DIVISION ALGEBRAS, GLOBAL FORMS OF THE INVERSE FUNCTION THEOREM, DIFFERENTIABLE MANIFOLDS, AND FIXED POINT THEOREMS

by

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ABSTRACT

In this undergraduate thesis, we use results from Topology and Analysis, including but not limited to the Banach Fixed Point Theorem, in order to establish some global forms of the Inverse Function Theorem. As an application that brings together three different branches of mathematics, we prove a basic, yet important, result in Algebra: there is no commutative division algebra (not necessarily associative) that is isomorphic to $\mathbb{R}^n$ for all n greater than or equal to 3.

Furthermore, we will develop enough theory regarding differentiable manifolds to discuss and prove the Brouwer Fixed Point Theorem and the Schauder Fixed Point Theorem along with an application to game theory and economics. We will then compare and contrast the applications of the Banach Fixed Point Theorem and the Schauder Fixed Point Theorem to the field of differential equations. These applications are two famous theorems commonly known as the Picard-Lindelöf Theorem and Peano's Theorem, respectively.

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0 Introduction

Some of the biggest branches of Mathematics are Analysis, Topology, and Algebra. Topology is somewhat similar to Analysis in that the notions we learn in Analysis are much more generalized to deal with different types of spaces and different types of contexts. However, when learning more about each field in the respective classes, it doesn't seem as though there is much overlap between Analysis and Algebra. Sure, there are basic concepts from the other branches that we may assume to be true in order to move on, but at first, it doesn't seem like there is something that really ties them together. Of course, there is a branch that ties Topology and Algebra together fittingly called Algebraic Topology. However, in this paper, we will show that the three branches are connected even in places one may not think.

To even begin to understand, we need to know different basic definitions in Topology and Analysis. We will use these to prove the Banach Fixed Point Theorem, and in turn, we will use this to prove the (local) Inverse Function Theorem. The proof of the Banach Fixed Point Theorem is vital not only in this section but in the coming sections as well, when we will turn our focus fom the Inverse Function Theorems to differentiable manifolds and fixed point theorems. The point of proving these two theorems is not only because they are important in general but because they will be used in later chapters to establish more results, all of which will lead up to global forms of the Inverse Function Theorem.

After this chapter, we will briefly cover the basics of covering spaces. We will visit various definitions and examples in order to prove a theorem about when a covering space is a homeomorphism. The importance of this theorem may not immediately be seen, but it is used in proofs that Plastock [14] gives regarding different forms of the Global Inverse Function Theorem.

The final chapter before shifting our focus will be about the global forms of the Inverse Function Theorem. We will cover different theorems about which conditions are necessary to turn C^1 local diffeomorphisms/homeomorphisms between Banach Spaces into global diffeomorphisms/homeomorphisms. In these theorems, different conditions are used in order to arrive at a diffeomorphism/homeomorphism, but we will remark that these conditions are actually equivalent. At this chapter's culmination, we cover an application to Algebra due to W. Gordon [5]:

Theorem. *For $n \geq 3$ there is no operation of multiplication on a commutative division algebra (not necessarily associative) that is isomorphic to $\mathbb{R}^n$.*

Once we have proven this application to Algebra, we will turn back to building up our analytical and topological foundation. Similar to the first two chapters, we will only cover what is necessary in order to move onto differentiable manifolds and the fixed point theorems.

The differentiable manifolds chapter is vital to understand the Brouwer Fixed Point Theorem. We will cover a few definitions and a lot of lemmas and theorems that keep building upon one another. Many of the lemmas rely on each other, leading to results upon which the proof of the Brouwer Fixed Point Theorem relies.

For the remainder of the paper, we will talk about fixed point theorems, namely Brouwer's Fixed Point Theorem, Schauder's Fixed Point Theorem (which is an extension of Brouwer's to Banach spaces), and Peano's Theorem. We will note the differences not only between Brouwer's Fixed Point Theorem and Schauder's Fixed Point Theorem, but also between Peano's Theorem and the Picard-Lindelöf Theorem, the latter being an application of the Banach Fixed Point Theorem.. The following are the statements of these theorems:

Theorem. (Brouwer Fixed Point Theorem). *Let D^n be the closed unit n-ball. Then, any continuous function $G : D^n \to D^n$ has a fixed point.*

Theorem. (Schauder Fixed Point Theorem). *Let M be a nonempty, compact, convex subset of a Banach space X, and suppose $T : M \to M$ is a continuous operator. Then T has a fixed point.*

Theorem. (Peano's Theorem). *Consider the following initial value problem:*

$$x'(t) = f(t, x(t)), x(t_0) = y_0.$$

If f is continuous and bounded on a rectangle $\subset \mathbb{R}^2$, then there exists a local solution to the initial value problem.

Theorem. (Picard-Lindelöf Theorem). *Consider the following initial value problem:*

$$x'(t) = f(x, y(t)), y(x_0) = y_0.$$

If f is continuous with respect to t and Lipschitz continuous in the second variable on a rectangle $\subset \mathbb{R}^2$, then there exists unique local solution to the initial value problem.

These theorems talk about functions f from a space X into itself such that $f(x) = x$, $x \in X$. Whether or not this x is unique will depend on the strength of the conditions regarding f and X, but one thing both theorems guarantee is the *existence* of such an x in varying conditions.

Why are we so concerned about whether or not such an x exists? It turns out that these theorems have far and wide-reaching applications, varying from differential equations to game theory and economics, all of which we will discuss via applications of said theorems later. Our first application will be a more light-hearted and fun application. This will use the Brouwer's/Schauder's Fixed Point Theorem. We will consider Colorado and show that if we have a continuously-distorted picture of Colorado on a sheet of paper and proceed to place that paper on Colorado soil, then there is a point on the picture of Colorado directly above its location in mainland Colorado.

The second application will use Brouwer's/Schauder's Fixed Point Theorem to show that a mixed-strategy Nash equilibrium solution will always exist (though it may not be unique) in a finite game, regardless of whether a pure-strategy solution exists. Basically, a mixed-strategy assigns probabilities to choices from which players in a game can choose. If we suppose that all players are aware of the other players' current strategy, then the Nash equilibrium is the strategy "vector" (where the n^{th} component is the strategy of player n) in which no player can benefit by deviating from it.

The third application will be proving Peano's Theorem as a result of Schauder's Fixed Point Theorem. We will how that certain ordinary differential equations are guaranteed to have solutions. These solutions may or may not be unique. Because Peano's theorem only tells us about the existence of a solution, we cannot conclude anything about the uniqueness of such a solution without imposing stronger conditions.

The final application will be using a previously-proved theorem: the Banach Fixed Point Theorem. We will also apply this to the field of differential equations in order to highlight the differences between Peano's Theorem and the Picard-Lindelöf Theorem. Because the Picard-Lindelöf Theorem imposes stronger conditions on the function f, namely f must be Lipschitz-continuous, then f is not only guaranteed to have a solution but a unique one at that. Peano's Theorem does not assume such a condition, nor does it assume any other strong conditions either, and as such, we are left with a guarantee only of the existence of a fixed point.

1 The Inverse Function Theorem

1.1 Topological Preliminaries

The main goal of this section is to provide just enough notions from Topology to build up toward the Banach Fixed Point Theorem. There is nothing too in-depth with regard to Topology in this section, and we will assume a moderate knowledge of Calculus and vector spaces. These notions will be used at the end of the chapter to prove the Inverse Function Theorem. For a more comprehensive and rigorous treatment of Topology, see [10].

Though Euclidean n-spaces are the most basic and well-known spaces, we will be generalizing our results to metric and Banach spaces, the latter being special types of metric spaces. Before that, though, we need to define a sense of "distance" in our spaces. This notion of distance is called a *metric*, and is defined as follows:

Definition 1.1. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ with the following properties:

(1) $d(x,y) \geq 0$ for all $x, y \in X$; the equality holds if and only if $x = y$.

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Since a metric is considered as a distance between two points, it is very easy to define a set of points within a certain radius of a single point x. We use the notation $B(x,r)$ and say "the (open) ball of radius r centered at x." Formally, it is defined in a space X as

$$B(x,r) = \{y \in X \mid d(x,y) < r\}.$$

Note that in $\mathbb{R}$, a ball is simply an open interval with midpoint x. In $\mathbb{R}^2$, it is the open set of all the interior points sitting inside the circle of radius r centered at x. This notion is extended to an n-dimensional ball in $\mathbb{R}^n$, simply called a ball when the context is understood. We are also able to talk about norms on vector spaces, which induce a metric by: $d(x,y) = \left|x - y\right|$. We can easily check that this is indeed a metric. In fact, every normed vector space $(X, ||\cdot||)$ is a metric space by definition. Recall that the standard Euclidean norm of $\mathbb{R}^n$

$$||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

is not always used.

Example 1. Since we will be dealing with Banach spaces, which will be introduced later, let us consider $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = \left|x - y\right|$, where d is the standard absolute value function. It is fairly simple to check that d is indeed a metric.

(1) Clearly, $\left|x - y\right| > 0$ for $x \neq y$ by definition of d and when $x = y$, $\left|x - y\right| = |0| = 0$.

(2) Since $|-z| = |-1||z| = |z|$,

$$d(x,y) = \left|x - y\right| = \left|y - x\right| = d(y,x),\ \forall x,y \in \mathbb{R}.$$

(3) To prove that $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$, one need only consider the triangle inequality.

$$d(x,z) = |x - z| = \left|x - y + y - z\right| \leq \left|x - y\right| + \left|y - z\right| = d(x,y) + d(y,z).$$

Thus, $d(x,z) \leq d(x,y) + d(y,z),\ \forall x,y,z \in \mathbb{R}$.

Hence, d is indeed a metric.

Definition 1.2.

1) A *metric space* is a space X endowed with a metric d.

2) A *complete* metric space is a metric space (X,d) in which every Cauchy sequence is convergent.

3) A *Banach space* is a complete normed vector space.

Given any space, we need to be able to classify types of sets (collections of elements in that space.) This is extremely vital and elementary to many different proofs, not only in this paper but in Topology and Analysis as well. We will restrict our definitions to metric spaces, though they can be generalized to any Topological space.

Definition 1.3. Let (M,d) be a metric space with a set $U \subseteq M$.

1) U is *open* in M if, given a point $x \in U$, $\exists \epsilon > 0$ such that $B(x,\epsilon) \subset U$.

2) F is *closed* in M if $M \setminus F$ is open. Equivalently, F is *closed* if every convergent sequence (x_i) in F converges to a point in F.

Though it seems that if U is open, it can't be closed (and vice versa), this is not true. If U is both open and closed, we say U is *clopen*. We have to be careful, though, in our frame of reference. Every set U is clopen in U, as will be discussed in Example 2, but U is not necessarily clopen in V for $U \subset V$. In this paper, however, our frame of reference will be on the metric space in the context.

Example 2. Let (X,d) be a metric space. Note that $\emptyset, X$ are both clopen in X. X is open because every ball in X is by definition a subset of X. On the other hand, $\emptyset$ vacuously satisfies the definition of an open set; that is, there is no $x \in \emptyset$ in the first place, and hence because it does not fail the definition, it satisfies it. X is closed because $X \setminus X = \emptyset$ is open. Furthermore, $\emptyset$ is clearly closed since we have already stated that $X \setminus \emptyset = X$ is open.

Definition 1.4. Let (X, d_1) and (Y, d_2) be metric spaces.

1) A *Lipschitz map* is a continuous function $f : X \to Y$ for which there exists a constant $C \in \mathbb{R}^+$ such that $\forall x, y \in X$, we have

$$d_2(f(x), f(y)) \le C d_1(x, y).$$

C is called a *Lipschitz constant*.

2) A *contraction mapping*, or simply a *contraction*, is a continuous function $g : (X, d) \to (X, d)$ such that $\exists k \in \mathbb{R}$ with $0 \le k < 1$ such that $\forall x, y \in (X, d)$, we have

$$d(g(x), g(y)) \le k d(x, y).$$

Note that k is merely a Lipschitz constant constrained to values in $[0, 1)$. Furthermore, Lipschitz constants are independent of the points that we choose; that is, there exists at least one $k \in [0, 1)$ such that k works for any pair of points that we choose.

Example 3. Let $X = [1, \infty)$. We will show that $f : X \to X$ defined by $f(x) = \sqrt{x}$ is a contraction. Let $x, y \in X$. Without loss of generality, let $y > x$. We need to show that $\exists k \in [0, 1)$ such that $\forall x, y \in X$, $\sqrt{y} - \sqrt{x} < k(y - x)$. This is equivalent to showing that $\frac{\sqrt{y}-\sqrt{x}}{y-x} < k$, for all $y > x$. We note that

$$\frac{\sqrt{y} - \sqrt{x}}{y - x} = \frac{\sqrt{y} - \sqrt{x}}{(\sqrt{y} + \sqrt{x})(\sqrt{y} - \sqrt{x})} = \frac{1}{\sqrt{y} + \sqrt{x}} < \frac{1}{\sqrt{1} + \sqrt{1}} = \frac{1}{2},$$

and see that $k = 1/2$ does the job.

■

It is important to note that $g : [0, \infty) \to [0, \infty)$ given by $g(x) = \sqrt{(x)}$ is not a contraction since $\lim_{y \to 0^+} \frac{\sqrt{y}-0}{y-0} = +\infty$. As a concrete counterexample to g being a contraction, note that $g(.7) - g(.01) = \sqrt{.7} - \sqrt{.1} > .7 - .01 = .69$. So, there is no $k \in (0, 1)$ such that $g(.7) - g(.01) \le k(.7 - .01)$. Thus, g cannot be a contraction. In the next section, we will see another reason as to why g is not a contraction.

1.2 The Banach Fixed Point Theorem

This section will be almost entirely devoted to the Banach Fixed Point Theorem and its proof. Though we will not cover here its application to first-order differential equations and the uniqueness/existence of solutions (see Section 9.2), the BFPT does apply to this and to other areas in mathematics as well. However, we will limit our scope in the following sections to the Inverse Function Theorem.

Definition 1.5. A point $\bar{x} \in X$ is a *fixed point* of a map $f : X \to X$ if $f(\bar{x}) = \bar{x}$.

Theorem 1.6. (The Banach Fixed Point Theorem). *Let (X,d) be a non-empty complete metric space. If $T : X \to X$ is a contraction, then T admits a fixed-point $x^* \in X$, and furthermore, x^* is unique.*

Remark 1.7. Recall that since (X,d) is a non-empty complete metric space, that means X has a metric d, X has at least one point $x \in X$, and every Cauchy sequence (x_i) convergesto a point in X.

Proof of Theorem 1.6. [16] Fix a point $x \in X$ and call $C \in [0,1)$ the Lipschitz constant of T. Consider the sequence $(T^k x)_{k=0}^{\infty}$. We need to show that this is Cauchy in the space (X,d) in order to find the candidate for our fixed point. So, without loss of generality, let $n \geq m > 0$ be nonnegative integers and note that because $T^k x = T(T^{k-1}x)$ for any $k \in \mathbb{Z}^+$, then

$$d(T^n x, T^m x) = d(T(T^{n-1}x), T(T^{m-1}x) \leq Cd(T^{n-1}x, T^{m-1}x) \leq \ldots \leq$$

$$C^m d(T^{n-m}x, x) = C^m d(x, T^{n-m}x).$$

Because d is a metric, then by the Triangle Inequality (property 3), we know that $d(x, T^{n-m}x) \leq d(x, T^{n-m-1}x) + d(T^{n-m-1}x, T^{n-m}x)$, so by induction on the second term, we see that

$$d(x, T^{n-m}x) \leq d(x, T^{n-m-1}x) + d(T^{n-m-1}x, T^{n-m}x) \leq d(x, T^{n-m-2}x) +$$

$$d(T^{n-m-2}x, T^{n-m-1}x) + d(T^{n-m-1}x, T^{n-m}x) \leq \ldots \leq \sum_{k=1}^{n-m} d(T^{k-1}x, T^k x).$$

So,

$$d(T^n x, T^m x) \leq C^m \sum_{k=1}^{n-m} d(T^{k-1}x, T^k x) \leq C^m \sum_{k=1}^{n-m} C^{k-1} d(x, Tx) =$$

$$C^m d(x, Tx) \sum_{k=1}^{n-m} C^{k-1} = C^m \frac{1 - C^{n-m-1}}{1 - C} d(x, Tx),$$

since $\sum_{k=1}^{n-m} C^{k-1}$ is a geometric series. So, because $C < 1$, we see that

$$C^m \frac{1 - C^{n-m-1}}{1 - C} \to 0 \text{ as } m \to \infty$$

and hence $d(T^n x, T^m x) \to 0$. So, $(T^k x)_{k=0}^{\infty}$ is indeed a Cauchy sequence, and because (X,d) is complete, $(T^k x)_{k=0}^{\infty}$ converges in X, say to u. Now that we have our candidate for a fixed point, we need to show that it actually is a fixed point; that is, $Tu = u$. By definition of a contraction, T is continuous (and hence we can bring it inside/outside limit signs), so

$$d(Tu,u) = d(T(\lim_{k\to\infty} T^k x), \lim_{k\to\infty} T^k x) = d(\lim_{k\to\infty} T^{k+1} x, \lim_{k\to\infty} T^k x) = d(u,u) = 0.$$

Since $d(Tu,u) = 0$, one has that $Tu = u$, so u has a fixed point. We have thus shown existence and now need to show uniqueness. Suppose $v \in X$ satisfies $Tv = v$ as well. Then,

$$d(u,v) = d(Tu,Tv) \leq Cd(u,v),$$

but since $C < 1$, it must be that $d(u,v) = 0$, else if $d(u,v) = p \neq 0$, then $p \leq Cp$, which is a false statement. Since d is a metric, $d(u,v) = 0 \iff u = v$, hence u is unique.

■

Remark 1.8. Now we return to Example 3. Recall that $g : [0,\infty) \to [0,\infty]$ was defined by $g(x) = \sqrt{x}$. We stated that g was not a contraction by finding values to disprove it. However, in more general cases, it may be harder to show that a function is not a contraction using this method. Thus, here are some observations:

(1) $[0,\infty)$ is clearly a metric space with metric $|\cdot|$ defined as the standard absolute value.

(2) $[0,\infty)$ is complete as a closed subset of the complete metric space $\mathbb{R}$.
Proof: Clearly $X = [0,\infty)$ is closed since $\mathbb{R} \setminus X = (-\infty, 0)$ is open. Let (x_i) be a Cauchy sequence in X (and hence, in $\mathbb{R}$). Since $\mathbb{R}$ is complete, (x_i) converges in $\mathbb{R}$, say to $x \in \mathbb{R}$. But, X is closed and thus contains all of its limit points by definition 1.3, $x \in X$. We see, then, that every Cauchy sequence in X converges in X, so $X = [0,\infty)$ is complete. Note that this conclusion holds for any closed subset of a complete metric space.

(3) The Banach Fixed Point Theorem, assuming that (X,d) is a non-empty complete metric space, is equivalent to the following: If there is no fixed point or more than one fixed point of the function $T : X \to X$, then T is not a contraction.

Thus, showing that g is not a contraction boils down to simply noting that $g(0) = 0$ and $g(1) = 1$, hence both 0 and 1 are different fixed points. By (3), the contrapositive of Theorem 1.6, g is not a contraction.

1.3 Analytic Preliminaries

In this section, we will assume knowledge about continuity, inverses, and derivatives. We will also assume a basic knowledge of matrices and determinants, from Linear Algebra. The point of this section is to provide enough terminology and knowledge to understand new theorems, lemmas, and definitions in later chapters and sections. Our focus, then, will be restricted to homeomorphisms, diffeomorphisms, and the Jacobian Matrix.

Definition 1.9. Let X, Y be metric spaces. Then, $f : X \to Y$ is a *homeomorphism* if f is continuous, a bijection, and has a continuous inverse.

What a homeomorphism is really doing is stating when two metric spaces (more generally, topological spaces) are really the "same" if we allow them to be "deformed" and "bent" from one into the other.

Definition 1.10. Let X, Y be metric spaces. Then, $f : X \to Y$ is a *diffeomorphism* if f is differentiable, a bijection, and has a differentiable inverse.

Notice the similarity between homeomorphisms and diffeomorphisms. The difference is that a homeomorphism is not necessarily differentiable for all $x \in X$ nor is its inverse necessarily differentiable for all $y \in Y$. However, because differentiability implies continuity, every diffeomorphism is a homeomorphism, but not every homeomorphism is a diffeomorphism.

Definition 1.11. Let X, Y be open subsets of some Euclidean space $\mathbb{R}^n$. Then, $f : X \to Y$ is differentiable at $a \in X$ if there exists a linear transformation $\lambda : X \to Y$ such that

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

Theorem 1.12. ([16], p. 16) *If such a λ exists, then the derivative of f at a is unique.*

Proof of Theorem 1.12. To see this, suppose μ satisfies the above condition as well. If $d(h) = f(a+h) - f(a)$, then

$$\lim_{h \to 0} \frac{||\lambda(h) - \mu(h)||}{||h||} = \lim_{h \to 0} \frac{||\lambda(h) - d(h) + d(h) - \mu(h)||}{||h||} \leq$$

$$\lim_{h \to 0} \frac{||\lambda(h) - d(h)||}{||h||} + \lim_{h \to 0} \frac{||d(h) - \mu(h)||}{||h||} = 0.$$

If $x \in X$, then $tx \to 0$ as $t \to 0$. Hence, for $x \neq 0$, we have

$$0 = \lim_{t \to 0} \frac{||\lambda(tx) - \mu(tx)||}{||tx||} = \frac{||\lambda(x) - \mu(x)||}{||x||}.$$

So, $\lambda(x) = \mu(x)$.

■

Definition 1.13. Let X, Y be open subsets of a Euclidean-n space, and $f : X \to Y$ be differentiable at $a \in X$. Then, the matrix associated to the unique λ as in the previous definition is called the *Jacobian matrix* of f at a and is denoted by $f'(a)$ or $Df(a)$.

Though we will not show this, if f as before has the form $f = (f_1, f_2, \dots, f_n)$, then

$$f'((x_1, x_2, \dots, x_n)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

where all the partial derivatives are computed at $(x_1, \dots, x_n)$.

Lemma 1.14. Chain Rule. ([16], p. 19) *If $f : X \to Y$ is differentiable at $a \in X$, and $g : Y \to Z$ is differentiable at $f(a)$, then the composition $g \circ f$ is differentiable at a, and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$

Lastly, we will define the norm of a matrix. This will be used in the Inverse Function Theorem.

Definition 1.15. Let X, Y be finite-dimensional vector spaces and A a matrix representing a linear transformation from X to Y. Then, the *norm of A* is defined as

$$\|A\| = \sup\{\|Ah\| : \|h\| = 1\}.$$

That is, the norm of A is the supremum of the lengths of the images of all the unit vectors $h \in X$.

1.4 Inverse Function Theorem

Theorem 1.16. (The Inverse Function Theorem). [2] *Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ a differentiable function of class C^1. Suppose $x_0 \in U$ is a point where $f'(x_0)$ is invertible. Then, there exists neighborhoods $W \subset U$ of x_0 and V of $y_0 = f(x_0)$ such that $f|_W : W \to V$ is a bijection. Furthermore, the inverse $g : V \to W$ is differentiable of class C^1, with derivative $g'(y) = f'(g(y))^{-1}$.*

Remark 1.17. We will discuss informally what this theorem is saying and how we go about proving it. Since U is an open set, we can take nonempty open subsets of U around different points. By picking W around x_0 such that $f'(x_0)$, that is the Jacobian matrix of f at x_0 is invertible (determinant is nonzero), then we can make W small enough so that it only contains points at which $f'(x)$ is invertible. We are able to do this because the partial derivatives of f are continuous, meaning f is of class C^1. Furthermore, since W is chosen so that it contains only invertible points, there does not exist any relative extrema and thus $f|_W$ is injective. Since $V = f(W)$, $f|_W$ is surjective as well, thus bijective. We then choose W in such a way so that $f^{-1} = g$ is differentiable and thus continuous as well.

Lemma 1.18. ([2], p. 1) *Suppose* $\|f'(x)\| \leq M$ *on some disk* D*. Then for any points* $x, x+h \in D$*,*

$$\|f(x+h) - f(x)\| \leq M\|h\|.$$

Proof of Theorem 1.16. (The Inverse Function Theorem). We may assume that $x_0 = y_0 = 0$ and $f'(0) = I$. If not, we can use a linear function and a linear change of coordinates to get this normalization. Since $f \in C^1$, let

$$D = \{x \in \mathbb{R}^n \mid \|x\| \leq \delta\},$$

where δ is chosen small enough to make $D \subset U$ and $\|f'(x) - I\| \leq \frac{1}{2}$ for all $x \in D$. Let $w(x) = f(x) - x$, so $w'(x) = f'(x) - I$, and by Lemma 1.18, $\|w(x+h) - w(x)\| \leq \frac{1}{2}\|h\|$. So,

$$\|f(x+h) - f(x) - h\| \leq \frac{1}{2}\|h\|. \tag{1.1}$$

We will now choose our V as in the theorem. Let $V = \{y \in \mathbb{R}^n \mid \|y\| < \frac{1}{2}\delta\}$. From this point on, δ has been chosen as such. Let $g(y) = x$.

Let $u : D \to D$ be defined by $u(x) = x + (y - f(x))$. Note that $y = f(x) \iff u$ is a fixed point. If we can eventually check that the map u is a contraction, then we know a fixed point x exists and is unique by Theorem 1.6 and thus a point where $y = f(x)$. First, we must check that u is well-defined, that is, $u(D) \subset D$. Let $x = 0$ in Equation 1.1 and see that

$$\|f(h) - h\| \leq \frac{1}{2}\|h\| \leq \frac{1}{2}\delta, \ \forall h \in D.$$

Rename h as x, and we see that

$$\|u(x)\| \leq \|y\| + \|f(x) - x\| \leq \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

Now, we will check that u is a contraction. So,

$$u(x+h) - u(x) = x + h + y - f(x+h) - x - y + f(x) = -f(x+h) + f(x) + h.$$

Using Equation 1.1 again,

$$\|u(x+h) - u(x)\| = \|f(x+h) - f(x) - h\| \leq \frac{1}{2}\|h\|.$$

Indeed, u is a contraction (and hence continuous) with a Lipschitz constant of $\frac{1}{2}$.

Note that D is the closed ball in $\mathbb{R}^n$, and since $(\mathbb{R}^n, \|\cdot\|)$ is a complete metric space, D is a complete metric space (see Observation (2) in Remark 1.8). So, u is a contraction and thus yields a fixed point, say $x \in D$. Since $\|u(x)\| = \|x\| < \epsilon$, we see that $x \in \mathring{D}$. Now, we define W as in the theorem as $W = f^{-1}(V) \cap \mathring{D}$. $f|_W$ is bijective onto V since $f|_{D \cap f^{-1}(V)}$ is a bijection. Thus, we have shown that there exist neighborhoods W of x_0 and V of $y_0 = f(x_0)$ such that $f|_W : W \to V$ is a bijection. Let g be the inverse of $f|_W$.

We now need to check that our $g : V \to W$ is continuous. Let $y, y+k \in V$ and let $g(y) = x,\ g(y+k) = x+k$. By Equation 1.1 once again, $\|k-h\| \leq \frac{1}{2}\|h\|$. So,

$$\|h\| = \|h-k+k\| \leq \|k-h\| + \|k\| \leq \frac{1}{2}\|h\| + \|k\|.$$

Thus, $\frac{1}{2}\|h\| \leq \|k\|$ and hence $\|h\| \leq 2\|k\|$. Simply let $\|k\| \leq \frac{1}{2}\epsilon$ and see that g is indeed continuous.

We need only show that g is differentiable with a derivative $g'(y) = f'(g(y))^{-1}$. So, since f is differentiable at x, we have that

$$f(x+h) = y+k = f(x) + Ah + e(h) = y + A[g(y+k) - g(y)] + e(h),$$

where A is the Jacobian matrix of f at x. So, we can apply A^{-1} to the previous equation to obtain

$$g(y+k) = g(y) + A^{-1}k - A^{-1}e(h).$$

Recall that $\|h\| \leq 2\|k\|$ and see that

$$\frac{\|A^{-1}e(h)\|}{|k\|} \leq 2\|A^{-1}\|\frac{\|e(h)\|}{\|h\|} \longrightarrow 0$$

as $k \to 0$ since $\frac{\|e(h)\|}{\|h\|} \to 0$ as $h \to 0$ as $k \to 0$ from the differentiability of f and what we recalled. Thus, g is by definition differentiable with derivative $g'(y) = A^{-1} = f'(g(y))^{-1}$.

■

2 Covering Spaces

2.1 Basic Theory

This section will cover basic definitions and review examples in order to gain an intuition regarding covering spaces and loops. Though seemingly disjoint from the previous sections on Topology and Analysis, our goal here is to build up to Hadamard's Global Inverse Function Theorem which makes use of both topics.

Definition 2.1. A *path* is a continuous map $f : [0,1] \to X$. We say that $f(0)$ and $f(1)$ are called the *initial point* and *terminal point* respectively.

Definition 2.2. A path f is called *closed* if $f(0) = f(1)$.

Definition 2.3. A space X is called *path connected* if $\forall x, y \in X$, there exists a path $p : I \to X$ such that $p(0) = x$ and $p(1) = y$.

Definition 2.4. Let $f, g : I \to X$ be paths such that $f(0) = g(0)$ and $f(1) = g(1)$. We say f and g are *equivalent* or *homotopic relative to the endpoints*, denoted by $f \sim g$, if there exists a continuous map $h : I \times I \to X$ such that for all $s, t \in I$:
(1) $h(t,0) = f(t)$
(2) $h(t,1) = g(t)$
(3) $h(0,s) = f(0) = g(0)$
(4) $h(1,s) = f(1) = g(1)$

Proof of Equivalence Relation in Definition 2.4. Recall that an equivalence relation is reflexive, symmetric, and transitive.

(1) *Reflexivity*: Take h to be $h(t,s) = f(t)$. Then, $h(t,0) = f(t)$, $h(t,1) = f(t)$, $h(0,s) = f(0)$, and $h(1,s) = f(1)$. Clearly, h is continuous since f is continuous.

(2) *Symmetry*: Let $f \sim g$ with homotopy h_0. So, consider $h_1 = h_0(t, 1-s)$. So, note that $h_1(t,0) = h_0(t,1) = g(t)$, $h_1(t,1) = h_0(t,0) = f(t)$, $h_1(0,s) = h_0(0,s) = f(0) = g(0)$, and $h_1(1,s) = h_0(1,s) = f(1) = g(1)$. Thus, h_1 is the homotopy from g to f, so $g \sim f$.

(3) *Transitivity*: Let $f \sim g$ and $g \sim h$ with respective homotopies j, k. Clearly, $f(0) = h(0)$, $f(1) = h(1)$ by transitivity. Define $l : I \times I \to X$ as:

$$l(t,s) = \begin{cases} j(t,2s), & \text{if } s \in [0, \frac{1}{2}] \\ k(t,2s-1), & \text{if } s \in [\frac{1}{2}, 1] \end{cases} \tag{2.1}$$

Well, $l(t,0) = j(t,0) = f(t)$, $l(t,1) = k(t,1) = h(t)$. Also, $l(0,s) = f(0) = g(0) = h(0)$, and $l(1,s) = f(1) = g(1) = h(1)$. Also, l is continuous since $l(t, \frac{1}{2}) = j(t,1) = k(t,0) = g(t)$.

■

Definition 2.5. A *covering space* of a metric space X is a pair $(\tilde{X}, p)$ of a space $\tilde{X}$ and a continuous, surjective map $p : \tilde{X} \to X$ such that each $x \in X$ has a path-connected neighborhood U with the property that each path-connected component of $p^{-1}(U)$ is mapped homeomorphically onto U by p.

Remark 2.6. Note that p as in the previous definition is indeed an open map, but it is much stronger than that. It states that for each point in the codomain, there is a corresponding neighborhood around it such that the neighborhood is the image of the union of disjoint open sets, where each open set in the union is mapped homeomorphically *onto* the neighborhood by p.

Example 4. Let $X = \{(\cos 2\pi t,\ \sin 2\pi t, t) \mid t \in \mathbb{R}\}$. Let $p : X \to S^1$ be the natural projection given by $p(\cos 2\pi t,\ \sin 2\pi t, t) = (\cos 2\pi t,\ \sin 2\pi t)$. Note that if we choose any nonempty, open subset sufficiently small in S^1, then these neighborhoods are elementary neighborhoods, meaning they are the "U" in our covering space definition. If we let the neighborhood be S^1, then note that $p^{-1}(S^1) = X$, but p is clearly not injective, thus it does not map X homeomorphically onto S^1.

2.2 The Covering-Homeomorphism Theorem

Every covering map is a surjective, continuous, and open map, but because of its strong property as noted in Remark 2.6, a covering map is not typically injective, but we can find a property such that it makes it one. This property requires that the codomain be simply connected, which in turn causes the covering map to be injective and hence a global inverse exists.

We will be making use of a lemma in our proof, but the lemma will be taken as true without proof. For a full proof, a development of two other lemmas is necessary and can be found in Massey's "Algebraic Topology: An Introduction." ([8], p. 151)

Lemma 2.7. *Let $(\tilde{X}, p)$ be a covering space of X and let $g_0, g_1 : I \to \tilde{X}$ be paths in $\tilde{X}$ which have the same initial and terminal points. If $pg_0 \sim pg_1$, then $g_0 \sim g_1$.*

Theorem 2.8. *Let $f : M \to N$ be a covering of N. If N is simply connected, then f is a homeomorphism.*

Proof of Theorem 2.8. Recall that a homeomorphism is a bijective continuous function with a continuous inverse. We know that since (M, f) is a covering map, then f is continuous and surjective by definition. Likewise, because of its special topological property, it is an open map as well, thus its inverse, if it exists, is continuous. Because the existence of an inverse is guaranteed by a bijection, we need only prove that it is injective.

Let $g_0 : I \to M$ be a path defined as $g(0) = a$, $g(1) = b$, where $a \neq b$. Suppose $f(a) = f(b)$, by contradiction. So, fg_0 is a closed loop. Let $g_1 : I \to M$ also be the constant path defined as $g_1(x) = a$, $\forall x \in I$. Note that fg_1 is the trivial loop at $f(a) \in N$.

Since N is simply connected, we know that fg_0 is homotopic rel. endpoints to the base point $f(a)$, so $fg_0 \sim fg_1$. By Lemma 2.7, that means $g_0 \sim g_1$, so $g_0(1) = g_1(1)$. But $g_0(1) = a \neq b = g_1(1)$, thus a contradiction. So, $f(a) \neq f(b)$. Therefore, f is injective and hence a homeomorphism.

■

3 Global Inverse Function Theorems

3.1 Banach Space Homeomorphisms

In this section, we will cover different theorems that can be found in [14] describing homeomorphisms (diffeomorphisms) between Banach spaces. For more information on global forms of the Inverse Function Theorem, see [6]. [12], and [14].

Definition 3.1. Let $D \subseteq X$ is open and connected with X, Y being Banach spaces. One says that $F : D \to Y$ *lifts lines* in $F(D)$ if for each line $L(t) = (1-t)y_1 + ty_2$ in $F(D)$, $t \in [0,1]$, and for every point $x_\alpha \in F^{-1}(y_1)$ there is a path $P_\alpha(t)$ such that $P_\alpha(0) = x_\alpha$ and $F(P_\alpha(t)) = L(t)$, for *every* $t \in [0,1]$.

Definition 3.2. Let $F : D \to Y$ be continuous and suppose $D \subseteq X$ is open and connected with X, Y being Banach spaces. One says that F satisfies *Condition (L)* if whenever $P(t)$, $0 \le t < b$, is a path satisfying $F(P(t)) = L(t)$ (where $L(t) = (1-t)y_1 + ty_2$ is any line in Y), then there is a sequence $t_i \to b$ as $i \to \infty$ such that $\lim_{i\to\infty} P(t_i)$ exists and is in D.

Lemma 3.3. ([14], p. 170) *Let $D \subseteq X$ be open and connected, with X, Y as Banach spaces and $F : D \to Y$ a local homeomorphism. Then, (D, F) covers $F(D)$ if and only if F lifts lines in $F(D)$.*

Theorem 3.4. ([14], p. 170) *Let $F : D \subseteq X \to Y$ be a local homeomorphism. Then condition (L) is necessary and sufficient for F to be a homeomorphism.*

Proof of Theorem 3.4. If F is a homeomorphism already, then clearly (L) is satisfied. So, suppose that condition (L) is satisfied first. Let $L(t)$ be any line in $F(D)$, with $L(0) = \bar{y}$. Let $\bar{x} \in F^{-1}(\bar{y})$. Since, F is a local homeomorphism, then for $\bar{x}$ we can find an $\epsilon > 0$ and a path $P(t)$ such that $P(0) = \bar{x}$ and $F(P(t)) = L(t)$ for $0 \le t < \epsilon$. Let K be the largest number for which $P(t)$ can be extended to a continuous path for $0 \le t < K$ and satisfying $F(P(t)) = L(t)$ for $0 \le t < K$.

Since F satisfies condition (L), let $z = \lim_{t_i \to K} P(t_i)$. By continuity of F, $F(z) = L(K)$. Let W be a neighborhood of z on which F is a homeomorphism. Then, $\exists N$ such that $P(t_i) \in W$ for $i \ge N$. Also, $\exists \delta > 0$ and a path $Q(t)$ defined for $K - \delta < t < K + \delta$ such that $Q(t_M) = P(t_M)$, where M is such that $M \ge N$ and $K - \delta < t_M < K$, and $F(Q(t)) = L(t)$ for $K - \delta < t < K + \delta$. So, $P(t)$ can be extended once again and relabeled to a continuous path $P(t)$ on $0 \ge t < K + \delta$, $P(0) = \bar{x}$, and $F(P(t)) = L(t)$, $0 \le t < K + \delta$. However, recall that we chose K to be maximal with respect to extending $P(t)$ to a continuous path, so $K = 1$. Therefore, we see that this is true for all $t \in F^{-1}(\bar{y} = [0,1)$, so F lifts lines.

So, (D, F) covers $F(D)$. We now need to show that $F(D) = Y$. Clearly, $F(D) \subseteq Y$, so we will only show $Y \subseteq F(D)$. Let $\bar{y} \in Y$. Choose $y_1 \in F(D)$ and let $L(t) = (1-t)y_1 + t\bar{y}$. Thus, we can find a path $P(t)$, $0 \le t \le 1$, so that $F(P(t)) = L(t)$. Thus, $F(P(1)) = L(1) = \bar{y}$, hence $\bar{y} \in F(D)$. Therefore, $Y \subseteq F(D)$, so $F(D) = Y$. Since Y is a Banach space, we know that Y is simply connected. Thus, by Theorem 2.8, we see that F is indeed a homeomorphism. ■

The following theorem is originally due to Banach and Mazur [1].

Theorem 3.5. ([14], p. 174) *Let X, Y be Banach spaces, $F : X \to Y$ be a local homeomorphism. Then, F is a homeomorphism if and only if it is proper.*

Proof of Theorem 3.5. If F is a homeomorphism, then F^{-1} is continuous and thus maps compact sets into compact sets. Therefore, F is proper. So, suppose that F is proper. If we show that F satisfies (L), then by Theorem 3.4, F will be a homeomorphism. So, let $P(t)$ for $0 \leq t < b$ satisfies $F(P(t)) = L(t)$ on the same interval. Let $t_i \to b$. Let $S = \{L(t)\}_{0 \leq t \leq 1}$ and note that S is compact, therefore $F^{-1}(S)$ is compact by properness. Also, $F^{-1}(S)$ contains $P(t_i)$. By compactness, we can find a subsequence $t_{i_j} \to b$ such that $P(t_{i_j}) \to \bar{x}$, so (L) is satisfied. Therefore, by Theorem 3.4, F is a homeomorphism. ■

Note that under the assumptions of Theorem 3.5,

$$F \text{ satisfies condition } (L) \iff F \text{ is a homeomorphism} \iff F \text{ is proper.}$$

3.2 Application to Algebra

The point of this section is to bring everything that we have learned to a single, yet important, application. Though this application deals with an area of mathematics called Algebra, to use the theorem, we have had to gain a better understanding and intuition of ideas and theorems from Topology and Analysis. Essentially, we have tied together three big branches in mathematics together with one topic: the Global Inverse Function Theorem.

Lemma 3.6. *A proper local homeomorphism $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ lifts lines in $F(D)$.*

We will assume this lemma to be true, for its proof is very similar to the maximality argument used in Theorem 3.4. This lemma is vital in proving the important theorem which follows.

Theorem 3.7. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a proper local diffeomorphism. If $F(D)$ is simply connected, then F is a diffeomorphism from D onto $F(D)$.*

Proof of Theorem 3.7. By Lemma 3.6, we know that F lifts lines in $F(D)$. By Lemma 3.3, we know that (D, F) is a covering of $F(D)$. Since $F(D)$ is simply connected, by Theorem 2.8, F is homeomorphism from D onto $F(D)$. ■

This theorem is different from Theorem 3.5 in that it doesn't require F to go from the whole Banach space but only a subset of it. This will be important in our proof as we will not take the whole space as the domain. Again, it is interesting to note that topological and analytical notions and theorems are being brought together to prove an important result in Algebra: there is no commutative division algebra (not necessarily associative) that is isomorphic to $\mathbb{R}^n$ for $n \geq 3$. So, in other words, any finite-dimensional, commutative division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. If we allow the division algebra to be noncommutative, then it can be isomorphic to either

$\mathbb{H}$ (the quaternions) or $\mathbb{O}$ (the octonions) as well. The quaternions were discovered in 1843 by William Hamilton whereas the octonions were discovered later that year by John Graves, inspired by Hamilton's discovery. It is important to note that the octonions are not associative. By allowing the division algebra to be noncommutative and restricting it to be associative, we get the equivalent of Frobenius' theorem [15], which states that every finite-dimensional associative division algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Therefore, we will restrict it to be not necessarily commutative nor necessarily associative. For more information on the following theorem, see [5] and ([12], p. 85). The rigorous statement is as follows:

Theorem 3.8. *For $n \geq 3$, there is no operation of multiplication on $\mathbb{R}^n$ which satisfies:*

(1) $x(\lambda y) = \lambda xy$, where $\lambda \in \mathbb{R}$ is a scalar.

(2) $x(y+z) = xy + xz$ (distributivity)

(3) $xy = 0 \implies x = 0$ or $y = 0$ (no zero divisors)

(4) $xy = yx$ (commutativity)

Because we will assume a multiplication on $\mathbb{R}^n$, we need to define how multiplication works in relation to the Euclidean norm.

Lemma 3.9. *There exists two positive constants C_1, C_2 such that for $x \in \mathbb{R}^n$,*

$$C_1\|x\|^2 \leq \|x^2\| \leq C_2\|x\|^2.$$

Proof of Lemma 3.9. Note that if $x \neq 0$, then by (1),

$$x^2 = \|x\|^2\Big(\frac{x}{\|x\|}\Big)\Big(\frac{x}{\|x\|}\Big).$$

Hence,

$$\|x^2\| = \|x\|^2 \left\|\Big(\frac{x}{\|x\|}\Big)\Big(\frac{x}{\|x\|}\Big)\right\|.$$

Consider $V : S^{n-1} \to [0,\infty)$ defined by $V(u) = \|u \cdot u\|$. This is a continuous function over the compact set S^{n-1}, so therefore there exists a $u_0 \in S^{n-1}$ such that,

$$V(u_0) = \|u_0 \cdot u_0\| = \inf_{u \in S^{n-1}} \|u \cdot u\|.$$

Since there are no zero divisors by (3), $\|u_0 \cdot u_0\| \neq 0$, and so $C_2 = V(u_0)$. So, we see that

$$C_2\|x\|^2 \leq \|x^2\|.$$

Likewise, there exists a $u_1 \in S^{n-1}$ such that,

$$V(u_1) = \|u_1 \cdot u_1\| = \sup_{u \in S^{n-1}} \|u \cdot u\|.$$

Since there are no zero divisors by (3) and V is a continuous function acting on a compact set, $\|u_1 \cdot u_1\| < \infty$ and $\|u_1 \cdot u_1\| \neq 0$, thus $C_1 = V(u_1)$. Thus,

$$\|x^2\| \leq C_1 \|x\|^2,$$

and we see that

$$C_2\|x\|^2 \leq \|x^2\| \leq C_1 \|x\|^2.$$

■

Proof of Theorem 3.8. We will prove that this cannot be true by contradiction. Suppose that there does exist a multiplication in $\mathbb{R}^n$ for $n \geq 3$ that satisfies conditions (1) through (4). Let F be the map from $\mathbb{R}^n - \{0\}$ to $\mathbb{R}^n - \{0\}$ defined by $F(x) = x^2$. This is well-defined because there are no zero divisors according to condition (3), so we know that if $xy = 0 \in \mathbb{R}^n - \{0\}$, then either $x = 0$ or $y = 0$. If $x, y \notin \mathbb{R}^n - \{0\}$, then $xy \notin \mathbb{R}^n - \{0\}$. Thus, F is well-defined.

It is fairly simple to check that the condition below is equivalent to properness:

$$\text{if } x_n \to \begin{cases} 0 \\ \infty \end{cases}, \text{ then } F(x_n) = x_n^2 \to \begin{cases} 0 \\ \infty \end{cases}.$$

We see that by continuity, $F(x_n) \to 0$ as $x_n \to 0$. By Lemma 3.9, $F(x_n) \to \infty$ as $x_n \to \infty$. Thus, F is indeed proper.

Now, in order to check that F satisfies the hypothesis of Theorem 3.7, let us consider the differential of F, supposing it exists. We claim that the following holds:

$$dF_x v = \lim_{h \to 0} \frac{1}{h}(F(x + hv) - F(x)) = xv + vx = 2xv,$$

where the last equality follows from (4). To see this, let us consider the following:

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - dF_x(h)\|}{\|h\|} = \lim_{h \to 0} \frac{\|x^2 + 2xh + h^2 - x^2 - 2xh\|}{\|h\|} = \lim_{h \to 0} \frac{\|h^2\|}{\|h\|} = 0,$$

where the last inequality follows by Lemma 3.9. Indeed, F is differentiable by Definition 1.11, and we know dF_x is unique by Theorem 1.12, and so $dF_x v = 2xv$. In particular, F is continuous. Also, $2xv \neq 0$ for all $x, v \in \mathbb{R}^n \setminus \{0\}$ by (4). Thus, dF_x is non-singular, and so by the Inverse Function Theorem (1.16), F is a local diffeomorphism.

Now, we draw attention to the fact that $n \geq 3$. In order to use Theorem 3.7, the codomain must be simply connected. However, $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{0\}$ are not simply connected. When $n \geq 3$, $\mathbb{R}^n \setminus \{0\}$ is simply connected, and so we see that 3 is the smallest integer for which Theorem 3.7 applies.

Therefore, F is a diffeomorphism. However, F is not even a injective since $(-x)^2 = x^2$, $\forall x \in \mathbb{R}^n \setminus \{0\}$. Thus, we have a contradiction, so our assumption that there does exist a multiplication as previously stated is false. Therefore, there is no multiplication in $\mathbb{R}^n$ for $n \geq 3$ satisfying conditions (1) through (4); that is, by combining results and notions from Analysis and Topology, we have found that there is no commutative division algebra (not necessarily associative) that is isomorphic to $\mathbb{R}^n$ for $n \geq 3$.

■

4 Further Analytical and Topological Notions

From here on out, there will be somewhat of a shift in gears. We have talked about global forms of the inverse function theorem and its application to division algebras, but let's recall the Banach Fixed Point Theorem. It states that any contraction from a space into itself has a unique fixed point. In this second half, we want to study various fixed point theorems and look at the different conditions necessary for the result to occur.

Before we can move on, it is precisely because there are various different conditions, applied not only to the spaces but to the functions as well, that we need more definitions and ideas regarding functions and how they interact with certain spaces. In this section, we will limit our focus to the setting of a metric space as the space to which our sets belong.

Definition 4.1. A set A is *bounded* if $\exists M \in \mathbb{R}$ such that $d(x,y) \leq M$ for all $x, y \in A$.

Definition 4.2. Let f be a function from A to $\mathbb{R}$. We say f is *bounded* if the image $f(A)$ is bounded.

Example 5. Consider the following function $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$. $\lim_{x\to 1} \frac{1}{x} = 1$, but $\lim_{x\to 0} \frac{1}{x} = \infty$. Note carefully that $(0,1]$ is clearly bounded since $\max(d(x,y)) = 1$, $\forall x, y \in (0,1]$. However, $f((0,1])$ is not bounded because $\lim_{x\to 0} d(f(x), f(1)) = \infty$; this is due to the fact that f is not continuous on the interval $(0,1]$. Geometrically, it is clear to see since there is an asymptote at $x = 0$. Though $(0,1]$ is bounded, because $f((0,1])$ is not, we say that f is not bounded on the set $(0,1]$.

With the introduction of bounded sets, one may ask when boundedness in both the domain and the image is guaranteed. With boundedness alone, we will have to extra conditions to the function apart from mere continuity. However, if we add more conditions to the domain rather than the function, we get a whole new set of ideas.

Definition 4.3. An *open cover* of A is any collection of open sets $C = \{\bigcup U_a \mid a \in A\}$ such that $A \subseteq C$. A *subcover* of C is any subset $B \subseteq C$ that still covers A.

Definition 4.4. A set A is *compact* if every open cover of A has a finite subcovering.

Recall that a closed set is a set which contains all of its limit points, that is, the limit of every convergent sequence in a closed set is also contained in the closed set. In Euclidean spaces, it turns out that the idea of compactness is equivalent to the combination of being both closed and bounded.

Theorem 4.5. (Heine-Borel Theorem). *For $A \subset \mathbb{R}^n$, A is compact if and only if A is closed and bounded.*

We will not delve into the proof of this. With the introduction of this concept of compactness, we return to our idea guaranteeing boundedness in both the domain and the image. In fact, we can do much better than this. We can guarantee compactness in both the domain and image simply by having a continuous function, which, by the Heine-Borel Theorem, means that the domain and image are not only bounded but closed as well in the context of a Euclidean space.

Lemma 4.6. *The image of a continuous function f over a compact space X is compact.*

Proof of lemma 4.6. Given that X is compact and thus has a finite subcover for any cover of X, we must show that any cover of $f(X)$ has a finite subcover as well. Let C be an open cover of $f(X)$. Note that for any open set $U \in C$, because f is continuous, $f^{-1}(U)$ is open in X. Given any $x \in X$, $f(x)$ must be in some $V \in C$, thus $f^{-1}(f(x))$ contains $x \in f^{-1}(V)$. So, $f^{-1}(C)$ is an open cover of X. Because X is compact, there exists a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_n)\}$ which covers X. Given any $x \in X$, x surely belongs to $f^{-1}(U_i)$ for some $1 \leq i \leq n$. Thus, by continuity $f(x) \in U_i$. Since x was arbitrary in X, this applies for any $f(x) \in f(X)$, thus $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $f(X)$. Therefore, $f(X)$ is compact.

■

Lemma 4.7. *A continuous bijection f is a homeomorphism if and only if it is open (if U is open, then $f(U)$ is as well), or equivalently, if and only if it is closed (if W is closed, then $f(W)$ is as well).*

Proof of lemma 4.7. First, suppose f is open. Consider any open set U and note that $f(U)$ is open. As a switch in notation, we say $f(U) = U'$ and $U = f^{-1}(U')$. Thus, we have that whenever $f^{-1}(U')$ is open, then U' is open, which shows that f^{-1} is continuous. Thus, f is a homeomorphism.

Next, suppose f is a homeomorphism. Thus, f^{-1} is continuous. So, let V be an open set in the domain of f^{-1}. By continuity, $f^{-1}(V)$ is open. Let $f^{-1}(V) = V'$ and $V = f(V')$. Thus, whenever V' is open, so is $f(V')$. Therefore, f is open.

The proof is similar for closed sets.

■

5 Manifold Preliminaries

Some of the key ideas in understanding the Brouwer Fixed Point Theorem and the Schauder Fixed Point Theorem are the notions of differentiable manifolds and different theorems surrounding them. As such, we need to introduce some more definitions and lemmas. The following definitions and lemmas are to be found in Milnor [9].

Definition 5.1. Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open sets. A function $f : U \to V$ is called a *smooth function* if all of its partial derivatives exist and are continuous. Alternatively, if U and V are not necessarily open, then a map $f : U \to V$ is called a *smooth map* if for each $x \in X$, there exists an open set $W = U \cap X$ containing x such that there is a smooth function $F : U \to \mathbb{R}^l$ coinciding with W.

We have defined what a diffeomorphism is previously, but now we define a diffeomorphism as a homeomorphism f where f and f^{-1} are both smooth.

Definition 5.2. A subset $M \subset \mathbb{R}^k$ is called a *smooth manifolds of dimension m* if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the euclidean space $\mathbb{R}^m$.

The key word here is *diffeomorphic*. It is not enough to simply be homeomorphic but rather diffeomorphic. If the diffeomorphic property is not satisfied, then the smoothness of the manifold cannot be guaranteed. Furthermore, the fact that each point has a neighborhood diffeomorphic to an open subset of U and the fact that it is called a *smooth* manifold hint at the idea that the manifold is not jagged and sort of flowey geometrically.

The main focus of this will not be to study a certain manifold individually but rather to study functions from one manifold to another. If we have a smooth map $f : M \to N$, where M, N are smooth manifolds, then there needs to be a sense of a (partial) derivative.

Definition 5.3. Let $f : M \subset \mathbb{R}^k \to N$ be a smooth map between smooth manifolds. Then, we denote the *tangent space* at a point $x \in M$ as the linear subspace $TM_x \subset \mathbb{R}^k$ of dimension m formed by the tangent vector at x of all smooth curves lying in M. Then, df_x is the linear mapping $df_x : TM_x \to TM_{f(x)}$.

In the statements of the Brouwer and Schauder Fixed Point Theorems, each has a continuous function sending a set homeomorphic to a ball (finite-dimensional or infinite-dimensional) into itself. Thus, the domain and the range both have the same dimension. With this in mind, we turn in particular toward maps between smooth manifolds of the same dimension. Naturally, if we have a smooth map f with M, N as above, then $\dim(M) = \dim(N)$ implies that $\dim(TM_x) = \dim(TN_{f(x)})$.

We will introduce two definitions for some upcoming lemmas. These lemmas will be useful in proving the Brouwer Fixed Point Theorem.

Definition 5.4. Let $f : M \to N$ be a smooth map between manifolds of the same dimension. We say that $x \in M$ is a *regular point* of f if the derivative df_x is nonsingular. The point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points. Should df_x be singular, then x is called a *critical point* of f and $f(x)$ is called a *critical value*.

Definition 5.5. A subset $X \subset \mathbb{R}^k$ is called a *smooth m-manifold with boundary* if each $x \in X$ has a neighborhood $U \cap X$ diffeomorphic to an open subset $V \cap H^m$ of H^m, where $H^m = \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$.

These lemmas build upon each other in a linear fashion, so it is important that we cover each one in order to understand the subsequent one.

Lemma 5.6. ([9], p. 11) *If $f : M \to N$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

Proof of Lemma 5.6. Let $x \in f^{-1}(y)$. Since y is a regular value, the derivative df_x must map TM_x onto TN_y. Therefore, the null space of df_x will be an $(m-n)$-dimensional vector space.

If $M \subset \mathbb{R}^k$, we will define a linear map $L : \mathbb{R}^k \to \mathbb{R}^k$ that is nonsingular on this aforementioned subspace. We now define a map $F : M \to N \times \mathbb{R}^{m-n}$ by $F(\xi) = (f(\xi), L(\xi))$. By linearity, we know that $dF_x(v) = (df_x(v), L(v))$. So, dF_x is nonsingular due to the nonsingularity of L on the subspace. By the Inverse Function Theorem, this F maps some neighborhood U of x diffeomorphically onto a neighborhood V of $(y, L(x))$. So, F maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times \mathbb{R}^{m-n} \cap V)$, so $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.

■

Lemma 5.7. ([9], p. 12) *Let M be a manifold without boundary and let $g : M \to \mathbb{R}$ have 0 as a regular value. The set of $x \in M$ with $g(x) \geq 0$ is a smooth manifold, with boundary equal to $g^{-1}(0)$.*

Proof of Lemma 5.7. By Lemma 5.6, if $x \in M$ gets mapped to 0, then the set of such x is a smooth manifold since 0 is a regular value. This means that $g^{-1}(0)$ is a smooth manifold of dimension $m - 1$. So, $g^{-1}(0)$ is merely the boundary of M. Therefore, the set of all $x \in M$ with $g(x) \geq 0$ is a smooth manifold.

■

Lemma 5.8. ([9], p. 13) *Consider a smooth map $f : X \to N$ from an m-manifold with boundary to an n-manifold, where $m > n$. If $y \in N$ is a regular value, both for f and for the restriction $f \restriction_{\partial X}$, then $f^{-1}(y) \subset X$ is a smooth $(m-n)$-manifold with boundary, and furthermore, the boundary $\partial(f^{-1}(y))$ is precisely equal to $f^{-1}(y) \cap \partial X$.*

Proof of Lemma 5.8. Since we are not dealing with a global condition but rather a local one (i.e. with sets dependent upon regular values $y \in N$) and since every smooth $(m-n)$-manifold with boundary is locally diffeomorphic to H^m, we will simply consider the case of $f : H^m \to \mathbb{R}^n$ with regular values $y \in \mathbb{R}^n$. Let $\bar{x} \in f^{-1}(y)$. If $\bar{x}$ is an interior point, then as before $f^{-1}(y)$ is a smooth manifold in the neighborhood of $\bar{x}$ by Lemma 5.6.

Now, suppose that $\bar{x}$ is a boundary point. We choose a smooth map $g : U \to \mathbb{R}^n$ defined throughout the neighborhood of $\bar{x}$ coinciding with f on $U \cap H^m$, where g has no critical points (we can adjust U if necessary). Hence, $g^{-1}(y)$ is a smooth manifold of dimension $m - n$ by definition.

Let $\pi : g^{-1}(y) \to \mathbb{R}$ be given as $\pi((x_1, x_2, \ldots, x_m)) = x_m$. We claim that 0 is a regular value for π. Indeed, the tangent space of $g^{-1}(y)$ at a point $x \in \pi^{-1}(0)$ is equal to the null space of $dg_x = df_x : \mathbb{R}^m \to \mathbb{R}^n$. However, $f \restriction \partial H^m$ is regular at x, so we know that the null space cannot be completely contained in $\mathbb{R}^{m-1} \times \{0\}$. By Lemma 5.7, the set of all $x \in g^{-1}(y)$ with $\pi(x) \geq 0$ is a smooth manifold with boundary equal to $\pi^{-1}(0)$.

∎

Lemma 5.9. ([9], p. 14) *Let X be a compact manifold with boundary. There is no smooth map $f : X \to \partial X$ that leaves ∂X point-wise fixed.*

Proof of Lemma 5.9. [7] Suppose there were such a map f leaving ∂X point-wise fixed (i.e. $f(x) = x$, $\forall x \in \partial X$). We can surely find a regular value $y \in \partial X$. Note that since y is a regular value of f, clearly y is a regular value of the identity map, which is simply $f \mid \partial X$. So, by Lemma 5.8, then $f^{-1}(y)$ is a smooth $(m - (m-1)) = 1$-manifold with boundary $f^{-1}(y) \cap \partial X = \{y\}$. This is impossible because $f^{-1}(y)$ is compact, and we claim the only 1-manifolds are finite disjoint unions of circles and segments, so that $\partial f^{-1}(y)$ must consist of an even number of points. Such a proof of this claim can be found in the appendix of Milnor's *Topology from the Differentiable Viewpoint* ([9], p. 55). Thus, such a map cannot exist.

∎

Note how strikingly similar the following lemma is to the Brouwer Fixed Point Theorem:

Lemma 5.10. ([9], p. 14) *Any smooth map $g : D^n \to D^n$ has a fixed point.*

While the Brouwer Fixed Point Theorem requires only continuity, this statement imposes the stronger condition of smoothness, which in and of itself includes continuity as well. Thus, the Brouwer Fixed Point Theorem is a matter is relaxing this condition while maintaining the result.

Proof of Lemma 5.10. Suppose such a map g has no fixed point. We will create a new map which maps the manifold D^n into ints boundary. For $x \in D^n$, let $f(x) \in S^{n-1}$ be the intersection of S^{n-1} with the ray from $g(x)$ in the direction of x (see Proof of Proposition 6.3). The map f is smooth, as is claimed by Milnor. So, whenever $x \in S^{n-1}$, $x = f(x)$, thus f leaves $S^{n-1} = \partial D^n$ point-wise fixed, which contradicts Lemma 5.9.

∎

As we can see, this set of lemmas is like a pyramid, constantly building on top of the previous one. However, we want to place our focus specifically on the last lemma. The statement is extremely similar to the Brouwer Fixed Point Theorem, with the difference being the smoothness of the function. Our goal in proving the Brouwer Fixed Point Theorem will rely on an appoximation of a continuous function with a smooth function, during which time we will call upon Lemma 5.10 for its conclusion.

6 The Brouwer/Schauder Fixed Point Theorems

Now that we have covered enough material to understand not only the statements of the Brouwer and Schauder Fixed Point Theorems but the proofs as well, let us recall the statements of the theorems.

Theorem 6.1. (Brouwer Fixed Point Theorem). *Any continuous function $G : D^n \to D^n$ has a fixed point.*

In a more general sense, the Brouwer Fixed Point Theorem can be stated as follows:

Reformulation: *Let M be a nonempty, compact, convex subset of a Euclidean space $\mathbb{R}^n$. Then, any continuous function $G : M \to M$ has a fixed point.*

How do we know that the Brouwer Fixed Point Theorem and the reformulation are equivalent? We only need to show that the fixed-point property is invariant under homeomorphisms; that is, if two spaces X, Y are homeomorphic to each other and any continuous function from one space into itself always has a fixed point, then the other space has this property as well. This is rather simply to prove. Indeed, Let X, Y be homeomorphic, denoted by a function g, and let X have the fixed point property. Take $h : Y \to Y$ to be continuous and define $j : X \to X$ by $j = g^{-1} \circ h \circ g$. As a composition of continuous functions, j is continuous as well, thus j permits a fixed point, say $x_0 \in X$. So, $x_0 = j(x_0) = g^{-1}(h(g(x_0)))$, thus due to g being a homeomorphism, we have $g(j(x_0)) = g(x_0) = h(g(x_0))$. Therefore, $g(x_0)$ is a fixed point of our arbitrary h. Y, then, has the fixed-point property as well, and we see that such a property is invariant under homeomorphism.

Theorem 6.2. (Schauder Fixed Point Theorem). *Let M be a nonempty, compact, convex subset of a Banach space X, and suppose $T : M \to M$ is a continuous operator. Then T has a fixed point.*

Comparing the reformulation of Theorem 6.1 and Theorem 6.2, we can definitely see that the Schauder Fixed Point Theorem is an extension of the Brouwer Fixed Point Theorem into infinite dimensions. In a sense, because Euclidean spaces are Banach spaces as well, the Schauder Fixed Point Theorem is the Brouwer Fixed Point Theorem with a more general set M. Though we have yet to show that theorem 6.1 is true, showing that Theorem 6.1 and the reformulation are equivalent is simply a matter of proving there is a homeomorphism between the compact, convex set M and some n-ball.

Now, let $K \subset \mathbb{R}^m$ be a compact, convex set and let n be the minimum dimension an affine space needs to contain K. Let A be the affine space such that $K \subset A$.

Proposition 6.3. The n-ball D^n is homeomorphic to K, i.e. there exists a function $f : D^n \to K$ such that f is a continuous bijection.

Lemma 6.4. (Two Interior Points). *Let $a, b \in K^{\circ}$. Then the line segment L with endpoints a, b is fully contained in K°.*

Proof of Lemma 6.4. Suppose not. By convexity of K, $L \subset K$. Thus, there exists a point $c \in \partial K$ such that $c \in L$ as well. Take three neighborhoods of equal radius, V_a of a, V_b of b, and V_c of c, all small enough such that $V_a, V_b \subset K$. By definition, there will always exist a point $c^* \in (A \setminus K) \cap V_c$. Translate L by a vector (call this translation L^*) such that $c^* \in L^*$ and the endpoints of L^* are contained in V_a, V_b respectively. We have a line segment L^* connecting two interior points $a^*, b^* \in K^{\circ}$ with a point $c^* \in L^*$ outside of K while $L^* \subset K$. Thus, a contradiction. Therefore, no such c exists. ■

Though Lemma 6.4 will not help with the proof of Proposition 6.3, it did give inspiration to the following lemma.

Lemma 6.5. (Interior and Boundary Points). *Let $a \in K^{\circ}, b \in \partial K$. Then, the line segment $L \setminus \{b\}$, whose L is given by $L = \{ta + (1-t)b \mid t \in [0,1]\}$, is completely contained in K°.*

Proof of Lemma 6.5. Suppose not. By convexity of K, $L \subset K$. Thus, it must be that there exists a point $c \in L$ such that for some $t_0 \in (0,1)$, we have $t_0 * a + (1-t_0) * b = c$ and such that $c \in \partial K$. We will take two neighborhoods, V_a of the interior point a and V_c of the (middle) boundary point c. Any neighborhood V_c contains a point outside of K (label it c^*), so shrink V_c until there exists an $a^* \in V_a$ such that a^*, c^*, and b are colinear. Note that this is possible due to the following. Let x be the radius of V_c. Consider the line segment L_x given by $L_x = \{t * a_x + t * b \mid t \in [0,1]$ with $c_x = t_0 * a_x + t_0 * b$ and $c_x \notin K$. Note that a_x lies somewhere in the direction of going from b to c_x. Note also that as $x \to 0$, $a_x \to a$ and $c_x \to c$. Thus, for a small enough x, we have our a^* and c^*, which, respectively, are $a_x \in V_a$ with $c_x \in V_c$. Since $a^* \in K$ and $b \in K$, then L^* is surely in K. This is a contradiction since $c^* \in L^*$ lies outside of K. Thus, no such c exists. ■

Proof of Proposition 6.3. Recall that $n \leq m$. Without loss of generality, let any point x in the affine space A be written as $x = (x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_{m-n})$, where a_i are constants, for $1 \leq i \leq m-n$. We write $k \in K \implies k = (k_1, k_2, \ldots, k_n, a_1, a_2, \ldots, a_{m-n})$. Let $K_n \subset \mathbb{R}^n$ be given as $\{(k_1, k_2, \ldots, k_n) \mid (k_1, k_2, \ldots, k_n, a_1, a_2, \ldots, a_{m-n}) \in K\}$. Consider the following function:

$$p : K \subset \mathbb{R}^n \times \{a_1\} \times \{a_2\} \times \cdots \times \{a_{m-n}\} \to K_n \subset \mathbb{R}^n,\ p((k_1, \ldots, k_n, a_1, \ldots, a_{m-n})) = (k_1, \ldots, k_n).$$

p is clearly a homeomorphism. If $0 \notin K_n$, then we can translate K_n by a vector (call this translation K_0) in such a way that $0 \in {K_0}^{\circ}$, and clearly K_n and K_0 are homeomorphic, say, by a function

$$q : K_n \to K_0.$$

Because $0 \in {K_0}^{\circ}$, we can find a neighborhood of radius ϵ such that this neighborhood is contained in ${K_0}^{\circ}$. Consider the set $\{\epsilon x \mid x \in D^n\} = D^n_{\epsilon} \subset \mathbb{R}^n$, where D^n is the closed unit n-ball and D^n_ϵ is the closed n-ball of radius ϵ. Note that $D^n_\epsilon \subseteq K_0$ The function

$$s : D^n \to D^n_\epsilon,\ s(x) = \epsilon x$$

is clearly a homeomorphism. Thus, we need only find a homeomorphism

$$r : K_0 \to D^n_\epsilon,$$

and we will simply let

$$f = p^{-1} \circ q^{-1} \circ r^{-1} \circ s \tag{6.1}$$

to find a homeomorphism between D^n and K. We will do so in a radial manner. Let the notation $\overrightarrow{0:k}$ be $\{nk,\ n \in [0,\infty)\}$. This is merely the set of points in the ray starting at 0 in the direction of point $k \in K_0$. Fix a point $k_0 \in \partial K_0$. Consider the function $r : K_0 \to D^n_\epsilon$ given by

$$r(k) = \begin{cases} \frac{\epsilon k}{\left\|(\overrightarrow{0:k} \cap \partial K_0)\right\|} & k \neq 0 \\ 0 & k = 0 \end{cases} \tag{6.2}$$

In order for r to be well-defined, we must ensure that for each $k \neq 0$, $(\overrightarrow{0:k} \cap \partial K_0)$ produces only one value, i.e. that $(\overrightarrow{0:k} \cap \partial K_0)$ is a single point. By Lemma 6.5, we conclude that r is well-defined. The claim that r is a continuous bijection is supported in Borwein's and Luis's *Convex Analysis and Nonlinear Optimization* ([4], p. 184), where $\left\|(\overrightarrow{0:k} \cap \partial K_0)\right\| = \|k\|^{-1}\,\gamma_{K_0}(k)$. We only need to show that r^{-1} is continuous. Given any closed set $V \subset K_0$, V is compact since K_0 is compact. Thus, by lemma 4.6, $r(V)$ is compact is well. By theorem 4.5, $r(V)$ is closed and bounded, thus the image of a closed set under r is closed. So, r is a closed map and is thus a homeomorphism. We have found a homeomorphism r, and therefore f is also a homeomorphism. ■

We have shown that the Brouwer Fixed Point Theorem and its reformulation given above are equivalent because we have found a homeomorphism. As such, we will proceed on to proving the Brouwer Fixed Point Theorem. We will assume the validity of the Weierstrass approximation theorem which states that any continuous function on a compact subset of $\mathbb{R}^n$ can be uniformly approximated as closely as desired by a polynomial function. The following proof is due to Milnor ([9], p. 14-15).

Proof of the Brouwer Fixed Point Theorem. Because we want to show that any continuous function $G : D^n \to D^n$ has a fixed point, we will start off by approximating G by a smooth mapping. Given $\epsilon > 0$, we can find a polynomial $P_1 : \mathbb{R}^n \to \mathbb{R}^n$ with $\|P_1(x) - G(x)\| < \epsilon$, for all $x \in D^n$. However, this P_1 is not guaranteed to send points $x \in D^n$ to points inside D^n, since the distance between $P_1(x)$ and $G(x)$ may be ϵ whilst $P_1(x)$ is outside D^n. To correct this, we will let $P(x) = P_1(x)/(1+\epsilon)$. Suppose we have a point x_0 mapped to a point $P_1(x_0)$ outside of D^n. So, $\|P(x_0)\| = \|P_1(x_0)/(1+\epsilon)\| <$

$\|(1+\epsilon)/(1+\epsilon)\| = 1$, thus $P(D^n) \subseteq D^n$. Also,

$$\|P(x) - G(x)\| \;\leq\; \|P(x) - P_1(x)\| + \|P_1(x) - G(x)\| \;<\; \left|1 - \frac{1}{1-\epsilon}\right| \|P_1(x)\| + \epsilon \;\leq\; 2\epsilon.$$

Suppose now that $G(x) \neq x$ for all $x \in D^n$. Then the continuous function $\|G(x) - x\|$, by compactness of D^n, must take on a miminum $\mu > 0$ on D^n. Choosing $P : D^n \to D^n$ as above, and ϵ small enough such that $\|P(x) - G(x)\| < 2\epsilon < \mu$ for all x, then $P(x) \neq x$. Thus, P is a smooth map from D^n to itself without a fixed point, which contradicts Lemma 5.10.

■

In our proof, we used a polynomial to approximate our continuous function $G : D^n \to D^n$ as closely as we would like. Because P is smooth and its domain and image are D^n, then P must have a fixed point by the lemma mentioned.

Our method in proving Schauder's Fixed Point Theorem will not be similar, but rather, we will reduce our infinite-dimensional problem to a finite-dimensional one in order that we may use the Brouwer Fixed Point Theorem. Recall that the Schauder Fixed Point Theorem states that any continuous function T from a non-empty, compact, and convex subset $M \subset X$, where X is a Banach space, into itself has a fixed point.

Proof of the Schauder Fixed Point Theorem. ([1], p. 57-58) By continuity of T and compactness of M, we know that $T(M)$ is compact as well. For each $n \in \mathbb{N}$, there are elements $y_i \in T(M)$, $i = 1, 2, \ldots, N$ such that $\min_i \left\|Tx - y_i\right\| < 1/n$, $\forall x \in M$. We will let M_n be the convex hull of points y_i, $i = 1, 2, \ldots N$. Because M is convex itself, then $M_n \subseteq \text{co}(T(M)) \subseteq M$, where the first inequality follows from M_n being the smallest set containing all of the y_i and $\text{co}(T(M))$ being the smallest convex set containing $T(M)$ which contains all of the y_i, and where the second inequality follows from the fact that $T(M) \subseteq M$, and since M is convex, then $\text{co}(T(M)) \subseteq M$. Note that because N is finite, M_n is compact in addition to convex.

We will define a function $P_n : M_n \to M_n$ by

$$P_n(x) = \frac{\sum_{i=1}^N a_i(x) y_i}{\sum_{i=1}^N a_i(x)},$$

where $a_i(x) = \max(n^{-1} - \left\|Tx - y_i\right\|, 0)$. Note that

$$0 \leq \frac{a_i(x)}{\sum_{j=1}^N a_j(x)} \leq 1, \quad \sum_{i=1}^N \frac{a_i(x)}{\sum_{j=1}^N a_j(x)} = 1,$$

thus $P_n(x)$ is a convex combination of $y_1, \ldots, y_N$. Hence, $P_n(M_n) \subset M_n$. Also, $M_n \subseteq \mathbb{R}^N$. So, $\text{span}\{y_1, \ldots, y_N\}$ is a subspace of $\mathbb{R}^N$. Also note that because a_i do not all vanish simultaneously by our selection of y_i, $i = 1, 2, \ldots, N$, then $P_n : M_n \to M_n$ is a continuous function.

We will also point out that for all $x \in M$,

$$\|P_n x - Tx\| = \left\| \frac{\sum_i a_i(x)(y_i - Tx)}{\sum_i a_i(x)} \right\| \leq \frac{\sum_i a_i(x) n^{-1}}{\sum_i a_i(x)} \leq n^{-1}.$$

We see that as $n \to \infty$, we have $P_n \to T$. for all $x \in M$.

We see that the reformulation of the Brouwer Fixed Point Theorem applies, and so there is a fixed point $x_n = P_n(x_n)$, where $x_n \in M_n \subseteq M$. By compactness of M, there is a convergent subsequence, denoted by (x_n), such that $x_n \to x$ as $n \to \infty$. This x is the desired fixed point, as can be seen by letting $n \to \infty$ in $\|x_n - Tx\| \leq \|P_n x_n - Tx_n\| + \|Tx_n - Tx\|$. ■

7 Application of the Brouwer Fixed Point Theorem to Real-World Maps

In the previous section, we discussed two important theorems: the Brouwer Fixed Point Theorem and its extension to infinite-dimensional spaces, the Schauder Fixed Point Theorem. A fixed point is guaranteed when we have a nonempty, compact, convex subset of a Banach space and a continuous map from that subset into itself, whether the subset be finite-dimensional (Brouwer) or infinite-dimensional (Schauder). Our first application will be a fun, casual application to start things off.

Theorem 7.1. *Consider the rectangular state of Colorado and a sheet of paper on which is completely printed a (possibly continuously-distorted) map of Colorado. Then, with the map on the ground, there is at least one point on the paper directly on top of the point in Colorado which is indicated by the paper.*

Remark 7.2. In the real world, Colorado is not a two-dimensional rectangle. Rather, it is some shape in 3-space due to the curvature of the Earth. However, for our purposes, we will assume it is flat and a nice rectangle.

Proof of Theorem 7.1. In this proof, "map" will always refer to the piece of paper and not a function. Note that the sheet of paper/map (denote as M) and Colorado (denote as C) are rectangular in nature and thus are convex sets. Because they are two-dimensional and thus can be considered as subsets of $\mathbb{R}^2$, they are both homeomorphic to a disk. Because we placed the map on top of Colorado, we can think of these two two-dimensional sets as subsets of $\mathbb{R}^3$ parallel to each other.

Denote the downward projection of M into C by $k : M \to C$. Note that k is continuous, and $k(M) \subseteq C$ (if $k(M) = C$, then that is one huge piece of paper!). Note that the map of Colorado is itself the image of a continuous function $j : C \to M$ (possibly even continuously-distorted) which sends points in actual Colorado to points on the sheet of paper.

Consider a function $f : C \to C$ by $f = k \circ j$ and note that f is a continuous map of Colorado into itself, thus it has a fixed point by the Brouwer Fixed Point Theorem since it is homeomorphic to the disk D^2. This fixed point c_0 can be traced back by $k^{-1}(c_0)$ (the preimage, not the inverse function). Since k is clearly injective, $k^{-1}(c_0)$ is just a set containing one point, namely $j(c_0)$.

Since k was strictly the downward projection, then $j(c_0)$ lies directly on top of $c_0 \in C$. Thus, there is at least one point on the paper/map directly on top of the point in Colorado which is indicated by the paper/map itself. ∎

8 Application of the Brouwer Fixed Point Theorem to Nash Equilibriums

We discussed an application of the Brouwer Fixed Point Theorem to the real world in a physical sense in the previous chapter. While it is certainly appealing to visualize such applications, the Brouwer Fixed Point Theorem is a very important result in mathematics and has various applications. The second and last application we will be discussing is its application to game theory and economics.

Theorem 8.1. *Every two-person finite game has a Nash equilibrium in mixed strategies.*

This is a completely different field, so it will be useful to define our terms.

Definition 8.2. ([18]) A *game* is a situation involving gains or losses in which at least two players each have partial control over the outcomes.

Definition 8.3. ([3], p. 88) A *strategy* of player n is an element of the set of choices X_n, or the *strategy set*, from which player n can choose in the game. The product of all strategy sets is called the set of strategy vectors.

One can think of a possible outcome as the strategy vector $x =< x_{1,a}, x_{2,b}, ..., x_{n,m} >$, where $x_{i,j}$ is the j-th choice/strategy in the strategy set X_i (the set of options for player i).

Definition 8.4. ([20]) A *Nash equilibrium* is a strategy vector x for which no player has an incentive to deviate from their current strategy given the choices of the other players.

It is best to understand what these ideas mean when they come together, so we will use our own example of the famous problem of the prisoner's dilemma.

Example 6. Consider the following situation in which Prisoner A's choices are given in the left-most rows and Prisoner B's choices are given in the upmost columns:

$$\begin{bmatrix} & \text{Silent} & \text{Testify} \\ \text{Silent} & \text{Both get 1 year} & \text{A: 10 years, B: free} \\ \text{Testify} & \text{A: free, B: 10 years} & \text{Both get 7.5 years} \end{bmatrix}$$

This situation tells us of two prisoners. Both have the options either to testify or to remain silent, but how much jail time they receive is dependent not only on their individual choice but on the other prisoner's choice as well. From this example, it is obvious that the best choice that takes everyone into account is for both prisoners to work together by staying silent; this way, they will each only get half a year. However, what's interesting is that the Nash equilibrium does not necessarily work this way. Suppose A is silent. Then B has the option of getting 1 year or going free, and he will clearly want to go free. We will "mark" $< Silent, Testify >$ to show that this is B's strategy given a certain strategy for A. Now, suppose A testifies. Then B will either get 10 years of jail, or he will get 7.5 years. He will clearly choose to testify and receive 7.5 years, so we will mark $< Testify, Testify >$.

Now, let's switch perspectives. Suppose B remains silent. Then A can testify and go scot-free, or A can remain silent and get 1 year. Because B's choice is given, A will choose to testify to receive less jail time. So, we mark $< Testify, Silent >$. Finally, suppose B testifies. Then, A will want to testify as well because A prefers 7.5 years of jail time over 10. So, we mark $< Testify, Testify >$.

Notice that the only cell that we marked twice (twice because the number of players $n = 2$) is $< Testify, Testify >$. This is the Nash equilibrium. Neither player has the incentive to deviate from their current strategy individually given the other player's strategy. Even though $< Testify, Testify >$ is not the best option for the players collectively, when they are treated individually, neither player will have the incentive to change.

Our example has yet to take into account one more variable in the theorem: the notion of mixed-strategies.

Definition 8.5. ([19]) A *mixed-strategy* is a collection of moves together with a corresponding set of weights which are followed probabilistically in the playing of a game. Alternatively, for every $1 \leq i \leq n$ there are probability variables $p_{i,j}$ attached to each strategy $x_{i,j} \in X_i$ for player i such that $\sum_{j=1}^{m} p_{i,j} = 1$.

Our example showed the existence of a Nash equilibrium in a pure-strategy game, that is, there are no probabilities that one can continually vary assigned to the game; there are probabilities, namely 100% or 0%, but these cannot be varied; they are fixed.

However, in a mixed-strategy game, players do not have to go all-in and are allowed to vary their strategy choices with probabilities. By contrast, a Nash equilibrium in a pure strategy will be a strategy vector that will be picked 100% of the time. In a mixed-strategy game, a Nash equilibrium will be a vector that includes the strategies *and* their probabilities for each player. If you change the probabilities, then the vector is possibly not a Nash equilibrium (there can be more than one).

Proof of Theorem 8.1 ([17]) In order to proceed, we need notation. We will label the strategies of player 1 by $1, 2, \ldots, m$ and those of player 2 by $1, 2, \ldots, n$. We define the k-simplex by:

$$\Delta_k = \{x \in \mathbb{R}_+^{k+1} \mid \sum_{i=1}^{k+1} x_i = 1\}. \tag{8.1}$$

Because the mixed-strategy game has probabilities for each player that add up to 1, we can think of any mixed strategy of player 1 as a point in Δ_{m-1} and a mixed strategy of player 2 as a point in Δ_{n-1}.

Let $u_l(i,j)$ be the value of player l's choice numerically given strategy $i \leq m$ and $j \leq n$. We can define matrices A and B by $a_{ij} = u_1(i,j)$ and $b_{ij} = u_2(i,j)$ respectively.

Given strategies $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$, the expected payoff $Eu_l(p,q)$ for player l is given by the formula $Eu_l(p,q) = \sum_{i=1}^{m} \sum_{j-1}^{n} p_i q_j u_l(i,j)$, or alternatively, $Eu_1(p,q) = p'Aq$ and $Eu_2(p,q) = p'Bq$, where p and q are written as column vectors and p' and q' are written as row vectors.

Now, we will denote the i-th column of A by A_i and the j-th column of B by B_j. So, $A_i q$ gives the expected payoff of player one when playing the pure strategy i against player two's mixed strategy q, while $p'B_j$ gives the expected payoff of player one when playing the pure strategy j against player one's mixed strategy p.

With the preparation out of the way, we will now proceed with our proof. Because the Nash equilibrium says no player is incentivized to deviate from their current strategy given the other players' strategies, we can interpret this as saying the gain of deviating from a pure strategy to a mixed strategy is not positive.

We want to capture this idea with formulas, so we will define two. Let $c_i(p,q) = \max(A_i q - p'Aq, 0)$ for $1 \leq i \leq m$ and $d_j(p,q) = \max(p'B_j - p'Bq, 0)$ for $1 \leq j \leq n$. Remember that $A_i q$ and $p'B_j$ represent the payoff for a pure strategy, whereas $p'Aq$ and $p'Bq$ represent mixed strategies, so functions c_i and d_j represent the gain (if any) of deviating from a pure strategy to a mixed strategy. Note that c_i and d_j are both continuous functions as the maximum of two continuous functions.

With c_i and d_j defined, we will define new functions $P((p,q)) = (P_1(p,q), \ldots, P_m(p,q))$ and $Q((p,q)) = (Q_1(p,q), \ldots, Q_n(p,q))$, where each P_i and Q_j is defined in the following manner:

$$P_i(p,q) = \frac{p_i + c_i(p,q)}{1 + \sum_{k=1}^{m} c_k(p,q)},$$

$$Q_j(p,q) = \frac{q_j + d_j(p,q)}{1 + \sum_{k=1}^{n} d_k(p,q)}.$$

Similarly to c_i and d_j, P_i and Q_j are also continuous since each of their denominators is strictly positive and each numerator is continuous. Note the following sums:

$$\sum_{i=1}^{m} P_i(p,q) = \frac{\sum_{i=1}^{m} p_i + \sum_{i=1}^{m} c_i(p,q)}{1 + \sum_{k=1}^{m} c_k(p,q)} = \frac{1 + \sum_{i=1}^{m} c_i(p,q)}{1 + \sum_{k=1}^{m} c_k(p,q)} = 1,$$

$$\sum_{j=1}^{n} Q_j(p,q) = \frac{\sum_{j=1}^{n} q_j + \sum_{j=1}^{n} d_j(p,q)}{1 + \sum_{k=1}^{n} d_k(p,q)} = \frac{1 + \sum_{j=1}^{n} d_j(p,q)}{1 + \sum_{k=1}^{n} d_k(p,q)} = 1.$$

Therefore, $P(\Delta_{m-1}) \subseteq \Delta_{m-1}$ and $Q(\Delta_{n-1}) \subseteq \Delta_{n-1}$ by Equation 8.1. Now, we will define a function $T : \Delta_{m-1} \times \Delta_{n-1} \to \Delta_{m-1} \times \Delta_{n-1}$ by $T(p,q) = (P(p,q), Q(p,q))$. Because P and Q are both continuous, T is continuous as well. Also, because simplices are convex hulls of a finite number of points, then $S = \Delta_{m-1} \times \Delta_{n-1}$ is compact and convex as well. Therefore, using the Brouwer Fixed Point Theorem, there is a fixed point $(p^*, q^*) = T(p^*, q^*)$. We claim that this fixed point is the Nash equilibrium. Therefore, we claim that $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$, meaning that the gain from deviating from a pure strategy p^* for player one is 0. Suppose this claim is false. Because (p^*, q^*) is a fixed point, we have

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^m c_k(p^*, q^*)}$$

for every i. Therefore, $c_i(p^*, q^*) = p_i^*[\sum_k c_k(p^*, q^*)]$, so from this we arrive at two relationships: $p_i^* = 0$ whenever $c_i(p^*, q^*) = 0$ and $\sum_{i=1}^m p_i^* = \frac{\sum_{i=1}^m c_i(p^*, q^*)}{\sum_{k=1}^m c_k(p^*, q^*)} = 1$. Because $\sum_i p_i = 1$, we know that there is at least one i, denote it by s, for which $c_i(p^*, q^*) > 0$, meaning that $A_s q^* > u_1^*$. Thus, $p_s^* A_s q^* > p_s^* u_1^*$. Summing over all such possible s, we get

$$u_1^* = \sum_{i=1}^m p_i^* A_s q^* \geq \sum_s p_s^* A_s q^* > \sum_s p_s^* u_1^* = u_1^*.$$

This is a contradiction, which means that our supposition that our claim is false is false itself. Therefore, our claim that $\sum_{k=1}^m c_k(p^*, q^*) = 0$ is true. A similar argument works for a pure strategy q^* of player 2 in response to player one's mixed strategy p^*. Therefore, our fixed point (p^*, q^*) is a Nash equilibrium.

■

9 Differential Equations and Fixed Point Theorems

9.1 Peano's Theorem

In a previous section, we discussed the Banach Fixed Point Theorem and used it to prove the Inverse Function Theorem. Because we are on the topic of fixed point theorems, it will be good to remember the statement of the Banach Fixed Point Theorem in order to contrast it with Peano's Theorem.

Theorem 9.1. (Banach Fixed Point Theorem). *Let (W, d) be a non-empty complete metric space. If $T : W \to W$ is a contraction, then T admits a fixed point $w^* \in W$, and furthermore, w^* is unique.*

Theorem 9.2. (Peano's Theorem). *Consider the following initial-value problem:*

$$x'(t) = f(t, x(t)),\ x(t_0) = y_0. \tag{9.1}$$

Let there be given real numbers t_0 and y_0, and the rectangle

$$Q_b = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \le a,\ |x - y_0| \le b\},$$

where a and b are fixed positive numbers. Suppose that $f : Q_b \to \mathbb{R}$ is continuous and bounded with

$$|f(t, x)| \le K \text{ for all } (t, x) \in Q_b,$$

and fixed $K > 0$. Set $c = \min(a, K/b)$. Then, the initial-value problem (4.1) has a continuously differentiable solution on $[t_0 - c, t_0 + c]$.

In our proof of Peano's Theorem, we will rewrite (9.1) as an intergral, and the intergral as the operator equation $x = Tx$. We will not prove the following, but it is important to note that if we let $X = C[t_0 - c, t_0 + c]$, then X is a complete metric space, whose norm is given by $\|x(s)\| = \max_{t_0 - c \le s \le t_0 + c} |x(s)|$, for all $x \in X$. Also, recall that a contraction is a continuous function for which there is a non-negative number $k < 1$ such that the distance between the images of two points x, y is less than or equal to k times the distance between x, y.

Although Peano's Theorem is a direct application to differential equations while the Banach Fixed Point Theorem itself is not, we can highlight the difference between the two. The Banach Fixed Point Theorem tells us that a contraction from a complete metric space into itself will yield a unique fixed point. On the other hand, Peano's Theorem does not require Lipschitz continuity in its function; it only requires regular continuity. Because of that, the assumptions in Peano's Theorem only leads to the guarantee of the existence of a fixed point, not the uniqueness.

In the next section, we will see this contrasted much more when we use the Banach Fixed Point Theorem in order to prove the Picard-Lindelöf Theorem. In the Picard-Lindelöf Theorem, the function is Lipschitz continuous in the second variable. Because this theorem imposes on the function conditions that are stronger, namely that it must be Lipschitz continuous, than the conditions in theorem 9.2, which only requires a continuous function, we are able to squeeze out more than existence and get uniqueness.

Consider the following proof of Peano's Theorem, and afterward, we will apply the Banach Fixed Point Theorem to ordinary differential equations as well to contrast the two once again.

Proof of Theorem 9.2. ([21], p. 57-58) From calculus, recall that (4.1) is equivalent to finding the solution to

$$x(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{9.2}$$

We will write this as $x = Tx$, $x \in M \subseteq X$, where we will let $X = C[t_0 - c, t_0 + c]$, $M = \{x \in X \mid \left\|x - y_0\right\| \leq b\}$, and $\|x\| = \max_{t_0 - c \leq t \leq t_0 + c} |x(t)|$. So, X is the space of continuous functions on a small interval $[t_0 - c, t_0 + c]$, and M is the set of all the functions in X such that their max distance between the function itself and the point y_0 is less than or equal to b on the aforementioned interval, hence M is closed. Also, M is convex and bounded in X. If $x \in M$, then $\left\|x - y_0\right\| \leq b$. Because we see that

$$\left\|Tx - y_0\right\| = \max_{t \in [t_0 - c, t_0 + c]} \left| \int_{t_0}^{t} f(s, x(s))ds \right| \leq cK \leq b,$$

then $T(M) \subseteq M$.

Claim: If M is a closed, bounded, convex subset of a Banach space X and $T : M \to M$ is compact (i.e. T(M) has a compact closure), then T admits a fixed point.

Proof: Let $A = \overline{\text{co}}(T(M))$. Then, $A \subseteq M$ since $T(M)$ is a subset of the convex set M. Additionally, A is compact and convex by definition. Clearly, $T(A) \subseteq A$, so the restriction $T : A \to A$ admits a fixed point by the Schauder Fixed Point Theorem. Since $A \subseteq M$, this is a fixed point on M as well.

By our claim, we have a fixed point for T in M. Therefore, we have a solution for (9.1), which is $x'(t) = f(t, x(t))$, $x(t_0) = y_0$.

■

9.2 The Picard-Lindelöf Theorem

We will start off in a situation resembling that of Theorem 9.2. However, whereas Peano's Theorem did not require Lipschitz continuity in the second variable, the Picard-Lindelöf Theorem does. In exchange for this stronger condition, we also get a stronger claim, namely a *unique* fixed point. Consider the following theorem.

Theorem 9.3. *Consider the following initial-value problem.*

$$y'(x) = f(x, y(t)),\ y(x_0) = y_0.$$

Suppose f is continuous in x and Lipschitz continuous with respect to y. Then, for some $\epsilon > 0$, there exists a unique solution $y(x)$ to the initial value problem on the interval $[x_0 - \epsilon, x_0 + \epsilon]$.

Proof of Theorem 9.3. [13] Let $A = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\} \subset \mathbb{R}^2$, and let $f : A \to \mathbb{R}$ be a Lipschitz continuous function in y. We want to show that $y' = f(x,y)$ not only has a solution $y = g(x)$ with $g(x_0) = y_0$ defined on an interval $[x_0 - \epsilon, x_0 + \epsilon]$, for some $\epsilon > 0$, but also that this $y = g(x)$ is unique. This set A is similar to Q_b as in Peano's Theorem, and our function f which takes A into $\mathbb{R}$ is similar to the f in theorem 9.2 that takes A into $\mathbb{R}$.

However, the f in this proof is Lipschitz continuous. Similar to the previous proof, we will again write note that the initial-value problem is equivalent to

$$g(x) = g(x_0) + \int_{x_0}^{x} f(t, g(t))dt. \tag{9.3}$$

Because f is Lipschitz continuous in the second variable, then by definition there exists a Lipschitz constant $q > 0$, i.e. a $q > 0$ such that $\left|f(x,y_1) - f(x,y_2)\right| \le q\left|y_1 - y_2\right|$ for all $(x,y_1),(x,y_2) \in A$. Note that these pair of points change with respect to the y-value, not the x–value. Because $A \subset \mathbb{R}^2$ is compact and f is continuous, f is bounded by some constant $M > 0$ on A.

We have yet to choose our ϵ, so we will choose an $\epsilon > 0$ such that $\epsilon < q^{-1}$, and we will define a new set

$$B = \{(x,y) \in \mathbb{R}^2 \mid |x - x_0| \le \epsilon, \left|y - y_0\right| \le M\epsilon\}.$$

Note that $B \subset A$. Let X be the subset of $(C([x_0 - \epsilon, x_0 + \epsilon]), d)$, with $d(\cdot,\cdot) = \|\cdot - \cdot\|_{L^\infty}$, of functions g satisfying $d(g, g(x_0)) \le M\epsilon$. We see that (X,d) is a closed subspace because it contains all of its limit points. Now, let $h = y_0 + \int_{x_0}^{x} f(t,g(t))dt$. Observe that

$$d(h, y_0) = \sup_{x\in[x_0-\epsilon, x_0+\epsilon]}\left|y_0 + \int_{x_0}^{x} f(t,g(t))dt - y_0\right| \le \sup_{x\in[x_0-\epsilon,x_0+\epsilon]} \int_{x_0}^{x} |f(t,g(t))dt|$$

$$\le \sup_{x\in[x_0-\epsilon,x_0+\epsilon]} \int_{x_0}^{x} M dt = M\epsilon.$$

Because we have that $d(h, y_0) \le M\epsilon$, then $h \in X$. Now, let us define a mapping $T : X \to X$ by $Tg = h = y_0 + \int_{x_0}^{x} f(t,g(t))dt$. Because of the previous set of inequalities, we know that if $g \in X$, then $d(Tg, y_0) = d(h, y_0) \le M\epsilon$, thus Tg is in X. So, T is well-defined.

We will show that T is a contraction mapping. Take any $g_1, g_2 \in X$ and see that

$$d(Tg_1, Tg_2) = \sup_{x\in[x_0-\epsilon,x_0+\epsilon]}\left|y_0 + \int_{x_0}^{x} f(t,g_1(t))dt - y_0 + \int_{x_0}^{x} f(t,g_2(t))dt\right|$$

$$
\begin{aligned}
&\leq \sup_{x\in[x_0-\epsilon,x_0+\epsilon]}\left|\int_{x_0}^{x} f(t,g_1(t))dt - \int_{x_0}^{x} f(t,g_2(t))dt\right| \\
&\leq \sup_{x\in[x_0-\epsilon,x_0+\epsilon]}\int_{x_0}^{x} |f(t,g_1(t))dt - f(t,g_2(t))dt| \\
&\leq \sup_{x\in[x_0-\epsilon,x_0+\epsilon]}\int_{x_0}^{x} q\,|g_1(t)-g_2(t)|\,dt \\
&\leq d(g_1,g_2)\sup_{x\in[x_0-\epsilon,x_0+\epsilon]}\int_{x_0}^{x} q\,dt \\
&\leq q\epsilon d(g_1,g_2) = kd(g_1,g_2),
\end{aligned}
$$

where $k \in [0,1)$ due to our choice of $\epsilon < q^{-1}$. Thus, T is a contraction, and therefore, by the Banach Fixed Point Theorem, T has a fixed point. ■

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