compacts sets of the form $K_1 \times K_2$ are co-final among compact subsets of $\mathbb{R}^i \times cL$. Moreover, we can take K_1 to be closed balls and K_2 to be closed cones. We have the commutative diagram up to sign below by Proposition 7.1.8.

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(Y, Y-(K_1\times K_2); F) \arrow[d, "\cap(\Gamma_{K_1}\times\Gamma_{K_2})"] & H^*(\mathbb{R}^i, \mathbb{R}^i - K_1; F) \otimes_F I_{\overline{p}}\overline{H}^*_\nu(cL, cL-K_2; F) \arrow[l, "\times"'] \arrow[d, "\cap\Gamma_{K_1}\otimes\cap\Gamma_{K_2}"] \\
I^{\overline{q}}H^\nu_*(Y; F) & H_*(\mathbb{R}^i; F)\otimes I^{\overline{q}}H^\nu_*(cL; F) \arrow[l, "\times"']
\end{tikzcd}$$

The top horizontal map is the cohomology cross product which is an isomorphism by Remark 7.1.7, the lower horizontal map is an isomorphism by Theorem 3.4.1, and the right vertical map is an isomorphism upon passage to the direct limit. Therefore, by commutativity the left vertical map is also an isomorphism upon passage to the direct limit by the case above already considered. However, $\Gamma_{K_1\times K_2} = \Gamma_{K_1}\times\Gamma_{K_2}$ by Proposition 5.5.2 so that we have the desired isomorphism in this case. What's more, the left vertical map is easily seen to be a map of $F[\pi]$-modules so that it's actually an isomorphism of $F[\pi]$-modules.

Finally, we have the following diagram of Mayer-Vietoris sequences. We prove commutativity of this diagram up to sign in the next subsection [8].

$$\begin{tikzcd}
{} \arrow[r] & I_{\overline{p}}\overline{H}^i_\nu(U\cap V; F) \arrow[r] \arrow[d, "\mathscr{D}"] & I_{\overline{p}}\overline{H}^i_\nu(U; F)\oplus I_{\overline{p}}\overline{H}^i_\nu(V; F) \arrow[r] \arrow[d, "\mathscr{D}\oplus-\mathscr{D}"] & I_{\overline{p}}\overline{H}^i_\nu(U\cup V; F) \arrow[r] \arrow[d, "\mathscr{D}"] & I_{\overline{p}}\overline{H}^{i+1}_\nu(U\cap V; F) \arrow[r] \arrow[d, "\mathscr{D}"] & {} \\
{} \arrow[r] & I^{\overline{q}}H^\nu_{n-i}(U\cap V; F) \arrow[r] & I^{\overline{q}}H^\nu_{n-i}(U; F)\oplus I^{\overline{q}}H^\nu_{n-i}(V; F) \arrow[r] & I^{\overline{q}}H^\nu_{n-i}(U\cup V; F)) \arrow[r] & I^{\overline{q}}H^\nu_{n-i-1}(U\cap V; F) \arrow[r] & {}
\end{tikzcd} \quad (1)$$

[8] It is enough for diagram (1) to commute up to sign as the proof of Theorem 2.4.7 uses Mayer-Vietoris long exact sequences to make a Zorn's lemma argument using the five lemma that two of the maps are isomorphisms. Since multiplying by -1 has no effect on whether the maps are isomorphisms, we can still apply Theorem 2.4.7 in this setting.

This completes the proof in the case X is normal by Theorem 2.4.7. Now consider the case X_{reg} is connected, but X is possibly not normal. Let $\mathbf{n} : X^N \to X$ be the normalization of X. Notice that because X_{reg} is connected we have that X^N is a connected normal pseudomanifold. Let $K \subset X$ be compact and let $K^N = \mathbf{n}^{-1}(K)$ which is compact since $\mathbf{n}$ is proper. We have the commutative diagram below.

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(X^N, X^N - K^N) \arrow[d, "\cap\Gamma_{K^N}"] & I_{\overline{p}}\overline{H}^*_\nu(X, X - K) \arrow[l, "\mathbf{n}^*"'] \arrow[d, "\cap\Gamma_K"] \\
I^{\overline{q}}H^\nu_*(X^N) \arrow[r, "\mathbf{n}_*"] & I^{\overline{q}}H^\nu_*(X)
\end{tikzcd}$$

The diagram commutes by Proposition 7.1.5 and because $\Gamma_K = \mathbf{n}_*\Gamma_{K^N}$ by definition. The horizontal maps are isomorphisms since they are isomorphisms on the level of chain complexes by applying Proposition 2.1.8 and because $\mathbf{n}$ obviously preserves deck transformation actions. The left vertical map is an isomorphism upon passage to direct limits since X^N is normal and by definition F is ν-good on X^N (Definition 6.2.4).

. Thus, the right vertical map is also an isomorphism upon passage to the direct limit which is what we wanted to show.

□

7.3 Commutativity of Diagram (1)

In this subsection we prove commutativity of Diagram (1) in the preceding theorem. The proofs of our results in this subsection follow those in (11, Section 6.1). We will suppress field coefficients from the notation. The proof of commutativity of Diagram (1) follows immediately from the lemma below.

Lemma 7.3.1. *Let $X = U \cup V$ be an open cover of X and let K and L be compact subsets of U and V; respectively. The following diagrams commute*

(a)

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(X, X-K\cap L) \arrow[r] \arrow[d, "\cong"', "exc"] & I_{\overline{p}}\overline{H}^*_\nu(X, X-K) \oplus I_{\overline{p}}\overline{H}^*_\nu(X, X-L) \arrow[d, "\cong"', "exc"] \\
I_{\overline{p}}\overline{H}^*_\nu(U\cap V, U\cap V-K\cap L) \arrow[d, "\cap\Gamma_{K\cap L}"] & I_{\overline{p}}\overline{H}^*_\nu(U, U-K) \oplus I_{\overline{p}}\overline{H}^*_\nu(V, V-L) \arrow[d, "\cap\Gamma_K \oplus -\cap\Gamma_L"] \\
I^{\overline{q}}H^\nu_{n-k}(U\cap V) \arrow[r] & I^{\overline{q}}H^\nu_{n-k}(U) \oplus I^{\overline{q}}H^\nu_{n-k}(V)
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(X, X-K) \oplus I_{\overline{p}}\overline{H}^*_\nu(X, X-L) \arrow[r] \arrow[d, "\cong", "exc"'] & I_{\overline{p}}\overline{H}^*_\nu(X, X-K\cup L) \arrow[dd, "\Gamma_{K\cup L}"] \\
I_{\overline{p}}\overline{H}^*_\nu(U, U-K) \oplus I_{\overline{p}}\overline{H}^*_\nu(V, V-L) \arrow[d, "\Gamma_K \oplus -\Gamma_L"] & \\
I^{\overline{q}}H^\nu_{n-k}(U) \oplus I^{\overline{q}}H^\nu_{n-k}(V) \arrow[r] & I^{\overline{q}}H^\nu_{n-k}(X)
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(X, X-K\cup L) \arrow[r, "\delta"] \arrow[d, "\Gamma_{K\cup L}"] & I_{\overline{p}}\overline{H}^{k+1}_\nu(X, X-K\cap L) \arrow[r, "\cong", "exc"'] & I_{\overline{p}}\overline{H}^{k+1}_\nu(U\cap V, U\cap V-K\cap L) \arrow[d, "\Gamma_{K\cap L}"] \\
I^{\overline{q}}H^\nu_{n-k}(X) \arrow[rr, "\partial"] & & I^{\overline{q}}H^\nu_{n-k-1}(U\cap V)
\end{tikzcd}$$

Proof. To prove part (a) it suffices to work one summand at a time. Let us consider the first summand. Commutativity here follows by the diagram

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^k_\nu(X, X - K \cap L) \arrow[rr] \arrow[d] \arrow[dr] & & I_{\overline{p}}\overline{H}^k_\nu(X, X - K) \arrow[d] \\
I_{\overline{p}}\overline{H}^k_\nu(U \cap V, U \cap V - K \cap L) \arrow[d, "\cap\Gamma_{K\cap L}"] & I_{\overline{p}}\overline{H}^k_\nu(U, U - K \cap L) \arrow[l] \arrow[r] \arrow[dr, "\cap\Gamma_{K\cap L}"] & I_{\overline{p}}\overline{H}^k_\nu(U, U - K) \arrow[d, "\cap\Gamma_K"] \\
I^{\overline{q}}H^\nu_{n-k}(U \cap V) \arrow[rr] & & I^{\overline{q}}H^\nu_{n-k}(U)
\end{tikzcd}$$

Each of the maps not labelled are inclusion maps. The upper half obviously commutes since they are inclusion maps. The lower half commutes by Proposition 7.1.3 and since the inclusion map $(U, U - K \cap L) \to (U, U - K)$ maps Γ_K to $\Gamma_{K\cap L}$ by Theorem 5.4.4. The second summand is similar.

To prove part (b) we can similarly proceed one summand at a time. Consider the following commutative diagram

$$\begin{tikzcd}
I_{\overline{p}}\overline{H}^*_\nu(X, X - K) \arrow[r] \arrow[d] \arrow[ddr, "\Gamma_K"] & I_{\overline{p}}\overline{H}^*_\nu(X, X - K \cup L)) \arrow[dd, "\Gamma_{K\cup L}"] \\
I_{\overline{p}}\overline{H}^*_\nu(U, U - K) \arrow[d, "\Gamma_K"] & \\
I^{\overline{q}}H^\nu_{n-k}(U) \arrow[r] & I^{\overline{q}}H^\nu_{n-k}(X)
\end{tikzcd}$$

Each triangle again commutes by Proposition 7.1.3 and the fact that the inclusion map $(X, X - K \cup L) \to (X, X - K)$ takes $\Gamma_{K \cup L}$ to Γ_K by Theorem 5.4.4. Commutativity for the second summand follows similarly.

Finally, to prove part (c) we first need a lemma which we prove at the end of the subsection.

Lemma 7.3.2. *Let $X = U \cup V$ be an open cover of X and let K and L be compact subsets of U and V; respectively. There exist chains*

$$\beta_{U-L} \in (I^{\overline{p}} S^{\nu}_{*}(U - L))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(U - L, U - K \cup L)$$

$$\beta_{U \cap V} \in (I^{\overline{p}} S^{\nu}_{*}(U \cap V))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(U \cap V, U \cap V - K \cup L)$$

$$\beta_{V-K} \in (I^{\overline{p}} S^{\nu}_{*}(V - K))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(V - K, V - K \cup L)$$

Such that $\beta_{U-L} + \beta_{U \cap V} + \beta_{V-K}$ represents $\widetilde{d}(\Gamma_{K \cup L}) \in H_*((I^{\overline{p}} S^{\nu}_{*}(X))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(X, X - K \cup L))$

Assuming the lemma for now we complete the proof of part (c) of Lemma 7.3.1. The proof follows (11) which in turn follows as in (14). First note that the inclusion $(X, X - K \cup L) \to (X, X - K \cap L)$ maps $\Gamma_{K \cup L}$ to $\Gamma_{K \cap L}$. This means that the image of $\beta_{U-L} + \beta_{U \cap V} + \beta_{V-K}$ in $H_*((I^{\overline{p}} S^{\nu}_{*}(X))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(X, X - K \cap L))$ represents the image of $\widetilde{d}(\Gamma_{K \cap L})$. But this is just $\beta_{U \cap V}$ since the other terms map to 0 in $(I^{\overline{p}} S^{\nu}_{*}(X))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(X, X - K \cap L)$. Thus, $\beta_{U \cap V}$ represents the class $\widetilde{d}(\Gamma_{K \cap L})$ in $H_*((I^{\overline{p}} S^{\nu}_{*}(U \cap V))^t \otimes_{F[\pi]} I^{\overline{q}} S^{\nu}_{*}(U \cap V, U \cap V - K \cap L))$

Now we let $\varphi \in I^{\bar{p}}\overline{S}_{\nu}^{*}(X, X-K \cup L)$. We calculate the image of $[\varphi]$ for the two ways going around the diagram in part (c). Let A and B denote $X-K$ and $X-L$; respectively. Thus, $\varphi \in I^{\bar{p}}\overline{S}_{\nu}^{*}(X, A \cap B)$. Now from the zig-zag construction of the connecting homomorphism we have that $\delta[\varphi] = [\delta\varphi_A]$ where $\varphi = \varphi_A - \varphi_B$ with $\varphi_A \in I^{\bar{p}}\overline{S}_{\nu}^{*}(X, A)$ and $\varphi_B \in I^{\bar{p}}\overline{S}_{\nu}^{*}(X, B)$. Moving on to $I^{\bar{q}}H_{n-k-1}^{\nu}(U \cap V)$ we obtain $[(1 \otimes \delta\varphi_A)(\beta_{U\cap V})]$. However, this is the same in homology as $(-1)^{k+1}[(1 \otimes \varphi_A)(\partial\beta_{U\cap V})]$ by the equation

$$\partial((1 \otimes \varphi_A)(\beta_{U\cap V})) = (1 \otimes \delta\varphi_A)(\beta_{U\cap V}) + (-1)^k(1 \otimes \varphi_A)(\partial\beta_{U\cap V}).$$

To see the above identity holds write $\beta_{U\cap V} = \sum_j y_j \otimes z_j$. Then we have

$$\begin{aligned}
(1 \otimes \varphi_A)(\partial\beta_{U\cap V}) &= \sum_j \big((1 \otimes \varphi_A)((\partial y_j) \otimes z_j + (-1)^{|y_j|} y_j \otimes \partial z_j)\big) \\
&= \sum_j \Big((-1)^{(|y_j|-1)k}\overline{\varphi_A(z_j)}\partial y_j + (-1)^{|y_j|}(-1)^{|y_j|k}\overline{\varphi_A(\partial z_j)}y_j\Big) \\
&= \sum_j \Big((-1)^k\partial(1 \otimes \varphi_A)(y_j \otimes z_j) + (-1)^{(k+1)(|y_j|)}\overline{-(-1)^k(\delta\varphi_A)(z_j)}y_j\Big) \\
&= (-1)^k\partial(1 \otimes \varphi_A)\left(\sum_j y_j \otimes z_j\right) + (1 \otimes \delta\varphi_A)\left(\sum_j y_j \otimes z_j\right) \\
&= (-1)^k\partial(1 \otimes \varphi_A)(\beta_{U\cap V}) - (-1)^k(1 \otimes \delta\varphi_A)(\beta_{U\cap V})
\end{aligned}$$

where on the third equality we have the Koszul sign convention for coboundary maps. The desired identity then follows by multiplying both sides by $(-1)^k$ and rearranging terms.

Now we go around the diagram in part (c) the other way. Let $\beta = \beta_{U-L} + \beta_{U\cap V} + \beta_{V-K}$. Then $[\varphi]$ first maps to $[(1 \otimes \varphi)(\beta)]$. From the zig-zag construction of the connecting homomorphism ∂ we write $(1 \otimes \varphi)(\beta)$ as the sum of a chain in U and a chain in V

$$(1 \otimes \varphi)(\beta) = (1 \otimes \varphi)(\beta_{U-L}) + ((1 \otimes \varphi)(\beta_{U\cap V}) + (1 \otimes \varphi)(\beta_{V-K}))$$

and then we take the boundary of the first term above obtaining $[\partial(1 \otimes \varphi)(\beta_{U-L})]$. To see this is the same as $(-1)^{k+1}[(1 \otimes \varphi_A)(\partial\beta_{U\cap V})]$ we have the following computation

$$\begin{aligned}
\partial(1 \otimes \varphi)(\beta_{U-L}) &= (-1)^k(1 \otimes \varphi)(\partial\beta_{U-L}) && \text{since } \delta\varphi = 0 \\
&= (-1)^k(1 \otimes \varphi_A)(\partial\beta_{U-L}) && \text{since } (1 \otimes \varphi_B)(\partial\beta_{U-L}) = 0 \\
& && \text{because } \varphi_B \text{ vanishes on chains in } B = X - L \\
&= (-1)^{k+1}(1 \otimes \varphi_A)(\partial\beta_{U\cap V})
\end{aligned}$$

The last equality follows from the equation $\partial\beta_{U-L} = \partial\beta - \partial\beta_{U\cap V} - \partial\beta_{V-K} = -\partial\beta_{U\cap V} - \partial\beta_{V-K}$ and because φ_A vanishes on chains in $A = X - K$.

$\square$

Proof of Lemma 7.3.2: Let $\mathcal{C}$ denote the category with objects $U - L, U \cap V, V - K$ and their intersections and inclusion maps as morphisms. To prove the lemma it suffices to show $\widetilde{d}(\Gamma_{K \cup L})$ is in the image of

$$\kappa : \varinjlim_{W \in \mathcal{C}} (I^{\overline{p}} S_*^{\nu}(W))^t \otimes_{F[\pi]} I^{\overline{q}} S_*^{\nu}(W, W - K \cup L) \to (I^{\overline{p}} S_*^{\nu}(X))^t \otimes_{F[\pi]} I^{\overline{q}} S_*^{\nu}(X, X - K \cup L)$$

Let $Y \subset X \times X$ denote the subspace

$$((U - L) \times (U - L)) \cup ((U \cap V) \times (U \cap V)) \cup ((V - K) \times (V - K))$$

and notice that $d(X - (K \cup L)) \subset Y - (X \times (K \cup L))$.

$$\begin{tikzcd}
H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{\overline{0}} S_*^{\nu}(X, X - K \cup L)) \arrow[r, "1 \otimes d"] \arrow[dr, "1 \otimes d"] & H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\overline{p},\overline{q}}} S_*^{\nu \times \nu}(X \times X, X \times (X - K \cup L))) & H_*((I^{\overline{p}} S_*^{\nu}(X))^t \otimes_{F[\pi]} I^{\overline{q}} S_*^{\nu}(X, X - (K \cup L))) \arrow[l, "\cong"'] \\
 & H_*(F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\overline{p},\overline{q}}} S_*^{\nu \times \nu}(Y, Y - (X \times (K \cup L)))) \arrow[u] & \\
 & H_*(\varinjlim_{W \in \mathcal{C}} F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\overline{p},\overline{q}}} S_*^{\nu \times \nu}(W \times W, W \times (W - K \cup L))) \arrow[u, "\lambda"] & H_*(\varinjlim_{W \in \mathcal{C}} (I^{\overline{p}} S_*^{\nu}(W))^t \otimes_{F[\pi]} I^{\overline{q}} S_*^{\nu}(W, W - K \cup L)) \arrow[l, "\mu"'] \arrow[uu, "\kappa"']
\end{tikzcd}$$

Notice $\widetilde{d}(\Gamma_{K \cup L})$ is the image of $\Gamma_{K \cup L}$ going across the top row. The map λ is an isomorphism by Corollary 6.1.4 and by the fact that $F^{\tau_\pi} \otimes_{F[\pi]}$ preserves chain homotopy equivalences over $F[\pi]$. If we can show that μ is an isomorphism, then this will show $\widetilde{d}(\Gamma_{K \cup L})$ is in the image of κ.

Let W_1, W_2, W_3 denote $U - L, U \cap V, V - K$; respectively. Let $\mathcal{C}'$ denote the subcategory of $\mathcal{C}$ with objects W_1, W_2, and $W_1 \cap W_2$. Let $\mathcal{C}''$ be the subcategory of $\mathcal{C}$ with objects $W_1 \cap W_3, W_2 \cap W_3$, and $W_1 \cap W_2 \cap W_3$. Note that for any functor $\mathbf{D}$ from $\mathcal{C}$ to chain complexes we have that

$$\varinjlim_{W \in \mathcal{C}} \mathbf{D}(W)$$

is the pushout of the diagram

$$\begin{tikzcd}
\varinjlim_{W \in \mathcal{C}''} \mathbf{D}(W) \arrow[r] \arrow[d] & \mathbf{D}(W_3) \\
\varinjlim_{W \in \mathcal{C}'} \mathbf{D}(W) &
\end{tikzcd}$$

We note that

$$\varinjlim_{W \in \mathcal{C}'} \mathbf{D}(W) \text{ and } \varinjlim_{W \in \mathcal{C}''} \mathbf{D}(W)$$

are also pushouts of the following respective diagrams

$$\begin{tikzcd}
\mathbf{D}(W_1 \cap W_2) \arrow[r] \arrow[d] & \mathbf{D}(W_1) \\
\mathbf{D}(W_2) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{D}(W_1 \cap W_2 \cap W_3) \arrow[r] \arrow[d] & \mathbf{D}(W_1 \cap W_3) \\
\mathbf{D}(W_2 \cap W_3) &
\end{tikzcd}$$

For any pushout diagram of chain complexes

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

we have that if $A \to B \oplus C$ is injective, then there is a short exact sequence $0 \to A \to B \oplus C \to D \to 0$ so there is a Mayer-Vietoris long exact sequence

$$\to H_i(A) \to H_i(B) \oplus H_i(C) \to H_i(D) \to H_{i-1}(A) \to .$$

The maps for our push out diagrams will be injective by part (ii) of Theorem 6.1.1 and because $\mathcal{C}''$ is a subcategory of $\mathcal{C}'$. Using this long exact sequence it follows from Proposition 6.2.12 and the five lemma that

$$\varinjlim_{W \in \mathcal{C}'} F^{\tau_\pi} \otimes_{F[\pi]} I^{\bar{p}} S_*^\nu(W) \otimes I^{\bar{q}} S_*^\nu(W, W - K \cup L) \longrightarrow \varinjlim_{W \in \mathcal{C}'} F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\bar{p},\bar{q}}} S_*^{\nu \times \nu}(W \times W, W \times (W - K \cup L))$$

and

$$\varinjlim_{W \in \mathcal{C}''} F^{\tau_\pi} \otimes_{F[\pi]} I^{\bar{p}} S_*^\nu(W) \otimes I^{\bar{q}} S_*^\nu(W, W - K \cup L) \to \varinjlim_{W \in \mathcal{C}''} F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\bar{p},\bar{q}}} S_*^{\nu \times \nu}(W \times W, W \times (W - K \cup L))$$

are quasi-isomorphisms.

Using the above isomorphisms, and Proposition 6.2.12 once more, we apply the Mayer-Vietoris sequence to the pushout diagram above for

$$\varinjlim_{W\in\mathcal{C}} \mathbf{D}(W)$$

and the five lemma to see that

$$\varinjlim_{W\in\mathcal{C}} F^{\tau_\pi} \otimes_{F[\pi]} I^{\bar{p}}S^{\nu}_*(W) \otimes I^{\bar{q}}S^{\nu}_*(W, W-K\cup L) \longrightarrow \varinjlim_{W\in\mathcal{C}} F^{\tau_\pi} \otimes_{F[\pi]} I^{Q_{\bar{p},\bar{q}}}S^{\nu\times\nu}_*(W\times W, W\times(W-K\cup L))$$

is a quasi-isomorphism as desired. □

7.4 Poincaré duality for finitely branched coefficient systems

For our final subsection we show how our proof of universal Poincaré duality for intersection homology may be modified to obtain a (non-universal) duality theorem for twisted coefficients defined using a locally finite unbranched regular cover ν. We begin by defining cohomology with twisted coefficients.

Definition 7.4.1. Let F be a field. Let X be a stratified pseudomanifold with X_{reg} connected, ν be a regular cover of X_{reg} with deck transformation group π, and A be a right $F[\pi]$-module. We recall that if A is a right $F[\pi]$ module, then we define A^t to be the left $F[\pi]$-module induced by the τ_π-twisted involution. Specifically, for $a \in A$ and $g \in \pi$, we define $g \cdot a := a\bar{g}$.

We define the intersection cochain complex with twisted coefficients by

$$I_{\overline{p}}\widetilde{S}^*_{\nu}(X;A^t) := \mathrm{Hom}_{F[\pi]}(I^{\overline{p}}S^{\nu}_*(X;F), A^t)$$

and denote the cohomology by $I_{\overline{p}}\widetilde{H}^*_{\nu}(X;A^t)$.

We note the proofs of Lemma 6.3.3, Proposition 6.3.4, and Proposition 6.3.5 remain valid for general intersection cohomology with twisted coefficients.

Next, we extend the cap product to our current setting.

Definition 7.4.2. Let X be a stratified pseudomanifold with X_{reg} connected and with perversities $\overline{p}, \overline{q}$, and $\overline{r}$ such that $D\overline{r} \geq D\overline{p} + D\overline{q}$. Let $U, V \subset X$ be open, let ν be a locally finite unbranched oriented regular cover of X_{reg}, and let F be a ν-good field. Let $\alpha \in I_{\overline{q}}\widetilde{H}^m_{\nu}(X, U; A^t)$ and $x \in I^{\overline{r}}H_n(X, U \cup V; F^{\tau})$. Write $\widetilde{d}(x) = \sum_i y_i \otimes z_i$ where $y_i \in (I^{\overline{p}}S^{\nu}_*(X, U; F))^t$ and $z_i \in I^{\overline{q}}S^{\nu}_*(X, V; F)$. Then we define

$$\alpha \cap x := \sum_i (-1)^{m|y_i|}\alpha(z_i) \otimes y_i \in I^{\overline{p}}\widetilde{H}^{\nu}_{m-n}(X, V; A)$$

where α evaluates to 0 for $|z_i| \neq m$. We note that this is actually an element of $I^{\overline{p}}\widetilde{H}^{\nu}_{m-n}(X, V; A)$ by the natural map $A \otimes_{F[\pi]} I^{\overline{p}}S^{\nu}_*(X, V; F) \to I^{\overline{p}}\widetilde{S}^{\nu}_*(X, V; A)$. Moreover, this is well defined because if $g_i \in \pi$, then $\sum_i y_i \cdot g_i \otimes g_i^{-1} z_i = \sum_i \overline{g_i} y_i \otimes g_i^{-1} z_i$ and we have

$$\begin{aligned}
\sum_i (-1)^{m|y_i|} \alpha(g_i^{-1} z_i) \otimes \overline{g_i} y_i &= \sum_i (-1)^{m|y_i|} g_i^{-1} \cdot \alpha(z_i) \otimes \overline{g_i} y_i \\
&= \sum_i (-1)^{m|y_i|} \alpha(z_i) \overline{g_i^{-1}} \otimes \overline{g_i} y_i \\
&= \sum_i (-1)^{m|y_i|} \alpha(z_i) (\overline{g_i})^{-1} \otimes \overline{g_i} y_i \\
&= \sum_i (-1)^{m|y_i|} \alpha(z_i) \otimes \overline{g_i}^{-1} \overline{g_i} y_i \\
&= \sum_i (-1)^{m|y_i|} \alpha(z_i) \otimes y_i
\end{aligned}$$

We have analogous results to Proposition 7.1.3, Proposition 7.1.4, and Proposition 7.1.6. Moreover, we have a cohomology cross product for intersection cohomology with twisted coefficients in the same special case as well as the analogous result to Proposition 7.1.8.

Next, we define the map which will yield the duality for twisted coefficients. Let $K \subset X$ be compact and let $\Gamma_K \in I^{\overline{p}} H_n(X, X - K; F^{\tau})$ be the twisted fundamental class over K. Let $\mathscr{D}_K : I_{\overline{p}} \overline{H}^i_{\nu}(X, X - K; A^t) \to I^{\overline{p}}\ \overline{H}^{\nu}_{n-i}(X; A)$ be defined by $\mathscr{D}_K(\alpha) = (-1)^{in} \alpha \cap \Gamma_K$.

As in Section 7.2, we define twisted intersection cohomology with compact supports to be the direct limit over compact subsets $K \subset X$. That is,

$$I^c_{\overline{p}} \widetilde{H}^*_{\nu}(X; A^t) := \lim_{\to} I_{\overline{p}} \widetilde{H}^*_{\nu}(X, X - K; A^t)$$

and just as in Section 7.2 we may define $\mathscr{D} : I^c_{\overline{p}} \widetilde{H}^*_{\nu}(X; A^t) \to I^{\overline{p}} \widetilde{H}^{\nu}_*(X; A)$ by

$$\mathscr{D} := \lim_{\to} \mathscr{D}_K$$

Theorem 7.4.3. *Let X be a stratified n-dimensional pseudomanifold with X_{reg} connected and let ν be a connected locally finite unbranched oriented regular cover of X_{reg} with deck transformation group π. Let F be a ν-good field and A be a right $F[\pi]$-module. Then*

$$\mathscr{D} : I^c_{\overline{p}}\widetilde{H}^*_{\nu}(X; A^t) \to I^{\overline{q}}\widetilde{H}^{\nu}_*(X; A)$$

is an isomorphism of F-vector spaces where $\overline{p}$ and $\overline{q}$ are dual perversities.

Proof. First assume X is a connected normal pseudomanifold. As in Theorem 7.2.1 we induct on depth(X) and apply Theorem 2.4.7. If depth$(X) = 0$, the theorem follows from manifold theory.

In the case $U \subset X_{reg}$ is homeomorphic to Euclidean space we have that U is evenly covered by ν so that the theorem in this case also follows from manifold theory.

Next, we need to show that if the theorem holds for a $(k-1)$-dimensional connected normal link L, then it also holds for cL. The proof follows exactly as in Theorem 7.2.1. We only note that in the case $i \geq k - \overline{p}(\{v\})$, where v is the cone vertex, we have the commutative diagram

$$\begin{tikzcd}
I_{\overline{p}}\widetilde{H}^*_{\nu}(L; A^t) \arrow[r] \arrow[d, "\cap\Gamma_L"'] & I_{\overline{p}}\widetilde{H}^*_{\nu'}(L; A^t) \arrow[d, "\cap\Gamma_L"'] \\
I^{\overline{q}}\widetilde{H}^{\nu}_*(L; A) & I^{\overline{q}}\widetilde{H}^{\nu'}_*(L; A) \arrow[l]
\end{tikzcd}$$

where the left cap product uses the algebraic diagonal map involving ν and the right cap product uses the algebraic diagonal map involving a connected component ν' of $\nu|_L$. The diagram commutes by the corresponding result to Proposition 7.1.4. The top horizontal map is an isomorphism of F-vector spaces by the analogous result of Lemma 6.3.3 and the bottom horizontal map is an isomorphism by Lemma 6.2.7. Now ν' is a connected unbranched oriented regular cover of L and because F is a ν-good field it is also a ν'-good field so that the right vertical map is an isomorphism by induction on depth. Thus, by commutativity of the diagram the left vertical map is also an isomorphism of F-vector spaces.

The next step in the proof of Theorem 7.2.1 was to show the theorem holds for open sets of the form $\mathbb{R}^i \times cL$ where L is compact $(n-i-1)$-dimensional pseudomanifold. As in Theorem 7.2.1 this case follows by using the cross product for cohomology in this special case.

Finally, we have the analogous Mayer-Vietoris long exact sequence of Theorem 7.2.1 whose commutativity follows using similar proofs to those in Section 7.3.

The case X_{reg} is connected, but X is possibly not normal also follows the argument of Theorem 7.2.1 by appealing to a normalization $\mathbf{n} : X^N \to X$.

$\square$

As a last corollary we give a generalization of (11, Theorem 6.3) to include cases the pseudomanifold is perhaps non-orientable. If F is a field of characteristic 2, then X is F-orientable and duality in this case follows by (11, Theorem 6.3). The corollary below covers the cases F is not of characteristic 2.

Corollary 7.4.4. *Let X be a stratified pseudomanifold and let F be a field with char$F \neq 2$. Let $\overline{0} \leq \overline{p} \leq \overline{t}$ be a perversity on X with dual perversity $\overline{q}$. Then we have F-vector space isomorphisms*

- $\mathscr{D} : I^c_{\overline{p}}\widetilde{H}^*_{\mathfrak{o}}(X; F^\tau) \to I^{\overline{q}}H_*(X; F)$
- $\mathscr{D} : I^c_{\overline{p}}H^*(X; F) \to I^{\overline{q}}H_*(X; F^\tau) = I^{\overline{q}}\widetilde{H}^{\mathfrak{o}}_*(X; F^\tau)$.

Proof. We will assume X is connected. The proof for the non-connected case follows by breaking X up into its connected components.

The first isomorphism follows by Theorem 7.4.3 with $\nu = \mathfrak{o}$, $A = F$, and because $I^{\overline{q}}\widetilde{H}^{\mathfrak{o}}_*(X; F) = I^{\overline{q}}H_*(X; F)$.

The second isomorphism follows also by applying Theorem 7.4.3 with $\nu = \mathfrak{o}$ and $A = F^\tau$. Note that $(F^\tau)^t \cong F$ as a left $F[\mathbb{Z}_2]$-module. This can be seen from the fact that $\overline{\tau} = -\tau$ so that $\tau \cdot f = f \cdot \overline{\tau} = -f\tau = -(-f) = f$. Hence, the action by τ under $(F^\tau)^t$ is trivial. If we can show

$$I^c_{\overline{p}}\widetilde{H}^*_{\mathfrak{o}}(X; F) \cong I^c_{\overline{p}}H^*(X; F)$$

we will be done. Let $p : E(\mathfrak{o}) \to X_{reg}$ be the covering map of $\mathfrak{o}$. Consider $p^* : \mathrm{Hom}_F(I^{\overline{p}}S_*(X, X - K; F), F) \to \mathrm{Hom}_{F[\mathbb{Z}_2]}(I^{\overline{p}}S^{\mathfrak{o}}_*(X, X - K; F), F)$ defined by $(p^*\alpha)(x) = \alpha(p_*(x))$, where $\alpha \in \mathrm{Hom}_F(I^{\overline{p}}S_*(X, X - K; F), F)$ and $x \in I^{\overline{p}}S^{\mathfrak{o}}_*(X, X - K; F), F)$. To see this is well-defined we need to show that $p^*\alpha \in \mathrm{Hom}_{F[\mathbb{Z}_2]}(I^{\overline{p}}S^{\mathfrak{o}}_*(X, X - K; F), F)$. To this end, it suffices to show the map is equivariant over $F[\mathbb{Z}_2]$. Observe,

$$
\begin{aligned}
(p^*\alpha)(\tau x) &= \alpha(p_*(\tau x)) \\
&= \alpha(p_*(x)) \\
&= \tau \cdot \alpha(p_*(x)) \\
&= \tau \cdot (p^*\alpha)(x)
\end{aligned}
$$

where the second equality follows because τ has the deck transformation action and the third equality follows because $\alpha(p_*(x)) \in F$ and F is given the trivial left $F[\mathbb{Z}_2]$-module structure. Thus, p^* is induces a well-defined map.

We will show p^* is an isomorphism of cochain complexes. Recall the map $\Psi : I^{\bar{p}}S_*(X, X - K; F) \to I^{\bar{p}}S^{\mathfrak{o}}_*(X, X - K; F)$ from the proof of Proposition 5.2.4 defined by $\Psi(x) = \frac{1}{2}\widetilde{x} + \frac{1}{2}\tau\widetilde{x}$ where $\widetilde{x}$ is a choice of a single lift of each simplex appearing in x (notice the expression $\frac{1}{2}\widetilde{x} + \frac{1}{2}\tau\widetilde{x}$ is independent of choices of lifts). Define $\Psi^* : \mathrm{Hom}_{F[\mathbb{Z}_2]}(I^{\bar{p}}S^{\mathfrak{o}}_*(X, X - K; F), F) \to \mathrm{Hom}_F(I^{\bar{p}}S_*(X, X - K; F), F)$ by $(\Psi^*\beta)(z) = \beta(\Psi(z))$, where $\beta \in \mathrm{Hom}_{F[\mathbb{Z}_2]}(I^{\bar{p}}S^{\mathfrak{o}}_*(X, X - K; F), F)$ and $z \in I^{\bar{p}}S_*(X, X - K; F)$. Notice that since β is equivariant over $F[\mathbb{Z}_2]$ it is also equivariant over F so the map Ψ^* is well-defined. Now since $p\Psi = \mathrm{id}$ we have that $\Psi^*p^* = \mathrm{id}$. We next show that $p^*\Psi^* = \mathrm{id}$ as well.

Let $x \in I^{\bar{p}}S_*^{\mathfrak{o}}(X, X-K; F)$ and $\alpha \in \operatorname{Hom}_{F[\mathbb{Z}_2]}(I^{\bar{p}}S_*^{\mathfrak{o}}(X, X-K; F), F)$. Let $z = p_*x$. Then,

$$\begin{aligned}
(p^*\Psi^*\alpha)(x) &= \alpha(\Psi p_* x) \\
&= \alpha(\Psi z) \\
&= \alpha(\frac{1}{2}\widetilde{z} + \frac{1}{2}\tau\widetilde{z}) \\
&= \alpha(\frac{1}{2}x + \frac{1}{2}\tau x) \\
&= \frac{1}{2}\alpha(x) + \frac{1}{2}\alpha(\tau x) \\
&= \frac{1}{2}\alpha(x) + \frac{1}{2}\tau\alpha(x) \\
&= \frac{1}{2}\alpha(x) + \frac{1}{2}\alpha(x) \\
&= \alpha(x).
\end{aligned}$$

The third equality is because z is clearly a lift of p_*x so that we must have $\frac{1}{2}x+\frac{1}{2}\tau x = \frac{1}{2}\widetilde{z}+\frac{1}{2}\tau\widetilde{z}$ since $\widetilde{z}$ is also a lift of p_*x. The second to last equality follows since the action by $F[\mathbb{Z}_2]$ is trivial. Thus, p^* is an isomorphism with inverse given by Ψ^*. Passage to the direct limit over compact subsets $K \subset X$ then gives an isomorphism of cohomology with compact supports. $\square$

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ABSTRACT

UNIVERSAL POINCARÉ DUALITY FOR THE INTERSECTION HOMOLOGY OF BRANCHED AND PARTIAL COVERINGS OF A PSEUDOMANIFOLD

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The work of Friedman and McClure shows that intersection homology satisfies a version of universal Poincaré duality for orientable pseudomanifolds. We extend their results to include regular covers defined solely over the regular strata. Our approach allows us to also prove a universal duality result for possibly non-orientable pseudomanifolds. We also show that for a special class of coefficient systems, which includes fields twisted by the orientation character, there is a non-universal Poincaré duality via cap products for intersection homology with twisted coefficients.