HIGH SCHOOL GEOMETRY STUDENTS’ INTERPRETATIONS OF THE EQUAL SYMBOL

by

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Submitted to the Graduate Faculty of the
College of Education
Texas Christian University
in partial fulfillment of the requirements for the degree of

Master of Education

December 2016
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Algebra is frequently referred to as the gateway to college and career readiness (Kilpatrick & Izsák, 2008; McNeil et al., 2010; Moses, Kamii, Swap, & Howard, 1989; Picciotto & Wah, 1993; Stephens et al., 2013). A proper understanding of equivalence and the equal symbol is foundational to algebra (Baroody & Grinsburg, 1983; Booth, 1992; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Linchevski, 1995; McNeil et al., 2006; Stephens et al., 2013; Welder, 2012). There are two main interpretations of the equal symbol: operational and relational. Students who have an operational interpretation view the equal symbol as a “do-something” indicator; that is, when they see the equal symbol in an equation, they expect the answer to be on the other side of the equal symbol (Baroody & Ginsburg, 1983; Capraro, Ding, Matteson, Capraro, & Li, 2007; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Students with this interpretation make computations on one side of the equal symbol and put the answer on the other side. This inaccurate view (caused by a multitude of factors) can hinder students as they move forward in their studies of mathematics. On the other hand, students who interpret the equal symbol as relational have an accurate view of the equal symbol. They recognize that the equal symbol indicates equivalence and that there must be balance in an equation.

Many students at the elementary level view the equal symbol operationally (Baroody & Grinsburg, 1983; Capraro, Ding, Matteson, Capraro, & Li, 2007; Kieren, 1981; Knuth et al., 2005; McNeil et al., 2006; Powell, 2015; Stephens et al., 2013; Welder, 2012). However, students who view the equal symbol relationally outperform students who view the equal symbol operationally or other ways (Knuth et al., 2005). In order for students to develop a relational understanding of the equal symbol, students must be exposed to non-standard equations (e.g., $7 = \_\_ + 3$) as opposed to standard equations (e.g., $3 + 4 = \_\_$) (Baroody & Ginsburg, 1983;
McNeil et al., 2006; Stacey & MacGregor, 1996; Powell, 2015; Stephens et al., 2013; Welder, 2012).

Misconceptions with the equal symbol carry over into the middle school grades and even into high school (Kieren, 1981; Welder, 2012). However, there are few studies that look at how high school students define and interpret the equal symbol as well as look at the misconceptions that arise after any formal algebra class (e.g., Kieren 1981). Therefore, the purpose of the present study is to see how students currently enrolled in a geometry class define and interpret the equal symbol. Since the students have taken Algebra I the year before, the present study will look at what conceptions (accurate and inaccurate) students have with respect to the equal symbol after completing a formal algebra class.

**Literature Review**

**The Importance of Algebra**

Algebra has often been interpreted as generalized arithmetic (Kilpatrick & Izsak, 2008). It enables one to create a mathematical formula or rule that is true for all possible cases. Usiskin (1999) stated that algebra allows a person to answer all the questions of a particular type at one time. Without an understanding of algebra, one may repeat mistakes when calculating answers as opposed to deriving a formula using algebra. That is, instead of inputting the same calculations in a calculator using different numbers, one could write an equation in a spreadsheet using algebra. This process could eliminate mistakes when making calculations one at a time.

Often referred to as a gatekeeper to higher mathematics and college (Kilpatrick & Izsak, 2008; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; McNeil, Weinberg, Hattikudur, Stephens, Asquith, Knuth, & Alibali, 2010; Moses, Kamii, Swap & Howard, 1989; Picciotto & Wah, 1993; Stephens, Knuth, Blanton, Isler, Gardiner, & Marum, 2013), algebra is crucial for
the advancement of one’s education in higher mathematics in high school as well as in college. Without a proper understanding of algebra, one will have a difficult time understanding concepts at these advanced mathematical levels. McNeil et al. (2010) and Moses et al. (1989) point out that algebra is not only a foundation to high school and higher order mathematics but is also important for science courses. Algebra is crucial for any of the sciences in school and in the workforce.

Not only is algebra important to schooling, it is also important outside of school for careers. Usiskin (1999) argues that a lack of knowledge of algebra limits one’s opportunities in programs, careers, and finances as well as one’s ability to understand ideas discussed in areas such as science, economics, and business. Without a functioning understanding of algebra, one may have difficulties keeping up with advances in such areas. For instance, spreadsheet software can be extremely helpful in businesses. However, one cannot utilize spreadsheet software’s full capabilities without a firm understanding of algebra to write formulas, making the workplace more efficient.

**Equivalence**

An understanding of the concept of equivalence is crucial for successful algebraic thinking (Mann, 2004). Equivalence is the concept that two things are equal, and this understanding should not start in an algebra course; this understanding must be developed early in schooling. Equivalence is a “crucial idea for developing algebraic reasoning in young children” (Falkner, Levi, & Carpenter, 1999, p. 232). Falkner et al. (1999) also mention, “children need this understanding to think about relationships expressed by number sentences,” and a poor understanding is “one of the major stumbling blocks for students when they move
from arithmetic to algebra” (p. 234). In other words, without a proper understanding of
equivalence, students will not be as successful in algebra.

The Equal Symbol

According to Falkner et al. (1999), the equal symbol is a convention; it is the symbol
chosen by mathematicians to represent the concept of equivalence. For this reason, in the
present study, “=” is referred to as the “equal symbol” rather than the “equal sign.” Even though
the equal symbol is just a convention, researchers agree that the equal symbol is one of the most
important and prevalent symbols in school mathematics (Knuth et al., 2008; McNeil, Grandau,
Knuth, Alibali, Stephens, Hattikudur, & Kril, 2006). The equal symbol is the cornerstone of
mathematical equivalence, and its presence in all levels of mathematics demonstrates its
importance (Powell, 2015; Stephens et al., 2013).

Origins of struggles with the equal symbol and equivalence. Misconceptions and
misinterpretations of equivalence and the equal symbol start early (Baroody & Ginsburg, 1983;
Knuth et al., 2008; McNeil et al., 2006; Stephens et al., 2013; Welder, 2012) and cannot be
remediated by one or two examples (Falkner et al., 1999). One source of misconceptions is how
students see the equal symbol in everyday life. The equal symbol is frequently used as a symbol
of an association rather than a relationship (Stacey & MacGregor, 1996). For example, a student
may see, “Math = Fun,” or, “Hard Work = Success,” and carry the meaning of the equal symbol
in that context into the context of mathematics.

Some misconceptions stem from only seeing standard equations (e.g., $4 + 3 = ___$)
(Baroody & Ginsburg, 1983; Welder, 2012). Students are not seeing enough variety of equations
when introduced to the equal symbol. For this reason, students may see an equation such as $7 +
5 = ___ + 4$ and think that 12 should go in the blank rather than an 8. Misconceptions with the
equal symbol are also a result of the use of calculators; the “=” button is used as an operational key that returns the answer (Baroudi, 2006; Welder, 2012). Using a calculator encourages the interpretation of the equal symbol to mean, “The answer is next,” since the answer comes only after the “=” button is pressed (Capraro et al., 2007; Stacey & MacGregor, 1996). Students often incorrectly write strings of equivalence (e.g., $13 + 5 = 18 – 7 = 11 + 20 = 31$) (Kieren, 1981), and using calculators could be a reason for this error.

Many of the previously stated sources of misconceptions of the equal symbol can be connected to the curriculum materials (Capraro et al., 2007; Knuth et al., 2005; McNeil et al., 2006; Stacey & MacGregor, 1996). An appropriate, more in-depth view of the equal symbol is not always an explicit focus of instruction (Knuth et al., 2005). For instance, one study showed that textbooks in the United States do not reinforce this view of the equal symbol whereas textbooks in China do (Capraro et al., 2007). Furthermore, if a curriculum does not vertically align to foster a relational view of the equal symbol then the students will not have the exposure necessary to develop a proper relational reasoning of the equal symbol.

**Defining and interpreting the equal symbol.** Since the notion of equivalence is a fundamental concept on which algebraic reasoning depends, (Knuth et al., 2005, p. 68) and because the equal symbol indicates this notion of equivalence, studying how students define and interpret the equal symbol is essential. Research shows that defining the equal symbol is considered a difficult task for students (Stephens et al., 2013). Students struggle with identifying a proper definition of the equal symbol and even identify poor or wrong definitions as being correct when given a choice of definitions from which to choose (McNeil et al., 2006). This could be due to the fact that the equal symbol is so widely used (Powell, 2015) that students do not know how to put into words what the equal symbol represents or means.
The difference between “defining” and “interpreting” is subtle. For the present study, “defining” is putting into words what the equal symbol means; that is, students write a definition. On the other hand, “interpreting” refers to how students use the equal symbol. The students’ use of the equal symbol provides insight into how the students interpret the equal symbol. Interpreting the equal symbol can also be a struggle for students; there are advanced interpretations and rudimentary interpretations. However, rudimentary interpretations of the equal symbol are often still visible in high school (Kieren, 1981). There are two main ways students view the equal symbol.

**Operational.** Students who have an operational view of the equal symbol perceive the equal symbol as an operator, a “do-something” symbol (Baroody & Ginsburg, 1983; Capraro, Ding, Matteson, & Capraro, 2007; Kieren, 1981; McNeil et al., 2006; Powell, 2015). When students with this view see the equal symbol, they believe it means the answer is next; that is, they believe the number after the equal symbol is the answer to the calculation that precedes it (Capraro et al., 2007; Falkner et al., 1999; Knuth et al., 2005; Welder, 2012). For this reason, an operational view of the equal symbol is also known as a unidirectional view. This wording implies that students read an equation from left to right as opposed to looking at the whole equation. In other words, when students see the equal symbol they may view it as a left-to-right directional symbol (Linchevski, 1995; Welder, 2012). This operational view of the equal symbol is considered to be unrefined, and those with this view have a deficiency in their understanding of the structure of equations (Knuth et al., 2005; Stephens et al., 2013).

**Relational.** Students with a relational interpretation of the equal symbol have a more sophisticated view of the equal symbol (Knuth et al., 2005). This interpretation of the equal symbol recognizes the fact that there is a relationship between the left and the right side of the
equal symbol and an equation is bidirectional. With respect to algebra, students with a relational view of the equal symbol have an easier time working with non-standard equations than do students with an operational view. (Standard equations are written in the form $x + 3 = 5$. Non-standard equations are written $5 = x + 3$.)

Viewing the equal symbol as a relational symbol needs to be emphasized to students (Booth, 1982; Powell, 2015). Baroody and Ginsburg (1983) mention that students with a relational view of the equal symbol have meaningful algebraic solution strategies; otherwise, students tend to solve algebraic equations by rote. Despite the importance of a relational view of the equal symbol, the findings by Knuth et al. (2008) indicate that by grade eight less than half (46%) of students have a relational understanding of the equal symbol.

Not having a relational view of the equal symbol can cause students to struggle when they advance to Algebra I since algebra requires a more thorough understanding of the equal symbol. Students are required to move from an operational mode of reading an equation (unidirectional view) to a relational (bidirectional) mode of viewing an equation (Booth, 1982; Knuth et al., 2005; Linchevski, 1995; McNeil et al., 2006). Research has shown the importance of developing a relational understanding of the equal symbol since students with this view outperform their peers who have only an operational view of the equal symbol (Knuth et al., 2008; Powell, 2015). Stephens et al. (2013) split the relational view into two subcategories.

Relational-computational. Stephens et al. (2013) point out that students with a relational view of the equal symbol recognize the relationship that both sides of the equal symbol have to each other. Students with a relational-computational view of the equal symbol, however, will perform each side of the equation to confirm equivalence. For example, when asked to confirm equivalence for $5 + 2 = 3 + 4$, a student will recognize that since $5 + 2 = 7$ and $3 + 4 = 7$ that
5 + 2 must be equal to 3 + 4. A student with this view solves each side individually to confirm equivalence (Stephens et al., 2013; Welder, 2012). This is a basic relational interpretation of the equal symbol, and students with this view may outperform students with an operational view of the equal symbol, but a relational-computational view of the equal symbol is only a step towards a more complex view of the equal symbol.

*Relational-structural.* A deeper understanding in the area of a relational interpretation of the equal symbol is a relational-structural interpretation; this view represents the most in-depth understanding of the equal symbol (Stephens et al., 2013). A relational-structural view of the equal symbol uses the structure of an equation to solve the equation or confirm equivalence. When given the equation 8 + 4 = ____ + 5, a student with a relational-structural interpretation of the equal symbol reasons that since 5 is one more than 4, the missing value must be one less than 8 (Stephens et al., 2013). As another example, when given two equations such as 2x + 15 = 31 and 2x + 15 – 9 = 31 – 9, a student with a relational-structural interpretation of the equal symbol reasons that the value of x in both equations must be the same; this student notices that equivalence is maintained since 9 is subtracted from both sides of the second equation.

**Fostering correct interpretations of the equal symbol.** Since struggles with the equal symbol start early on (Baroody & Ginsburg, 1983; Knuth et al., 2008; McNeil et al., 2006; Stephens et al., 2013; Welder, 2012), the equal symbol must be carefully introduced so not to foster misconceptions. Teachers can begin teaching a relational view of the equal symbol in elementary school (Baroody & Ginsburg, 1983). When seeing the equal symbol, a teacher can say “is the same as,” rather than “equals,” to help foster a correct interpretation of the equal symbol (Capraro et al., 2007; Powell, 2015); that is, students can develop a bidirectional (relational) view of the equal symbol. McNeil et al. (2006) point out that a relational view of the
equal symbol may need an abundance of contextual support. Students need to see a wide variety of ways that equations can be represented; that is, students must be exposed to both standard (e.g., \(4 + 3 = \_\)) and non-standard equations (e.g., \(4 + \_ = 7\) or \(\_ = 4 + 3\)) early on.

**The importance of the equal symbol in algebra.** The equal symbol itself is important, but it has particular significance for algebra. An inadequate understanding of the equal symbol negatively affects students at all levels including algebra (Baroody & Ginsburg, 1983; Knuth et al., 2008). Without a proper understanding of the equal symbol, one is hindered in the learning of algebra and thus of higher mathematics. However, a surface-level understanding of the equal symbol is not enough for understanding algebra; algebra requires a sophisticated understanding of the equal symbol (Booth, 1982; Linchevski, 1995).

**Methodology**

The purpose of this study is to analyze how students define and interpret (correctly and incorrectly) the equal symbol. Multiple studies have been conducted concerning elementary and middle school students’ interpretation of the equal symbol, but few studies have been conducted on how high school students interpret the equal symbol (Kieren, 1981; McNeil et al., 2006). For this reason, the present study will include students who have completed a formal Algebra I class. A traditional mathematics program of study in Texas requires four years of mathematics and usually begins with Algebra I in ninth grade. In tenth grade, students take Geometry, and, in eleventh grade, students take Algebra II. Students take a fourth mathematics course in twelfth grade.

**Participants**

The high school in which the researcher teaches is in an urban school district in the state of Texas. According to the Texas Education Agency’s 2014-2015 School Report Card (2015),
the high school had a total student population of 3,889 students, 97% of which were minorities, and 75.8% of which were economically disadvantaged. Of the student population, 16.6% were English language learners and 8.7% were classified as needing special education services.

The student population of this study was the researcher’s tenth grade Geometry students since most of the students had already successfully taken Algebra I. The inclusion criteria for participants were as follows: a student must have passed Algebra I and be taking Geometry for the first time. In other words, to be a participant, a student could not be taking Geometry and Algebra I concurrently or have previously failed Geometry. There were 22 total students who returned the permission and assent forms and met the inclusion criteria.

Data Collection

The foci of the data collection tools used were the students’ interpretation of the equal symbol itself as well as how students used the equal symbol in a variety of contexts. The researcher used a variable $x$ to represent unknown values in equations (rather than a space or a box) since the students had already taken Algebra I.

Research instrument. The research instrument (Appendix A) consisted of five items. The first item seen in Figure 1 allowed the researcher to identify how the students name and define the equal symbol; the item was taken from Knuth et al. (2005). The first question allowed students to name the equal symbol. Even the way a student identifies the symbol can give insight as to how the student interprets the equal symbol. For example, Baroody and Ginsburg (1983) state that “children appear to view ‘equals’ as an ‘operator’ symbol (a ‘write-something’ symbol)” (p. 198). The next question allowed the students to tell the researcher precisely what they think the equal symbol means, while the last question allowed for any other meanings. For example, Stacey and MacGregor (1996) point out that students in and out of the classroom
setting see the equal symbol used in a variety of contexts such as association (e.g., Math = Fun, Hard Work = Success) (p. 255).

1. The following questions are about this statement:

\[ 3 + 4 = 7 \]

a. The arrow points to a symbol. What is the name of the symbol?
b. What does the symbol mean?
c. Can the symbol mean anything else? If yes, please explain.

Figure 1. Item 1 of the research instrument. Students name and define the equal symbol.

The second item (Figure 2) began to identify whether or not the students have a relational understanding of the equal symbol. There are three possibilities that were inherently noticeable in how students could solve for the value of \( x \). The first could be to add 88 and 49, and then subtract 48; this method demonstrates a relational-computational view of the equal symbol. The second possibility is to recognize that 48 is one less than 49, forcing \( x \) to be one more than 88; this method shows a relational-structural view of the equal symbol. The third possibility is that students may add 88, 49, and 48 or 88 and 49; this demonstrates an operational view of the equal symbol since students add all the numbers and ignore the equal symbol, or they view the equal symbol as indicating an answer to the computation on the left side of the equation, respectively. The idea for this item came from Stephens et al. (2013) but was modified to include larger numbers. The larger numbers compel students to either demonstrate an operational or a relational interpretation of the equal symbol. That is, the larger numbers influence students to either add the numbers or demonstrate the relational-structural view of the equal symbol.
2. What value of $x$ will make the following number sentence true? Explain your reasoning.

$$88 + 49 = x + 48$$

*Figure 2.* Item 2 of the research instrument, identifying whether or not students interpret the equal symbol relationally.

The third item, seen in Figure 3, had two aspects. The researcher observed how the students viewed the equal symbol in one equation as well as determined whether or not students took the previous solution and applied it to a similar equation. Both contributed to understanding how the students view the equal symbol. The researcher created this item to include a non-standard equation as well as to see if students could relate two different equations. A possible solution method for the first equation is for students to divide both sides of the equation by 4, resulting in $x = 5$. However, a more astute method is to recognize that $x$ must be 5 since $4 \times 5 = 20$. A possible solution method for the second equation is for students to divide both sides of the equation by 5, resulting in $x = 4$. Another possibility is to recognize that $x$ has to be 4 since $20 = 5 \times 4$. While this would provide evidence of a relational perspective, a third possible method of solving the second equation would be still more developed. One could recognize that since $x = 5$ from the first equation results in $4 \times 5$, then $x$ in the second equation must equal 4 since that would result in $5 \times 4$, which holds the same value. The last method demonstrates a relational-structural understanding of the equal symbol across two different equations.
3. What value of $x$ will make the following number sentences true? Explain your reasoning.
   a. $4x = 20$
   b. $20 = 5x$

*Figure 3.* Item 3 of the research instrument. Students have the ability to demonstrate a deeper understanding of the equal symbol and how it relates to equivalency across two different equations.

Seen in Figure 4, the fourth item in the instrument was taken from Knuth et al. (2005) and was included to also determine how students view the equal symbol when equations are explicitly compared to each other. There are two possibilities of how students could determine equivalency between the two equations. The first possibility is to solve for $x$ in each equation and compare the results. (There are various methods of how students may solve for $x$, and that was noted, as well.) The other possible method of determining equivalency is to recognize that the only difference between the two equations is that in the second equation 9 is subtracted from both sides of the equation resulting in the same value for $x$ for both equations. Solving for the value of $x$ is not necessary to determine equivalency.

4. Is the value of $x$ the same number in the following two equations? Explain your reasoning.
   
   $2x + 15 = 31$
   $2x + 15 - 9 = 31 - 9$

*Figure 4.* Item 4 of the research instrument. This item helps identify how students view the equal symbol when assessing equivalence between two equations.
Item 5, as seen in Figure 5, allowed the researcher to observe the methods students used to solve algebraic equations and to compare these methods to the students’ views of the equal symbol. Kieren (1981) and Stephens et al. (2013) stated that how students solve equations shows a skewed view of the equal sign and equivalence (p. 324). This item determines how students solve this non-standard, two-step equation. Non-standard equations might be more difficult for students if they were not introduced to them earlier in their schooling (Baroody & Ginsburg, 1983; Powell, 2015). Powell (2015) mentions that students who view the equal symbol as operational often solve non-standard equations incorrectly. The researcher compared the methods the students used to solve this equation to how the students interpreted the equal symbol.

5. What value of x will make the following number sentence true? Explain your reasoning.

25 = 3x + 7

Figure 5. Item 5 of the research instrument. The researcher compared the methods which students used to solve this equation to how the other items had previously been coded according to the students’ interpretations of the equal symbol.

Interviews. After the instrument was given, the researcher conducted one-on-one interviews with a subset of students. The interviews consisted of similar yet different questions than those on the research instrument (Appendix B). Item 1, however, was identical to that of the research instrument. The other items were only different in the number values used and were carefully designed to have the same principle structure. The students explained their reasoning as they answered each question. The purpose of the interviews was to further understand the
reasons behind the methods the students used to answer the items as well as their interpretation of the equal symbol. The interviews also allowed two-way communication, as the researcher was able to ask clarifying questions to better understand the students’ reasoning.

**Procedures**

The researcher obtained permission from his principal, school district, and university IRB. The researcher sent a parent letter, parent permission forms, and assent forms home with the students. To avoid coercion, a colleague passed out the forms and explained the study. The colleague asked students to bring the letter explaining the research as well as the parent permission home to their parents/guardians. Through these documents, the parents/guardians and students were informed that participation is voluntary and that the students may withdraw from the study at any time. Further, no incentive was given. The decision to participate did not affect the students’ grades. To avoid any feeling of exclusion, the researcher sent these documents home with all of his students even if they did not meet the inclusion criteria. The students were asked to return signed parent permission forms and assent forms to the colleague. Twenty-two students returned the permission and assent forms and met the inclusion criteria.

Once the researcher received permission from the parents and assent from the students, all of the students completed the instrument designed to evaluate their understanding of the equal symbol. The instrument (Appendix A) was completed during class at a time that minimized interruptions to the regular course schedule. Prior to administering the instrument, the researcher read from a script (Appendix C). The researcher gave the instrument to all of his students in each of his geometry classes on the same day. Since the researcher did not want any student to feel excluded, he gave the instrument to all of his students who were present the day of
implementation. The researcher gave completed instruments to his colleague who then sorted out the students who returned their forms, cut the names off the instruments, and gave the researcher only the completed instruments of students who had all the forms submitted and met the inclusion criteria.

After the instrument was implemented, the researcher’s colleague gave the researcher the names of all students who were present the day of implementation and who met the inclusion criteria. These students received an assent form for the interview. Thirteen students assented to participate in the interview process. The researcher then conducted one-on-one interviews with the 13 assenting students of the participating population. The interview participants completed the interview instrument (Appendix B) in the presence of the researcher and explained their reasoning process for each item. The researcher took notes during and after each interview.

**Data Analysis**

The researcher coded each individual item for each student. The researcher used conventional content analysis to code Item 1 according to how the students defined the equal symbol. This coding remained open as to how the students responded, and the responses were categorized accordingly. For the remaining items, the researcher used directed content analysis since the same codes that Stephens et al. (2013) used in their research were employed to code how students interpreted the equal symbol. The first possible code the researcher used was the *operational* code. Since there are two types of relational interpretations of the equal symbol (Stephens et al., 2013), the relational coding was split into two. Thus, the second possible code was *relational-computational* and the third possible code was *relational-structural*. The fourth code that was used was *undetermined* for any interpretations of the equal symbol that the researcher could not identify. Student responses coded in this way either did not show work or
their reasoning was unclear. One final code, *misconception*, was used in the case when a student incorrectly answered a question and did not demonstrate a relational or operational view of the equal symbol.

Tables 1 and 2 show the coding that the researcher used throughout the initial instrument and the interviews. For Item 1, the coding for the initial instrument and the interviews was the same. However, the coding for Item 1 and the other items differed. One difference was that there were no misconceptions for Item 1. Another difference between Item 1 and the other items was that there was no difference between the types of relational thinking (relational-computational and relational-structural) since the participants are only defining the equal symbol rather than being given a task such as in Items 2 through 5. The researcher also created an additional code for Items 2 through 5 to indicate whether the students demonstrated a relational-structural understanding initially or after some questioning.

Table 1

*Coding Used for Item 1 for the Initial Instrument and the Interviews*

<table>
<thead>
<tr>
<th>Item 1 Coding</th>
<th>Initial Instrument</th>
<th>Interviews</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undeterminable</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Operational</td>
<td>op</td>
<td>op</td>
</tr>
<tr>
<td>Relational</td>
<td>r</td>
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</table>
Table 2

*Coding Used for Items 2 – 5 for the Initial Instrument and the Interviews*

<table>
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<th>Items 2 – 5 Coding</th>
<th>Initial Instrument</th>
<th>Interviews</th>
</tr>
</thead>
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<td>Misconception</td>
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<td>m</td>
</tr>
<tr>
<td>Undeterminable</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>Operational</td>
<td>op</td>
<td>op</td>
</tr>
<tr>
<td>Relational-Computational</td>
<td>rc</td>
<td>rc</td>
</tr>
<tr>
<td>Relational-Structural</td>
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<td>rs</td>
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<tr>
<td>Initially Relational-Structural</td>
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<td>rsi</td>
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**Results**

**Initial Instrument**

Table 3 shows the frequencies and relative frequencies for each code for Item 1 on the initial instrument, and Table 4 shows an item-by-item comparison for Items 2 through 5 for each code for the initial instrument.

Table 3

*The Frequencies and Relative Frequencies for Each Code for Item 1 of the Initial Instrument*

<table>
<thead>
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<th>Code</th>
<th>Item 1</th>
<th>Percent</th>
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</tr>
<tr>
<td>op</td>
<td>14</td>
<td>63.64%</td>
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<tr>
<td>r</td>
<td>4</td>
<td>18.18%</td>
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</table>

Table 4

*An Item-by-Item Comparison for Items 2 – 5 of the Initial Instrument*

<table>
<thead>
<tr>
<th>Code</th>
<th>Item 2</th>
<th>Item 3</th>
<th>Item 4</th>
<th>Item 5</th>
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<th>Percent</th>
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<td>12</td>
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</tbody>
</table>
For Item 1, just more than half (14) of the students demonstrated a relational understanding of the equal symbol while a majority of the responses (55 out of 88 for the other items across the 22 individuals demonstrated a relational understanding of the equal symbol. In particular, 53 responses across the individuals for Items 2, 3, 4, and 5 demonstrated a relational-computational understanding and two indicated a relational-structural understanding.

Items 2 and 3 each have 16 responses that were relational-computational (none are relational-structural), yet Items 4 and 5 dropped to 11 and 10 relational-computational responses, respectively. Item 4 had the most undeterminable responses (eight) and Item 5 had the most operational responses (eight). Twenty-five out of 110 total item responses were undeterminable due to no explanation present, lack of a clear path of reasoning, or illegible handwriting.

**Item 1.** This item asked students to define the equal symbol and whether it can mean anything other than how students defined. Four responses for Item 1 were relational, just more than half (14) were operational, and the four remaining responses could not be determined.

Responses that demonstrated a relational understanding included statements like, “[B]oth sides are the same,” and “What’s on the left of the [sign] is the same as what’s on the right.” These responses connected the left and the right side of the equal symbol and showed the relationship between the sides.

Responses that demonstrated an operational understanding included, “[I]t means the sum of the equation equal to the number,” and “What the problem/equation equals (total).” These responses showed that the students believe the equal symbol indicates an action is required or that the answer will come after the symbol. Therefore, students with responses such as the two exemplified were coded as operational.
**Item 2.** Item 2 asked what the value of \( x \) will be in the equation \( 88 + 49 = x + 48 \). A majority (16) of Item 2 responses were relational-computational. Two student responses were operational and four could not be determined. No student responses represented a relational-structural understanding for this item.

Figure 6 demonstrates a student response with relational-computational reasoning for Item 2. The student followed an addition algorithm to add 88 and 49 to get 137. Next, the student followed a subtraction algorithm to subtract 48 from 137 thus isolating \( x \). Even though there was an arithmetic error, this response was still coded as relational-computational since the student related the two sides as equal.

![Figure 6. An example of relational-computational reasoning for Item 2.](image)

The two examples of operational reasoning differed. One student ignored the 48 on the right side while the other student formed a string of equations. Figure 7 demonstrates the former. The student added 88 and 49 to get 137 for the value of \( x \), ignoring the 48 altogether.

![Figure 7. An example of operational reasoning for Item 2.](image)
The other example for operational reasoning on Item 2 is in Figure 8. Even though the student answered the question correctly, the addition and subtraction algorithms were combined (on the left side of the figure). This combination of algorithms is equivalent to the incorrect string of equations $88 + 49 = 137 - 48 = 89$ (except the student also added 89 back to 48 to get 137).

![Figure 8](image)

**Figure 8.** An example of operational reasoning for Item 2.

Student responses coded as undeterminable included the correct answer of 89 with no reasoning provided. On the other hand, the student who wrote down 137 was coded as operational because there is only one way to get that answer. There are multiple ways to obtain both answers 89 and 137. However, there is only one type of reasoning that leads to the answer of 137, and that is to not take into consideration the “+ 48” on the right side.

**Item 3.** This item showed two equations, $4x = 20$ and $20 = 5x$ and asked students to solve for the value of $x$ in each equation. A majority (16) of Item 3 responses demonstrated relational-computational reasoning. One student response was operational and five responses could not be determined. No student responses represented a relational-structural understanding for this item.
Figure 9 demonstrates a student response with relational-computational reasoning for Item 3. As shown in the students’ explanation, the student solved for $x$ by dividing both sides of the equation by the coefficient of $x$. The student then related the two sides of the equation stating that “20 divided by 4 is 5,” and that “$5 \times 4 = 20$.” The student had the same reasoning for part b, as well. While the student related both sides of the equation, the student did not transfer understanding from part a to part b and thus demonstrated relational-computational reasoning rather than relational-structural reasoning.

![Figure 9](image)

*Figure 9. An example of relational-computational reasoning for Item 3.*

The student response in Figure 10 demonstrates an example of operational reasoning for Item 3; this student followed an algorithm and divided twenty by four in part a and twenty by five in part b. However, the student only divided one side of the equation rather than both, thus demonstrating only an operational understanding of the equal symbol.
**Item 4.** This item showed two equations, $2x + 15 = 31$ and $2x + 15 - 9 = 31 - 9$, and asked students if the value of $x$ was the same for both equations. Note that the question did not ask students to solve for the value of $x$. Two responses demonstrated relational-structural reasoning for Item 4. Half (11) of the responses demonstrated relational-computational reasoning. One student response was operational and eight responses could not be determined. One response demonstrated the only misconception of the equal symbol for the initial instrument.

Figure 11 shows a student response with relational-structural reasoning for Item 4. The student explained that the second equation is the same as the first but “taking away 9.” The student also bracketed $2x + 15$ and 31 in the second equation and circles $-9$ on both sides of the second equation, demonstrating an understanding that the $2x + 15$ and the 31 are equivalent in the first and second equations.
Figure 11. An example of a relational-structural reasoning for Item 4.

An example of relational-computational reasoning for Item 4 is in Figure 12. This student solved for the value of \( x \) in both equations and compared the results. Since the student did not holistically look at the set of the equations, the student only demonstrated relational-computational understanding.

Figure 12. An example of a relational-computational reasoning for Item 4.
Figure 13 demonstrates an example of a student with an operational understanding of the equal symbol. While the student solved for $x$ in both equations and finds the correct value of $x$, the student used incorrect strings of equations, writing “$31 - 15 = 16 \div 2 = 8$” and “$8 \times 2 = 16 + 15 = 31$.” These strings of equations demonstrated an operational understanding of the equal symbol.

![Figure 13](image)

*Figure 13. An example of an operational reasoning for Item 4.*

The student work in Figure 14 demonstrates an example of a misconception about the equal symbol. This student said that the value of $x$ is the same in both equations “because [they] have 31.” The student recognized that each equation has 31 in it, and that is the reason they are the same. This is coded as a misconception because the student only identified the presence of 31 in both equations.

![Figure 14](image)

*Figure 14. An example of a misconception for Item 4.*
**Item 5.** Item 5 asked students to solve for the value of $x$ given the equation $25 = 3x + 7$. Under half (10) of Item 5 responses demonstrated relational-computational reasoning. Eight student responses were operational and four responses could not be determined. No student responses represented a relational-structural understanding for this item.

Figure 15 demonstrates a relational-computational understanding of the equal symbol. The student solved for the value of $x$ by showing the result of one step to the next. Interestingly, throughout the entire instrument, this student rewrites any nonstandard equation in standard form (e.g., for Item 2, this student rewrote the question $137 = x + 48$ as $-x = 48 - 137$), even if it is not the most efficient way to solve for the value of $x$.

![Figure 15](image)

*Figure 15.* An example of a relational-computational understanding for Item 5.

Figure 16 shows a student’s work with operational reasoning. Not only did the student perform operations to only one side of the equation, the student also stated that “six times 7 = 18 + 7 = 25.” This string of equations is incorrect, demonstrating an operational interpretation of the equal symbol.
Figure 16. An example of an operational understanding of the equal symbol for Item 5.

Interviews

The frequencies and relative frequencies for each code for Item 1 in the interviews is shown in Table 5, and Table 6 shows an item-by-item comparison for Items 2 through 5 for each code for the interviews.

Table 5

The Frequencies and Relative Frequencies for Each Code for Item 1 of the Interviews

<table>
<thead>
<tr>
<th>Code</th>
<th>Item 1</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>0</td>
<td>0.00%</td>
</tr>
<tr>
<td>op</td>
<td>4</td>
<td>30.77%</td>
</tr>
<tr>
<td>r</td>
<td>9</td>
<td>69.23%</td>
</tr>
</tbody>
</table>

Table 6

An Item-by-Item Comparison for Items 2 – 5 of the Interviews

<table>
<thead>
<tr>
<th>Code</th>
<th>Item 2</th>
<th>Item 3</th>
<th>Item 4</th>
<th>Item 5</th>
<th>Total</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1.92%</td>
</tr>
<tr>
<td>u</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.00%</td>
</tr>
<tr>
<td>op</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>11.54%</td>
</tr>
<tr>
<td>rc</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>32</td>
<td>61.54%</td>
</tr>
<tr>
<td>rs</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>9</td>
<td>17.31%</td>
</tr>
<tr>
<td>rsi</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>7.69%</td>
</tr>
</tbody>
</table>
For Item 1, nine of the 13 responses were relational, while 45 (out of 52 total) responses of the 13 individuals across Items 2 through 5 demonstrated a relational understanding of the equal symbol. Of these relational responses, three were relational-structural as an initial response and 10 were relational-structural after questioning. Thirty-two responses were relational-computational, six were operational, and one was a misconception.

A majority of responses (11 out of 13) for Item 2 were either operational or relational-computational while all of Item 3 responses were either relational-computational or relational-structural. A majority of responses for Item 4 (12 out of 13) were either relational-computational or relational-structural and all but one response for Item 5 were relational-computational.

Since the interviews were in person, no responses were undeterminable. If any reasoning was unclear, the researcher asked the student to clarify his or her reasoning. Most students felt more comfortable explaining their thought process orally rather than in writing, so the researcher documented the students’ thinking in his notes.

**Item 1.** Nine responses for Item 1 were relational and four were operational. An example of a relational response was a student’s statement that the two sides are the “same or equivalent.” Another example of a relational response was, “[B]oth sides are the same.” “[G]etting the total of something,” and “the sum of the numbers in front of the equal symbol,” are two responses that demonstrated an operational response.

**Item 2.** Two responses for Item 2 demonstrated an initial relational-structural understanding of the equal symbol, that is, before any questioning. No responses demonstrated a relational-structural understanding of the equal symbol after questioning. Six responses were relational-computational and five were operational.
Both students with a relational-structural understanding were able to identify the value of $x$ very quickly. They used the same reasoning: since 78 on the left side of the equation is one more than the 77 on the right side of the equation, the value of $x$ must be one more than 54, and thus $x$ must be 55.

Figure 17 shows an example of a relational-computational response for Item 2. The student first added 54 to 78 and rewrote the equation with that sum. Next, the student subtracted 77 from both sides, isolating $x$. This student identified the relationship between the two sides of the equal symbol and maintained the equality of the equation step-by-step.

\[ 54 + 78 = x + 77 \]
\[ 132 = x + 77 \]
\[ 132 - 77 = x \]
\[ x = 55 \]

*Figure 17.* An example of a relational-computational response in the interviews for Item 2.

An example of an operational response for Item 2 is in Figure 18. This student did not offer any other explanation of reasoning other than what is shown on their interview paper. The student combined addition and subtraction algorithms into one step. This combination of algorithms is equivalent to the incorrect string of equations $54 + 78 = 132 - 77 = 55$. 
**Item 3.** One response for Item 3 demonstrated a relational-structural understanding before interviewer questioning, while six responses demonstrated a relational-structural understanding after questioning, and the other six responses demonstrated relational-computational reasoning. No responses demonstrated an operational understanding.

The student who initially had relational-structural reasoning solved for $x$ in the first equation by dividing both sides of the equation by 8. The student then quickly answered the second part, saying, “If eight times seven is 56, then seven times eight is also 56.” The student was able to transfer the work from the first part and apply it to the second part.

The students with relational-structural reasoning after questioning recognized that the two parts of Item 3 have the same product (much like the student with an initial relational-structural understanding). Prior to questioning, the students tried to relate the two equations, but did not see the connection at first. The researcher prompted, “Looking at part a, what would you do?” From here, the students solved for $x$ in part a by either dividing both sides of the equation by 8 or
by recognizing that $8 \times 7 = 56$. For part b, the students were able to recognize that the answer must be eight because of their work from part a.

On the other hand, the students with relational-computational reasoning treated the two parts of Item 3 as separate problems. The students demonstrated a relational understanding of the equal symbol in that they divided both sides of the equation by the same value, but they did not transfer their findings from part a to part b as seen in Figure 19. This student treated the two parts as separate problems, first dividing both sides of the equation by eight for part a followed by dividing both sides of the equation by seven for part b.

![Figure 19](image)

*Figure 19.* An example of relational-computational reasoning in the interviews for Item 3.

**Item 4.** One response demonstrated a relational-structural understanding initially, while three responses were relational-structural after questioning. Eight responses demonstrated a relational-computational understanding, and one demonstrated an operational understanding.

The student with an initial relational-structural understanding of the equal symbol reasoned quickly that the value of $x$ is the same in both equations. The student explained that the second equation added a “subtract 6” to both sides of the equation. When asked if the student needed to solve for the value of $x$ to verify, the student responded that there is no need to solve for $x$. 
Figure 20 shows a student’s relational-structural reasoning after questioning for Item 4. The student initially started to solve for the value of $x$. The student said that in the first equation, the right side is 57 while in the second equation the right side is 51. When asked, “Where did the 51 come from?” the student thought for a minute and then circled $4x + 13$ and 57 in the second equation. The student proceeded to explain that the two equations are the same because the only difference in the second equation is the $-6$ on both sides.

![Image](image.png)

*Figure 20. An example of relational-structural reasoning after questioning for Item 4.*

The student response in Figure 21 is an example of relational-computational understanding for Item 4. The student solved for the value of $x$ in each equation and then verified equivalence. The student even got to the point of $4x = 44$ when solving the equation on the right but continued to solve for $x$ rather than verifying equivalence at that point.

![Image](image.png)

*Figure 21. An example of relational-computational reasoning for Item 4.*
The student with operational understanding for Item 4 said that the value of $x$ is not the same in both equations. The student’s reasoning was that in the first equation there is 57 on the right side of the equation, but in the second equation there is a 51 on the right side of the equation. Upon further questioning, the student maintained his initial conclusion.

**Item 5.** Twelve responses demonstrated relational-computational understanding of the equal symbol for Item 5, while one response demonstrated the only misconception for the interview items.

Figure 22 shows a student response with relational-computational understanding for Item 5. The student subtracted 12 from both sides of the equation and then divided both sides of the equation by 8 to get $x = 3$.

![Equation](image)

*Figure 22. An example of relational-computational reasoning for Item 5 in the interviews.*

The student with a misconception struggled to solve the equation for the value of $x$. The student first wanted to divide both sides of the equation by 8, but only the $8x$ and not the 12 on the right side. To help the student, the interviewer rewrote the equation in standard form ($8x +
\[ 12 = 36 \]

but the student kept the same incorrect method as before and could not solve for the value of \( x \).

**Comparing the Initial Instrument to the Interviews**

After the researcher coded the interviews, his colleague gave him the key to match up the 13 students’ anonymous initial responses with their interview responses. Table 7 shows the item codes for each student for the initial instrument and the interviews, and Figure 23 shows percentages of increases, decreases, no change, or undeterminable change in students’ level of reasoning. A student’s change was undeterminable if his or her initial reasoning was undeterminable since there was not a way to track coding changes. A majority (75%) of the students’ reasoning either stayed the same or increased. Eight percent of the student reasoning decreased and the remaining 17% of the changes in student reasoning was undeterminable.

Table 8 shows if the students’ reasoning changed or stayed the same for an item-by-item comparison. Overall, the only decrease in reasoning occurred on Items 1 and 2. For Item 1, five of the students showed the same reasoning during the interview as on the initial instrument, six students had an increase in their reasoning, one had a decrease, and one was undeterminable. For Item 2, five of the students showed the same reasoning, two students had an increase, four students had a decrease, and two students’ changes were undeterminable. For Items 3 through 5 there were no decreases in reasoning. For Item 3, five students had the same level of reasoning, six students had an increase, and the change in reasoning for two was undeterminable. Item 4 had the largest number of students (seven) with the same reasoning during the interview and on the instrument, while three had an increase in level of reasoning, and three had undeterminable changes. For Item 5, three students showed the same reasoning, seven students showed an
increase in reasoning (the largest number out of all of the items), and three students had undeterminable changes.

*Table 7*

The Reasoning of Each Student for the Initial Instrument and the Interview

<table>
<thead>
<tr>
<th>Student</th>
<th>Item 1</th>
<th>Item 2</th>
<th>Item 3</th>
<th>Item 4</th>
<th>Item 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>D - Initial</td>
<td>op</td>
<td>rc</td>
<td>rc</td>
<td>rs</td>
<td>op</td>
</tr>
<tr>
<td>D - Interview</td>
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<td>op</td>
<td>rs</td>
<td>rs</td>
<td>rc</td>
</tr>
<tr>
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<td>u</td>
<td>u</td>
<td>u</td>
</tr>
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<td>rsi</td>
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<td>u</td>
<td>u</td>
<td>u</td>
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<td>rc</td>
<td>op</td>
<td>m</td>
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<td>rc</td>
<td>op</td>
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<td>rc</td>
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</tr>
</tbody>
</table>
Figure 23. A chart showing the percentages of if the students’ level of reasoning increased, decreased, remained the same, or was undeterminable between the initial instrument and the interview.

Table 8

An Item-by-Item Count (Percentage) of If the Students’ Coding Changed or Stayed the Same

<table>
<thead>
<tr>
<th>Type of Change</th>
<th>Item 1</th>
<th>Item 2</th>
<th>Item 3</th>
<th>Item 4</th>
<th>Item 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same</td>
<td>5 (38%)</td>
<td>5 (38%)</td>
<td>5 (38%)</td>
<td>7 (54%)</td>
<td>3 (23%)</td>
</tr>
<tr>
<td>Increase</td>
<td>6 (46%)</td>
<td>2 (15%)</td>
<td>6 (46%)</td>
<td>3 (23%)</td>
<td>7 (54%)</td>
</tr>
<tr>
<td>Decrease</td>
<td>1 (8%)</td>
<td>4 (31%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Undeterminable</td>
<td>1 (8%)</td>
<td>2 (15%)</td>
<td>2 (15%)</td>
<td>3 (23%)</td>
<td>3 (23%)</td>
</tr>
</tbody>
</table>
Discussion

Initial Instrument

On the initial instrument, Item 1 had a majority of operational responses (64%), whereas, for Items 2 and 3, the majority (73%) of the responses were relational-computational. For Item 4, the majority of responses that were not undeterminable were coded as relational-computational (73%), and this was the only item for which there were relational-structural responses. For Item 5, 46% of students had relational-computational responses and 36% had operational responses; the other 18% were coded as undeterminable.

Items 2 and 3 had the most relational-computational responses. For Item 2, this could be because the item itself is set up to elicit that type of response. In other words, Item 2 could be structured in such a way that the students perceived that they had to compute an answer. The relational-computational responses for Item 3 showed that the students did not transfer their thinking from part a to part b. Item 4 had the most undeterminable responses and the only misconception. The reason for these responses could have been due to the fact that the students were not asked to solve for \(x\). However, this is the only item on the initial instrument that had relational-structural responses. Despite the possibility of unfamiliar instructions, the structure of Item 4 may have encouraged some students to display relational-structural reasoning more so than the other items. Seeing a pair of equations this way (and in a different context) may have brought out different understandings of the equal symbol (Baroody & Ginsburg, 1983; Knuth et al., 2005; McNeil et al., 2006; Powell, 2015; Stacey & MacGregor, 1996; Welder, 2012).

Of the responses that could be coded for Item 5, operational understandings and relational-computational understandings are within eleven percent of each other (44% and 55%, respectively). This item could have been coded one way or the other based on the work shown.
For example, a student could have written a string of equations (operational such as Figure 16) or followed an algorithm to solve for the value of $x$ (relational-computational as in Figure 15). However, this item produced the most operational responses of any other item. Kieren (1981) and Stephens et al. (2013) point out that how students solve equations shows a skewed understanding of the equal symbol. For example, if a student writes a string of equations to solve for the value of an unknown variable then the student may have an operational view of the equal symbol.

The purpose of Item 5 was to determine how students solve a non-standard, two-step equation. Since students who view the equal symbol operationally often solve non-standard equations incorrectly (Powell, 2015), the researcher wanted to compare how students solved Item 5 with how they defined the equal symbol for Item 1. As seen in Figure 24, one-fourth of the comparisons could not be determined since the responses for either Item 1 or Item 5 (or both) were undeterminable. Half of the responses stayed the same from Item 1 to Item 5, 21% increased, and 4% decreased in coding from Item 1 to Item 5. Therefore, for half of the students, their written definition of the equal symbol was a predictor of how the students might solve non-standard equations (correctly or incorrectly).
Overall, for the initial instrument, there was one misconception, 21 undeterminable responses, 12 operational responses, 53 relational-computational responses, and two relational-structural responses (Tables 3 and 4). There were no patterns across the items for each student. One-fourth of the responses were undeterminable; this could be because the students did not know what it means to explain their reasoning. The students may have explained what they did rather than explained their reasoning. In other cases, when students wrote their explanation, they may have over-explained and put in extraneous information that is actually incorrect. This could have caused some students’ reasoning to be coded as operational rather than relational-computational or even relational-structural. Another reason students may have reasoned computationally is that they may be reverting back to an algorithm of some sort when they see an algebraic equation. In other words, students may have been taught algorithms to solve equations and may have inappropriately used them when they saw the equations on the instrument.
Interviews

For the interviews, the majority (69%) of responses for Item 1 were relational. For Item 2, there was a mix between operational (39%) and relational-computational (46%) responses, and 15% of the responses were initially relational-structural. For Item 3, there was a split between relational-computational and relational-structural (46% each), although there was one (8%) initial relational-structural response. For Items 4 and 5, a majority of responses were relational-computational (61% and 92%, respectively), but Item 4 had one initial relational-structural, and Item 5 had the only misconception (Tables 5 and 6). For the entire interview process, there were no undeterminable responses since the researcher asked students to tell him more if he did not understand the students’ reasoning.

Most operational responses appeared in Items 1 and 2 (nine total), with one operational response for Item 4. As previously discussed, this could be because these items tended to elicit an operational response more so than the others. With the exception of Item 5 having one misconception, all other responses demonstrated some sort of relational understanding. Except for the misconception, all responses for Item 5 were relational-computational. This may be because an equation such as this, that is prevalent in Algebra I courses, encourages a procedural approach.

When comparing Items 1 and 5 for the interviews, as seen in Figure 25, 75% of the students maintained relational reasoning from Item 1 to Item 5. The remaining 25% had growth from operational reasoning to relational reasoning. There were no comparisons that could not be determined since the researcher asked students to clarify their thinking and was able to code the student responses appropriately. No students maintained an operational understanding from Item 1 to Item 5, nor were there any students who decreased in their reasoning from Item 1 to Item 5.
Hence, for three-quarters of the students, their oral definition of the equal symbol was a predictor of how the students might solve non-standard equations (correctly or incorrectly).

Figure 25. A comparison between the coding of Item 1 and Item 5 for the interviews.

Comparing the Initial Instrument and the Interviews

Overall, there was an increase in relational reasoning for Item 1 from 18.18% on the initial instrument to 69.23% on the interviews. As Stephens et al. (2013) mentioned, defining the equal symbol is considered a difficult task for students. There was also an increase for Items 2 through 5 in relational reasoning from 61.8% on the initial instrument (including relational-computational and relational-structural) to 86.54% on the interviews (including relational-computational, relational-structural, and initially relational-structural). The students in this study could have had a more difficult time explaining their definitions in writing than orally in person. There could have been more students with relational reasoning on the interviews since there was no reasoning coded as undeterminable. Further, the reason for an increase in student reasoning
could have been that the students felt more comfortable orally explaining their work and reasoning rather than writing down their reasoning. Students may also have not known the difference between showing their work and explaining their work, causing their initial reasoning to be lower than what was their true understanding.

When comparing Items 1 and 5 for the interviews, 75% of the students kept a relational reasoning; this is an increase from the initial instrument (50% of the students kept the same reasoning whether it was relational or operational). This further shows that the students had an easier time orally explaining their definition and reasoning than in writing.

**Conclusion**

Since algebra is important to higher education and career readiness (Kilpatrick & Izsák, 2008; McNeil et al., 2010; Moses, Kamii, Swap, & Howard, 1989; Picciotto & Wah, 1993; Stephens et al., 2013) and a proper understanding of equivalence and the equal symbol is foundational to algebra (Baroody & Grinsburg, 1983; Booth, 1992; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Linchevski, 1995; McNeil et al., 2006; Stephens et al., 2013; Welder, 2012), students need to have an accurate working understanding of the equal symbol. There are two main interpretations of the equal symbol: operational and relational. Students who have an operational interpretation view the equal symbol as a “do something” indicator (Baroody & Ginsburg, 1983; Capraro, Ding, Matteson, Capraro, & Li, 2007; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Students who interpret the equal symbol as relational, on the other hand, have an accurate view of the equal symbol. They recognize that the equal symbol indicates equivalence and that there must be balance in an equation.

There are two types of relational understandings of the equal symbol, relational-computational and relational-structural (Stephens et al., 2013; Welder, 2012). Students with a
relational-computational view of the equal symbol will use an algorithm to solve for a value or, when asked to verify equivalence, they will simplify each side of the equation to confirm equivalence. Students with a relational-structural understanding of the equal symbol, on the other hand, will use the structure of the equation itself to solve for the value or to verify equivalence.

In this study, students’ overall interpretation of the equal symbol was slightly higher for the interviews than on the initial instrument. For Item 1 of the initial instrument, most students (64%) were coded as operational whereas on the interviews most students (69%) were coded as relational. There was an increase in relational-structural responses from the initial instrument to the interviews (from 2% to 25%). This could be explained by the fact that there were no undeterminable responses for the interviews. This could also mean that students can explain their reasoning orally better than in writing.

One limitation of this study is during the initial instrument the researcher coded students based on the work shown. A student could in actuality have a different reasoning than he or she was coded for, but the researcher could only code the student based on what was on the initial instrument. Also, some students did not feel comfortable writing down their explanations during the interview process. Another limitation of the study is the small sample size; only 22 students (out of about 100 students), who met the inclusion criteria, agreed to be part of the study. Of these 22 students, only 13 students agreed to take part in the interview process. Due to the timing of the school year, the researcher moved forward with the study nonetheless. Therefore, the findings of the present study cannot make any claims for the population, but it does shed light on some teaching implications.
Students need to be exposed to the equal symbol used in a variety of scenarios to help bring out different and more advanced perspectives of the equal symbol (Baroody & Ginsburg, 1983; Knuth et al., 2008; McNeil et al., 2006; Powell, 2015; Stacey & MacGregor, 1996; Stephens et al., 2013). As students begin to work with a literal symbol (such as $x$), students should be exposed to items that elicit relational-structural thinking (such as Items 2 and 4), but they must be able to translate that thinking into other scenarios, as well (such as parts a and b of Item 3). This may be done by focusing on the meaning of the equal symbol and the structure of the equation itself (Stephens et al., 2013). This study also implies that students need to be given opportunities to explain their reasoning orally as well as in writing; students must be capable of explaining their reasoning regardless of means. Also, when writing explanations, students need to know the difference between showing work and explaining their reasoning.

The present study was limited in its sample size, but it still opens doors to future research. A study similar to this on a larger scale will be beneficial to students, teachers, and the research community. Further research on how students interpret the equal symbol orally as compared to their written interpretations would also be beneficial. More research can be done on the types of equations that elicit the different types of understandings of the equal symbol, as well.

Despite the limitations of this study, it does contribute to the field as it was conducted with students after they have already taken (and passed) Algebra I. Prior to the present study, most studies were conducted in the elementary grades (Baroody & Grinsburg, 1983; Capraro, Ding, Matteson, Capraro, & Li, 2007; Kieren, 1981; Knuth et al., 2005; McNeil et al., 2006; Powell, 2015; Stephens et al., 2013; Welder, 2012) and few studies have been conducted on how high school students interpret the equal symbol (Kieren, 1981; McNeil et al., 2006). This study shows that an operational view is still present in the higher grades. This emphasizes the
importance of fostering a relational perspective of the equal symbol across all grade levels as students move from the content strand of numbers and operations to algebra.
References


Usiskin, Z. (1999). Why is algebra important to learn (Teachers, this one’s for your students!). In B. Moses (Ed.), *Algebraic Thinking, Grades K-12: Readings from NCTM’s School-Based Journals and Other Publications* (22-30). Reston, VA: The National Council of Teachers of Mathematics, Inc.


Appendix A

Research Instrument

Name_______________________________________

1. The following questions are about this statement:
   
   \[ 3 + 4 = 7 \]

   ↑

   a. The arrow points to a symbol. What is the name of the symbol?

   b. What does the symbol mean?

   c. Can the symbol mean anything else? If yes, please explain.

2. What value of \( x \) will make the following number sentence true? Explain your reasoning.

   \[ 88 + 49 = x + 48 \]
3. What value of $x$ will make the following number sentences true? Explain your reasoning.
   
   a. $4x = 20$
   
   b. $20 = 5x$

4. Is the value of $x$ the same number in the following two equations? Explain your reasoning.

   $2x + 15 = 31 \quad 2x + 15 - 9 = 31 - 9$

5. What value of $x$ will make the following number sentence true? Explain your reasoning.

   $25 = 3x + 7$
Appendix B

Interview Questions

Name of Student ________________________

1. The following questions are about this statement:

   \[ 8 + 4 = 12 \]

   a. The arrow points to a symbol. What is the name of the symbol?

   b. What does the symbol mean?

   c. Can the symbol mean anything else? If yes, please explain.

2. What value of \( x \) will make the following number sentence true? Explain your reasoning.

   \[ 54 + 78 = x + 77 \]
3. What value of \( x \) will make the following number sentences true? Explain your reasoning.
   a. \( 8x = 56 \)
   b. \( 56 = 7x \)

4. Is the value of \( x \) the same number in the following two equations? Explain your reasoning.
   \[ 4x + 13 = 57 \quad \text{and} \quad 4x + 13 - 6 = 57 - 6 \]

5. What value of \( x \) will make the following number sentence True? Explain your reasoning.
   \[ 36 = 8x + 12 \]
Appendix C

Script of What to Say Prior to the Implementing the Instrument

Thank you for completing the assessment. I just want to remind you that your participation is voluntary and you can withdraw by letting me know. If you choose not to participate, you will complete the assessment; however, your responses will not be used for the study. Participation as well as your decision to participate will not affect your grade in any way whatsoever.

Please do not talk to each other while you complete the assessment. When you are done, please let me know and I will collect the assessment from you. Please do not discuss the questions with anyone outside of the classroom. Other students may not have taken it yet and I want to know their ideas about the questions.

Please answer each question. There is legitimately no wrong answer. You are just helping me understand what you already know. Explaining your reasoning for every item may get tiresome, but it will help me understand your ideas. So, please do your best to explain your reasoning in full.

Are there any questions? *Answer questions.

I will be passing out the assessment now. Again, please do not talk to each other during the assessment. Again, please do not speak with anyone about the questions outside of class.