SMALL EIGENVALUES OF HYPERBOLIC POLYGONS

by

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ABSTRACT

The hyperbolic geometric structure is a type of non-Euclidean geometry. We first examine the geodesics in hyperbolic space using the properties of Möbius transformations in the upper half-plane. We derive a distance formula and use it to determine the hyperbolic versions of the Pythagorean theorem, the Law of Sines, the Law of Cosines, Ceva's Theorem, and Menelaus's Theorem. We then examine the spectral properties of hyperbolic triangles. We determine a differential equation for a familly of triangles with constant first eigenvalue of the hyperbolic Laplacian with Dirichlet boundary conditions.

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1 Geodesic Formulas

1.1 Geodesic path and distance

Let $\mathbb{H} = \{z = x + iy : y > 0\}$. The hyperbolic metric is defined as follows. The inner product of two vectors at a point z_0 is defined as $< v, w >= \dfrac{v \cdot w}{(Im(z_0))^2}$([1]). Thus, the hyperbolic length of a path $\alpha : [a, b] \to \mathbb{H}$ is

$$L(\alpha) = \int_a^b \frac{|\alpha'(t)|}{Im(\alpha(t))} dt.$$

In ([1]), it is shown that distance-minimizing paths (geodesics) are either vertical straight line in $\mathbb{H}$ or semicircular arcs in $\mathbb{H}$ centered on the horizontal axis. Given the formula for the hyperbolic length of a path in the upper half-plane, we will derive formulas for the distance between two points in the hyperbolic plane.

We will consider a simple case. Let $z = i$ and $w = \sin(\theta_1) + i\cos(\theta_1)$. One reparametrization of a geodesic path from z to w is $a(t) = (\sin(t), \cos(t))$ for $0 \le t \le \theta_1$.

So $|a(t)| = 1$ and $a'(t) = (\cos(t), -\sin(t))$ The distance from z to w is

$$\begin{aligned}
\int_0^{\theta_1} \frac{|a'(t)|}{\cos(t)} dt &= \ln\left(\frac{1}{\cos(t)} + \tan(t)\right)\Bigg|_0^{\theta_1} \\
&= \ln\left(\frac{1+\sin(t)}{\cos(t)}\right)\Bigg|_0^{\theta_1} \\
&= \frac{1}{2}\ln\left(\frac{1+\sin(t)}{1-\sin(t)}\right)\Bigg|_0^{\theta_1} \\
&= \frac{1}{2}\ln\left(\frac{1+\sin(\theta_1)}{1-\sin(\theta_1)}\right) \\
&= \frac{1}{2}\ln\left(\frac{1+\mathrm{Re}(w)}{1-\mathrm{Re}(w)}\right)
\end{aligned}$$

We now consider the general case and calculate the distance between $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Let a be a geodesic path $a(t)$ such that $a(0) = z$ and $a(1) = w$. (Here we assume $z_1 \leq w_1$ to parametrize our path) If $z_1 = w_1$, we have

$$\text{dist}_{\mathbb{R}^2}(z, w) = |z - w|$$

Let c be the center of the Euclidean circle A passing through z and w on the real axis. Let L be the Euclidean line segment joining z and w. The midpoint of L is $\frac{1}{2}(z+w)$. The slope of L is $k = \dfrac{Im(w) - Im(z)}{Re(w) - Re(z)} = \dfrac{w_2 - z_2}{w_1 - z_1}$. The perpendicular bisector H of L passes through $\frac{1}{2}(z+w)$ and has slope $-\dfrac{1}{k} = \dfrac{\text{Re}(z) - Re(w)}{Im(w) - Im(z)} = \dfrac{z_1 - w_1}{w_2 - z_2}$, so H has the equation:

$$y - \frac{1}{2}(Im(w) + Im(z)) = \left[\frac{Re(z) - Re(w)}{Im(w) - Im(z)}\right]\left(x - \frac{1}{2}(Re(z) + Re(w))\right)$$

$$y - \frac{1}{2}(w_2 + z_2) = \left[\frac{z_1 - w_1}{w_2 - z_2}\right]\left(x - \frac{1}{2}(z_1 + w_1)\right)$$

The Euclidean center c is the x-intercept of H

$$\begin{aligned} c &= \left[-\frac{1}{2}(Im(z) + Im(w))\right]\left[\frac{Im(w) - Im(z)}{Re(z) - Re(q)}\right] + \frac{1}{2}(Re(z) + Re(w)) \\ &= \frac{1}{2}\left[\frac{(Im(z))^2 - (Im(w))^2 + (Re(z))^2 - Re(w))^2}{Re(z) - Re(w)}\right] \\ &= \frac{1}{2}\left[\frac{|z|^2 - |w|^2}{Re(z) - Re(w)}\right] \\ c &= \frac{1}{2}\left[\frac{|z|^2 - |w|^2}{z_1 - w_1}\right] \end{aligned}$$

The Euclidean Radius of A is

$$r = |c - p| = \frac{1}{2}\left|\left[\frac{|z|^2 - |w|^2}{z_1 - w_1}\right] - z\right|$$

Let D be the center of the semicircle passing through z and w. By shifting the circle horizontally by $-c$,we construct a formula for a geodesic path $b(t)$ from $z' = z - c$ to $w' = w - c$ such that $a(0) = z'$, $a(1) = w'$, and the center is at the origin O. Let Z', W' be the image of z, w after shifting the semicircle with center D. Let $t_1 = -\angle DOZ'$, $t_2 = \angle DOW'$. The formula for a geodesic path from z to w in this case is $b(t) = \big(r \sin \big[t(t_2 - t_1) + t_1\big], r \cos \big[t(t_2 - t_1) + t_1\big]\big)$.

Shifting the circle horizontally by c, the general formula for a geodesic path passing through z and w is $a(t) = b(t) + c = \big(r \sin \big[t(t_2 - t_1) + t_1\big] + c, r \cos \big[t(t_2 - t_1) + t_1\big]\big)$.

We construct the general formula for the geodesic path passing through z and w by parametrizing the Euclidean circle with center c and radius r passing through z and w. The geodesic distance from z to w is obtained by

$$\begin{aligned}
d_{\mathbb{H}}(z,w) &= \int_0^1 \frac{|a'(t)|}{Im(a(t))}dt \\
&= \int_0^1 \frac{\big|\big(r(t_2-t_1)cos(t(t_2-t_1)+t_1), -r(t_2-t_1)\sin(t(t_2-t_1)+t_1)\big|}{r\cos(t(t_2-t_1)+t_1)}dt \\
&= \int_0^1 \frac{t_2-t_1}{\cos\big[t(t_2-t_1)+t_1\big]}dt
\end{aligned}$$

Let $u = t(t_2-t_1)+t_1$

$$\begin{aligned}
d_{\mathbb{H}}(z,w) &= \int_{t_1}^{t_2} \sec(u)du \\
&= \ln|\sec(u)+\tan(u)|\Big|_{t_1}^{t_2} \\
&= \ln\left|\frac{\sin(u)+1}{\cos(u)}\right|\Bigg|_{t_1}^{t_2} \qquad (1.1.1)
\end{aligned}$$

We also have

$$\begin{aligned}
\sin(t_1) &= \frac{z_1-c}{r} \\
\cos(t_1) &= \frac{z_2}{r} \\
\sin(t_2) &= \frac{w_1-c}{r} \\
\cos(t_2) &= \frac{w_2}{r}
\end{aligned}$$

Substituting into (1.1.1)

$$d_{\mathbb{H}}(z,w) = \ln\left|\frac{z_2(w_1-c+r)}{w_2(z_1-c+r)}\right| \qquad (1.1.2)$$

1.2 Alternate Formula for Hyperbolic Distance

Let $z = z_1 + iz_2$ and $w = w_1 + iw_2$ are belonging to $C(O,1)$, the circle with center O and radius 1. That means

$$|z|^2 = z_1^2 + z_2^2 = 1, \quad |w|^2 = w_1^2 + w_2^2 = 1.$$

$$z = e^{it_1} = \cos t_1 + i \sin t_1, \qquad w = e^{it_2} = \cos t_2 + i \sin t_2.$$

A parametrization (counterclockwise) of $C(O,1)$ is

$$a(t) = (\cos t, \sin t).$$

The distance:

$$d_{\mathbb{H}}(z,w) = \int_{t_1}^{t_2} \csc(t) dt$$

Now consider the transformation for $c \in \mathbb{R}, r > 0$,

$$\gamma(u) = ru + c = (ru_1 + c, ru_2) = (r \cos t + c, r \sin t),$$

where $u = u_1 + iu_2$. For every $u, v \in \mathbb{C}$,

$$d_{\mathbb{H}}(\gamma(u), \gamma(v)) = \int_{\theta_1}^{\theta_2} \csc t dt = d_{\mathbb{H}}(u,v).$$

It is shown in ([1]) that all orientation-preserving isometries (distance-preserving maps) of $\mathbb{H}$ to itself are the Möbius transformation maps of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Thus γ is an isometry. We may take any circular geodesic arc and transform it to a $C(0,1)$ arc with a choice of such a γ.

For $z, w \in C(O,1)$. (In this case $c = 0,\ r = 1$), we have $d_{\mathbb{H}}(z,w) = \ln \left| \frac{(z_1+1)w_2}{(w+1)z_2} \right|$ by (1.1.2) We also have

$$\begin{aligned} \cosh d_{\mathbb{H}}(z,w) &= \frac{1}{2} \left| \frac{(z_1+1)w_2}{(w+1)z_2} + \frac{(w+1)z_2}{(z_1+1)w_2} \right| \\ &= \frac{[z_2^2 w_1^2 + 2z_2^2 w_1 + z_2^2 + w_2^2 z_1^2 + 2w_2^2 z_1 + w_2^2]}{2z_2 w_2 (z_1+1)(w_1+1)}. \end{aligned} \tag{1.2.1}$$

We substitude $z_2^2 = 1 - z_1^2,\ w_2^2 = 1 - w_1^2$ and $z_2^2 + w_2^2 = 1$ into (1.2.1); we obtain

$$\begin{aligned} \cosh d_{\mathbb{H}}(z,w) &= \frac{-z_1^2 w_1^2 + z_1 - z_1^2 w_1 + w_1 + z_1 w_1^2 + 1}{2z_2 w_2 (z_1+1)(w_1+1)} \\ &= \frac{(2 - 2z_1 w_1)(z_1+1)(w_1+1)}{2z_2 w_2 (z_1+1)(w_1+1)} \\ &= \frac{(2 - 2z_1 w_1)}{2z_2 w_2} \\ &= \frac{z_1^1 + z_2^2 + w_1^2 + w_2^2 - 2z_1 w_1}{2z_2 w_2} \\ &= 1 + \frac{|z-w|^2}{2z_2 w_2}. \end{aligned}$$

Note that if z, w are replaced bny $rz + c, rw + c$, the result is the same, so that this formula for hyperbolic distance works for any $z, w \in \mathbb{H}$. The formula works even for z, w such that they are contained in a vertical line.

1.3 The Pythagorean Theorem for Hyperbolic Right Triangles

The following formulas were derived from the hyperbolic distance formulas above

Theorem 1.1 (Hyperbolic Pythagorean Theorem)**.** *In a hyperbolic right triangle ABC (right angle at B), with a,b,c be the opposite sides to the angle at A, B, C respectively then*

$$\cosh(b) = \cosh(a)\cosh(c).$$

Proof. We conveniently choose 3 vertices of a right triangle at z_1, with $z_1 = (0, 1) = i$, $z_2 = (0, y) = yi$, $z_3 = (\cos(t), \sin(t)) = \cos(t) + i\sin(t)$ for $0 < t < \pi$.

The Möbius group acts transitively on $\mathbb{H}$. Given two triples (w_1, w_2, w_3) and (z_1, z_2, z_3) of distinct points in $\bar{\mathbb{C}}$, there exists a unique element m of Möb$^+$ so that $m(w_1) = z_1, m(w_2) = z_2, m(w_3) = z_3$. Möbius transformations preserve angles and also preserve the distance between two points in $\mathbb{H}$; that is, Möbius transformations of $\mathbb{H}$ are conformal and are isometries of $\mathbb{H}$. Thus, given any triangle ABC that has a right angle at B in $\mathbb{H}$ we can construct an isometry (Möbius transformation) that maps B to z_1, A to z_2 and C to z_3.
Let c = distance between z_1 and z_2, b = distance between z_2 and z_3, and a = distance between z_1 and z_3. By the hyperbolic distance formula,

$$\cosh(c) = 1 + \frac{(y-1)^2}{2y} = \frac{y^2+1}{2y}, \tag{1.3.1}$$

$$\cosh(a) = 1 + \frac{\left|\cos(t) + i(\sin(t)-1)\right|^2}{2\sin(t)} = 1 + \frac{\cos(t)^2 + (\sin(t)-1)^2}{2\sin(t)} = \frac{1}{\sin(t)}, \tag{1.3.2}$$

$$\cosh(b) = 1 + \frac{\left|\cos(t) + i(\sin(t)-y)\right|^2}{2y\sin(t)} = 1 + \frac{\cos(t)^2 + (\sin(t)-y)^2}{2y\sin(t)} = \frac{1+y^2}{2y\sin(t)}. \tag{1.3.3}$$

From (1.3.1), (1.3.2), (1.3.3) we have

$$\cosh(b) = \cosh(a)\cosh(c)$$

. □

1.4 Law of Sines

Theorem 1.2 (Hyperbolic Law of Sines)**.** *In hyperbolic geometry, the Law of Sines states that in a hyperbolic triangle ABC, with sides a,b,c be the opposite to the angle at A,B,C respectively*

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$$

Proof. First, we prove the formula that relates between angle and distance in a hyperbolic right triangle. Thus, given any triangle ABC that has a right angle at B in $\mathbb{H}$ we can construct an isometry (Möbius transformation) that maps B to z_1, A to z_2 and C to z_3, with $z_1 = (0, 1) = i$, $z_2 = (0, y) = yi$, $z_3 = (\cos(t), \sin(t)) = \cos(t) + i\sin(t)$ for $0 < t < \pi$.

Let c = distance between z_1 and z_2, b = distance between z_2 and z_3, and a = distance between z_1 and z_3. We have

$$\tan(A) = \frac{\tanh(a)}{\sinh(c)}, \tag{1.4.1}$$

$$\sin(A) = \frac{\sinh(a)}{\sinh(b)}, \tag{1.4.2}$$

$$\cos(A) = \frac{\tanh(c)}{\tanh(b)} \tag{1.4.3}$$

The points z_2 and z_3 lie on a unique geodesic, which is a semi-circle with center at $u \in \mathbb{R}$. The line segment from u to z_2 is the radius of the semi-circle, as is the line segment from u to z_3. Calculating the length of these line segments, we see that

$$y^2 + u^2 = (\cos(t) + u)^2 + \sin^2(t).$$

So $y^2 = 1 + 2u\cos(t)$. Using $u = \dfrac{y^2 - 1}{2\cos(t)}$, in the triangle with vertices at x, z_2, 0, we also have

$$\tan(A) = \frac{y}{u} = \frac{2y\cos(t)}{y^2 - 1}.$$

Using the fact that $(\cosh(t))^2 - (\sinh(t))^2 = 1$ and $\tanh(t) = \dfrac{\sinh(t)}{cosh(t)}$ for all $t \in \mathbb{R}$ we have from (1.3.2) and (1.3.3),

$\sinh(c) = \dfrac{y^2 - 1}{2y}$, so $\tanh(c) = \dfrac{y^2 - 1}{y^2 + 1}$.

$\sinh(a) = \dfrac{cos(t)}{sin(t)}$, so $\tanh(a) = cos(x)$.

Thus $\tan(A) = \dfrac{\tanh(a)}{\sinh(c)}$.

Note that

$$\begin{aligned}\cos^2(A) &= \frac{1}{1 + \tan^2(A)} \\ &= \frac{1}{1 + \frac{\tanh^2(a)}{\sinh^2(c)}} \\ &= \frac{\sinh^2(c)}{\sinh^2(c) + \tanh^2(a)}\end{aligned}$$

From the Pythagorean theorem, we see that

$$cos^2(A) = \frac{\sinh^2(c)}{\sinh^2(c) + 1 - \frac{\cosh^2(c)}{\cosh^2(b)}} = \frac{\tanh^2(c)}{\tanh^2(b)}$$

or $\cos(A) = \dfrac{\tanh(c)}{\tanh(b)}$.

To prove $\sin(A) = \sinh(c)/\sinh(b)$, we use the equation $\sin(A) = \cos(A)\tan(A)$ to obtain

$$\begin{aligned}\sin(A) &= \frac{\tanh(a)}{\sinh(c)}\frac{\tanh(c)}{\tanh(b)} \\ &= \frac{\sinh(a)}{\cosh(a)}\frac{1}{\sinh(c)}\frac{\sinh(c)}{\cosh(c)}\frac{\cosh(b)}{\sinh(b)} \\ &= \frac{\sinh(a)}{\sinh(c)}.\end{aligned}$$

Given a triangle ABC, we draw a hyperbolic line perpendicular to BC from A. Let H be the perpendicular foot. If H lies on B or C, the Law of Sines is true from (1.4.2).

Without loss of generality, We will consider the case where ABC has all three perpendicular foot lies on their respective segments and the case where a perpendicular foot is external (not on its opposite segment).

Suppose ABC has all three perpendicular foot lies on their respective segments. Since H lies between B and C, applying (1.4.2) to the right triangle ABH, we have

$$\sin(B) = \frac{\sinh(h)}{\sinh(c)}.$$

We can also express $\sinh(h)$ as

$$\sinh(h) = \sin(B)\sinh(c).$$

Similarly, applying (1.4.2) to the right triangle ACH, we have

$$\sinh(h) = \sin(C)\sinh(b).$$

Thus, we have

$$\sin(B)\sinh(c) = \sin(C)\sinh(b).$$

Dividing both sides by $\sinh(b)\sinh(c)$ yields

$$\frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$

If $\triangle ABC$ has a perpendicular foot that does not lie between on its opposite segment. In this case we assume it is H. Also, assume B lies between H and C, applying (1.4.2) to the triangle $\triangle ABH$, we have

$$\sinh(h) = \sin(\angle ABH)\sinh(C.)$$

Since $\angle ABH = \angle ABC = \angle B$, we have

$$\sinh(h) = \sin(B)\sinh(c).$$

Also. applying (1.4.2) to the right triangle $\triangle ACH$, we have

$$\sinh(h) = \sin(C)\sinh(b).$$

Thus, we have

$$\sin(B)\sinh(c) = \sin(C)\sinh(b),$$

which means

$$\frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$

Similarly, using the hyperbolic perpendicular line from B to AC, we can also prove

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(C)}{\sinh(c)}.$$

Hence, combine both statements above, we have

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$

□

1.5 Law of cosines

Theorem 1.3 (Hyperbolic Law of Cosines)**.** *In a hyperbolic triangle $\triangle ABC$, with sides a,b,c be the opposite to the angle at A,B,C respectively,*

$$\cosh(b) = \cosh(a)\cosh(c) - \sinh(a)\sinh(c)\cos(B).$$

Proof. Given a triangle $\triangle ABC$ we draw a hyperbolic perpendicular line to BC from A. Let H be the perpendicular foot. We consider the case where ABC has all three perpendicular foot lying on their opposite segments. Let $d_{\mathbb{H}}(B, H) = a_1$ and $d_{\mathbb{H}}(C, H) = a_2$. Applying the Pythagorean theorem to the right triangle $\triangle ACH$, we have

$$\cosh(b) = \cosh(a_2)\cosh(h).$$

By replacing a_2 with $a - a_1$, using the formula $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$, we have

$$\cosh(b) = \cosh(a)\cosh(a_1)\cosh(h) - \sinh(a)\sinh(a_1)\cosh(h).$$

Applying the Pythagorean theorem to the right triangle $\triangle ABH$, we have $\cosh(h) = \dfrac{\cosh(c)}{\cosh(a_1)}$. Replace this for $\cosh(h)$ in the formula above to get

$$\cosh(b) = \cosh(a)\cosh(c) - \sinh(a)\sinh(a_1)\frac{\cosh(c)}{\cosh(a_1)},$$

which is

$$\cosh(b) = \cosh(a)\cosh(c) - \sinh(a)\sinh(c)\frac{\tanh(a_1)}{\tanh(c)}.$$

Finally, we apply (1.4.3) to the right triangle $\triangle ABH$ to get

$$\cosh(b) = \cosh(a)\cosh(c) - \sinh(a)\sinh(c)\cos(B).$$

The case where H is not between B and C is proved similarly. □

Theorem 1.4 (Second Hyperbolic Law of Cosines)**.** *In a hyperbolic triangle $\triangle ABC$, with sides a,b,c be the opposite to the angle at A,B,C respectively*

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cosh(a).$$

Note

Proof. We use the notation of the previous proof. Applying the Pythagorean theorem for the triangles $\triangle ABH$ and $\triangle ACH$, and multiply them together, we obtain

$$\cosh(b)\cosh(c) = \cosh^2(h)\cosh(a_1)\cosh(a_2).$$

Multiplying both sides by $\cosh(a) = \cosh(a_1+a_2)$, using the formula $\cosh(x+y) = \cosh(x)\cosh(y)+\sinh(x)\sinh(y)$, we have

$$\cosh(b)\cosh(c)\big(\cosh(a_1)\cosh(a_2) + \sinh(a_1)\sinh(a_2)\big) = \cosh(a)\cosh^2(h)\cosh(a_1)\cosh(a_2).$$

We substitute $\cosh^2(h)$ with $1+\sinh^2(h)$ and divide both sides by $\cosh(a_1)\cosh(a_2)$ to obtain

$$\cosh(b)\cosh(c)\big(1 + \tanh(a_1)\tanh(a_2)\big) = \cosh(a)\big(1 + \sinh^2(h)\big).$$

Rearranging this formula, we have

$$\cosh(b)\cosh(c) - \cosh(a) = -\cosh(b)\cosh(c)\tanh(a_1)\tanh(a_2) + \cosh(a)\sinh^2(h).$$

By theorem 1.3, we substitute $\cosh(b)\cosh(c) - \cosh(a)$ with $\cos(A)\sinh(b)\sinh(c)$ and get

$$\cos(A)\sinh(b)\sinh(c) = -\cosh(b)\cosh(c)\tanh(a_1)\tanh(a_2) + \cosh(a)\sinh^2(h).$$

We divide both sides by $\sinh(b)\sinh(c)$ to get

$$\cos(A) = -\frac{\tanh(a_1)}{\tanh(c)}\frac{\tanh(a_2)}{\tanh(b)} + \frac{\sinh(h)}{\sinh(c)}\frac{\sinh(h)}{\sinh(b)}\cosh(a).$$

Applying (1.4.3) to $\triangle ACH$ and $\triangle ABH$ allows replacing $\dfrac{\tanh(a_1)}{\tanh(c)}$ with $\cos(B)$ and $\dfrac{\tanh(a_2)}{\tanh(b)}$ with $\cos(C)$, and applying (1.4.2) $\triangle ACH$ and $\triangle ABH$ allows us to replace $\dfrac{\sinh(h)}{\sinh(c)}$ with $\sin(B)$ and $\dfrac{\sinh(h)}{\sinh(b)}$ with $\sin(C)$. Finally, we have

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cosh(a).$$

□

2 The Theorems of Ceva and Menelaus for Hyperbolic Triangles

2.1 Menelaus's Theorem for Hyperbolic Triangles

Theorem 2.1 (Menelaus's Theorem for Hyperbolic Triangles)**.** *If L is a hyperbolic line that does not go through any vertex of a hyperbolic triangle $\triangle ABC$ such that L intersects AB at P, BC at Q, and CA at R.Here AB, BC, and CA denote the hyperbolic line segments from A to B, B to C, and C to A respectively. Then*

$$\frac{\sinh(d_{\mathbb{H}}(P,A))}{\sinh(d_{\mathbb{H}}(P,B))}\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(Q,C))}\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(R,A))}=1$$

Proof. Depending on the position of the hyperbolic line L relative to $\triangle ABC$, we have two cases: Either only one intersection is external (not on the line segment) or all three intersections are external. If only one intersection is external, without loss of generality, assume Q is external. Applying the Hyperbolic Law of Sines to the triangles $\triangle APR$, $\triangle BPQ$, and $\triangle CRQ$, we have

$$\begin{aligned}\frac{\sin(m\angle APR)}{\sinh(d_{\mathbb{H}}(R,A))}&=\frac{\sin(m\angle ARP)}{\sinh(d_{\mathbb{H}}(P,A))},\\ \frac{\sin(m\angle BPQ)}{\sinh(d_{\mathbb{H}}(Q,B))}&=\frac{\sin(m\angle BQP)}{\sinh(d_{\mathbb{H}}(P,B))},\\ \frac{\sin(m\angle CQR)}{\sinh(d_{\mathbb{H}}(R,C))}&=\frac{\sin(m\angle CRQ)}{\sinh(d_{\mathbb{H}}(Q,C))},\end{aligned}$$

or

$$\begin{aligned}\frac{\sin(m\angle APR)}{\sin(m\angle(ARP)}&=\frac{\sinh(d_{\mathbb{H}}(R,A))}{\sinh(d_{\mathbb{H}}(P,A))},\\ \frac{\sin(m\angle BPQ)}{\sin(m\angle(BQP)}&=\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(P,B))},\\ \frac{\sin(m\angle CQR)}{\sin(m\angle(CRQ)}&=\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(Q,C))}.\end{aligned}$$

Notice that

$$\begin{aligned}m\angle APR&=m\angle BPQ,\\ m\angle BQP&=m\angle CQR,\\ m\angle ARP&=\pi-m\angle CRQ.\end{aligned}$$

With a little arrangement, we have

$$\frac{\sinh(d_{\mathbb{H}}(P,A))}{\sinh(d_{\mathbb{H}}(P,B))}\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(Q,C))}\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(R,A))}=1.$$

The case where all intersections are external is proved similarly. □

2.2 Ceva's Theorem for Hyperbolic Triangles

Theorem 2.2 (Ceva's Theorem for Hyperbolic Triangles)**.** *If I is a point not on any side of a hyperbolic triangle $\triangle ABC$ such that AI intersects BC at Q, BI intersects AC at R, and CI intersects AB at P, then*

$$\frac{\sinh(d_{\mathbb{H}}(P,A))}{\sinh(d_{\mathbb{H}}(P,B))}\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(Q,C))}\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(R,A))}=1.$$

Proof. Applying Menelaus's Theorem to the hyperbolic triangle $\triangle AQC$ with the hyperbolic line passing through B,I,and R

$$\frac{\sinh(d_{\mathbb{H}}(I,A))}{\sinh(d_{\mathbb{H}}(I,Q))}\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(B,C))}\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(R,A))}=1.$$

Similarly, applying Menelaus's Theorem to the hyperbolic triangle $\triangle AQB$ with the hyperbolic line passing through C,I,and P

$$\frac{\sinh(d_{\mathbb{H}}(I,A))}{\sinh(d_{\mathbb{H}}(I,Q))}\frac{\sinh(d_{\mathbb{H}}(Q,C))}{\sinh(d_{\mathbb{H}}(B,C))}\frac{\sinh(d_{\mathbb{H}}(P,B))}{\sinh(d_{\mathbb{H}}(P,A))}=1.$$

Dividing these two expressions, we have

$$\frac{\sinh(d_{\mathbb{H}}(P,A))}{\sinh(d_{\mathbb{H}}(P,B))}\frac{\sinh(d_{\mathbb{H}}(Q,B))}{\sinh(d_{\mathbb{H}}(Q,C))}\frac{\sinh(d_{\mathbb{H}}(R,C))}{\sinh(d_{\mathbb{H}}(R,A))}=1.$$

□

3 Hyperbolic Tessellations

3.1 Tiling the hyperbolic plane with equilateral triangles

Suppose that there exists a regular tiling of $\mathbb{H}$ by equilateral triangles, so that k triangles meet at each vertex. Then $k > 6$, since each angle must be less than $\frac{\pi}{3}$. Let $\triangle ABC$ be a hyperbolic equilateral triangle with angles of $\dfrac{2\pi}{k}$ such that $k > 6$, $k \in \mathbb{Z}$. Let AH be the hyperbolic perpendicular line to the side BC with H being a point on BC. We let a be the length of each side. The Second Hyperbolic Law of Cosines states that,in a hyperbolic triangle $\triangle ABC$, with sides a,b,c opposite to the angles $\angle A, \angle B, \angle C$ respectively,

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cosh(a).$$

Applying the Second Hyperbolic Law of Cosines to the triangle AHB, we have

$$\begin{aligned}\cos\left(\frac{\pi}{k}\right) &= \sin\left(\frac{2\pi}{k}\right)\cosh\left(\frac{a}{2}\right)\\ &= 2\sin\left(\frac{\pi}{k}\right)\cos\left(\frac{\pi}{k}\right)\cosh\left(\frac{a}{2}\right).\end{aligned}$$

Since $k > 6$, we have $\sin\left(\frac{\pi}{k}\right) < \frac{1}{2}$. We conclude that

$$a = 2\,\text{arcCosh}\left(\frac{1}{2\sin\left(\frac{\pi}{k}\right)}\right).$$

From this equation, we have that the length of a side a is smallest when k is the smallest possible value, or $k = 7$.

Therefore, we have found that if an equilateral triangle tessellates the hyperbolic plane with k of these triangles meeting at each vertex, then the side length a of each triangle will be smallest with $a = 2\,\text{arcCosh}\left(\frac{1}{2\sin\left(\frac{\pi}{7}\right)}\right)$ when $k = 7$.

3.2 Tiling the hyperbolic plane with regular polygons

Let $A_1A_2A_3...A_n$ be a regular n-gon tiling the hyperbolic plane such that there are k n-gons meeting at each vertex, $n > 3$. Let O be the center of the regular n-gon. We can divide the regular n-gon into n hyperbolic isoceles triangles by drawing a hyperbolic line segment between O and each vertex. In $\triangle OA_1A_2$, construct a bisector of $\angle A_1OA_2$ from O that cuts A_1A_2 at H. We denote $a = a(n,k)$ as the side length of the n-gon. Using the Second Hyperbolic Law of Cosines in $\triangle OA_1H$, we have

$$\cos\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{k}\right)\cosh\left(\frac{a}{2}\right).$$

So

$$a(n,k) = 2\,\text{arcCosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{k}\right)}\right).$$

In addition, in $\triangle OA_1A_2$, we have, $m\angle OA_1A_2 + m\angle A_1OA_2 + m\angle A_1A_2O < \pi$. Thus, $\frac{2\pi}{k} + \frac{2\pi}{n} < \pi$, or $\frac{1}{k} + \frac{1}{n} < \frac{1}{2}$. Since n is fixed, we have $a(n,k)$ is smallest when k is the smallest possible value greater than 2 such that $k > \frac{2n}{n-2}$.

For instance, with

$$\begin{aligned}
n &= 4, k = 5 \\
n &= 5, k = 4 \\
n &= 6, k = 4 \\
n &= 7, k = 3 \\
n &\geq 7, k = 3.
\end{aligned}$$

For all $n \geq 7$, $2 < \frac{2n}{n-2} < 3$ since $\frac{2n-4}{n-2} < \frac{2n}{n-2} < \frac{3n-6}{n-2}$. We also have $k > \frac{2n}{n-2}$, so $k = 3$ is the smallest number of these n-gons meeting at each vertex.

Range of values of a: $a(n,k)$ is increasing with n, and

$$\lim_{n\to\infty} a(n,k) = \lim_{n\to\infty} 2\,\mathrm{arcCosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{k}\right)}\right)$$
$$= 2\,\mathrm{arcCosh}\left(\frac{1}{\sin\left(\frac{\pi}{k}\right)}\right).$$
$$\lim_{k\to\infty} a(n,k) = \lim_{k\to\infty} 2\,\mathrm{arcCosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{k}\right)}\right)$$
$$= \infty.$$

4 Periodic and Aperiodic Hyperbolic Tilings

In hyperbolic geometry, a **periodic hyperbolic tiling** is a tiling that has an infinite group of symmetries.
A uniform tiling is a vertex transitive tiling by regular polygons, which means for every two vertices, there exists an isometry of $\mathbb{H}^2$ mapping the tiling to itself such that the first vertex gets mapped to the second.
Symmetries are isometries of $\mathbb{H}^2$ that map the tiling to itself.

Theorem 4.1. *Every regular tiling by regular n-gons (k-regular n-gons meeting at each vertex) is a uniform tiling.*

Note: Every uniform tiling of $\mathbb{H}^2$ is a periodic tiling (but not vice versa).

Proof. Since we have infinite number of vertices and the tiling is uniform, there is an infinite number of symmetries.

□

A tiling of $\mathbb{H}^2$ is aperiodic if there does not exist a rearrangement of the set of tiles into another tiling of $\mathbb{H}^2$ that is periodic (in particular, the tiling is not periodic, using the identity as rearrangement). Note that for $\mathbb{R}^2$, a set of two tiles discovered by Roger Penrose was shown to be periodic in ([5]).

There exists a set of twenty-six tiles that can tile the hyperbolic plane such that the set of tiles does not admit a tiling with an infinite cyclic subgroup of symmetries; i.e. the tiling is aperiodic ([2]). This result, published in 2005, was the first example of an aperiodic set of tiles for the hyperbolic plane.

5 The Laplacian and its spectrum

5.1 Orthonomal bases of functions

Given a finite-dimensional complex vector space V with inner product, let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis. For any $v \in V$, we have $v = c_1e_1 + c_2e_2 + ... + c_ne_n$ with $c_i \in \mathbb{R}$ for $i = 1$ to n

$$
\begin{aligned}
&=> \langle v, e_j \rangle = \langle c_1e_1 + c_2e_2 + ...c_ne_n, e_j \rangle = c_j \langle e_j, e_j \rangle = c_j \\
&=> v = \sum_{j=1}^{n} \langle v, e_j \rangle e_j.
\end{aligned}
$$

Note that $\langle v, e_j \rangle e_j$ is the projection of v onto e_j.

Example: Parseval's Equality (or Parseval Identity) is

$$
\begin{aligned}
\langle v, v \rangle &= \left\langle \sum_{j=1}^{n} \langle v, e_j \rangle e_j, \sum_{k=1}^{h} \langle v, e_k \rangle e_k \right\rangle \\
&= \sum_{j=1}^{n} \left| \langle v, e_j \right|^2.
\end{aligned}
$$

Infinite-dimensional Hilbert spaces (Infinite-dimensional vector spaces with complete inner products) also have bases. For example, the vector space $L^2(S^1) = \{f : S^1 \to \mathbb{C} : \int_0^{2\pi} |f(x)|^2 d\theta < \infty\}$ has the inner product $\langle f, g \rangle = \int_0^{2\pi} f(\theta)\overline{g(\theta)} d\theta$. For this inner product, $\{e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}\}$ forms an orthonormal basis since

$$
\begin{aligned}
\langle e_k, e_l \rangle &= \int_0^{2\pi} \frac{1}{2\pi} e^{ikx} e^{-ilx} dx \\
&= \int_0^{2\pi} \frac{1}{2\pi} e^{i(k-l)x} dx.
\end{aligned}
$$

If $k = l$ then $\langle e_k, e_l \rangle = \displaystyle\int_0^{2\pi} \frac{1}{2\pi} dx = 1$.

If $k \neq l$ then

$$
\begin{aligned}
\langle e_k, e_l \rangle &= \frac{1}{2\pi} \frac{e^{ix(k-l)}}{i(k-l)} \Bigg|_0^{2\pi} \\
&= 0.
\end{aligned}
$$

The Fourier series of a $\mathbb{C}$-valued function f on S^1 can be obtained by expanding

$$
f = \sum_{k \in \mathbb{Z}} c_k e_k
$$

with $e_k = \frac{1}{\sqrt{2\pi}}e^{ikx}$. Then, as in the finite-dimensional case

$$\begin{aligned} c_k &= \int_{-\pi}^{\pi} f(x)\overline{e_k(x)}dx \\ &= \int_{-\pi}^{\pi} f(x)\overline{\frac{1}{\sqrt{2\pi}}e^{ikx}}dx \\ &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}f(x)e^{-ikx}dx. \end{aligned}$$

Similarly, every continuous function $u(x)$ on $[0, L]$ that vanishes at 0 and L can be written as $u(x) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}x\right)$ because $\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi x}{L}\right)\right\}_{k=1}^{\infty}$ forms an orthonormal basis of the vector space of such functions.
Check: If $m = k$

$$\begin{aligned} \int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi x}{L}\right)dx &= \int_0^L \frac{2}{L}\sin^2\left(\frac{k\pi x}{L}\right)dx \\ &= \int_0^L \frac{x}{L} - \frac{1}{2}sin\left(\frac{2k\pi x}{L}\right)dx \\ &= 1 - \frac{1}{2}sin(2k\pi) - 0 \\ &= 1 \end{aligned}$$

If $m \neq k$

$$\begin{aligned} \int_0^L \frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx &= \int_0^L \left(\cos\left(\frac{(m-k)\pi x}{L}\right) - \cos(\left(\frac{(m+k)\pi x}{L}\right)\right)dx \\ &= \left(\frac{L}{(m-k)\pi}\sin\left(\frac{(m-k)\pi x}{L}\right) - \frac{L}{(m+k)\pi}\sin(\left(\frac{(m+k)\pi x}{L}\right)\right|_0^L \\ &= 0 \end{aligned}$$

5.2 Spectrum of symmetric operators

Let V be a finite-dimensional vector space with complex inner product $\langle , \rangle$, and let A be a symmetric operator, meaning that $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$.
The Spectral Theorem follows:

Theorem 5.1 (Spectral Theorem)**.** *If A is a symmetric operator on the finite dimensional vector space V with complex inner product $\langle , \rangle$ then*

1. *All eigenvalues of A are real*

2. *A is diagonalizable, and*

3. You can choose eigenvectors of A so that they form an orthonormal basis of V,

4. Eigenvectors corresponding to different eigenvalues are automatically orthogonal.

Proof. 1.Suppose λ is an eigenvalue of A with eigenvector v then

$$Av = \lambda v, v \neq 0$$

$$\Rightarrow \langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle$$
$$\Rightarrow \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle$$

since $\langle v, v\rangle \neq 0, \lambda = \overline{\lambda}$, so λ is real.
4.Suppose v, w are eigenvectors of A corresponding to eigenvalues λ and μ where $\lambda \neq \mu$, then

$$\langle Av, w\rangle = \langle \lambda v, w\rangle = \lambda\langle v, w\rangle$$

However,

$$\langle Av, w\rangle = \langle v, Aw\rangle = \langle v, \mu w\rangle = \overline{\mu}\langle v, w\rangle = \mu\langle v, w\rangle.$$

So, $\langle v, w\rangle = 0$ To prove 2. and 3., take one eigenvalue λ with $Av = \lambda v, v \neq 0$. **Note**: every square matrix has at least one eigenvalue. Let $V^1 = \{w \in V : \langle w, v\rangle = 0\} \subseteq V$
Lemma: A maps V^1 to V^1 If $w \in V^1, \langle w, v\rangle = 0$. But then $\langle Aw, v\rangle = \langle w, Av\rangle = \langle w, \lambda v\rangle = \lambda\langle w, v\rangle = 0$. So if $w \in V^1, Aw \in V^1$.
Let W be the set of vectors orthogonal to v. Thus, we can start over with A being a linear transformation from W to itself.Then A restricted to W must have an eigenvalue, and for the same reason as before, it must be real. Then, if v is a unit eigenvector corresponding to that eigenvalue, the orthogonal complement of v inside W must be mapped to itself by the same reason as before. Thus, one can keep going until an orthonormal basis is formed of $\mathbb{C}^n$. □

The spectral theorem is also valid in infinite dimensions, but only if additional conditions are met.To construct all the eigenvalues and eigenvectors:
1) Find one eigenvector v, normalize it so $|v| = 1$. Restrict A to V^1.
2) Go back to 1) with V replaced by V^1.

The Laplacian is a differential operator on the vector space of functions that is actually symmetric. Intuitively, the Laplacian of f is a measure of the curvature or stress of a function f. It tells one how much the value of the function differs from its average value taken over the surrounding points. This is because it is the divergence of the gradient.
Laplacian $= \Delta f = -div(grad\ f)$.
In $\mathbb{R}^1$, the Laplacian is $\frac{-\partial^2}{\partial x^2}$.
In $\mathbb{R}^2$, the Laplacian is $\frac{-\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$.
In $\mathbb{R}^n$, the Laplacian is $\frac{-\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - ... - \frac{\partial^2}{\partial x_n^2}$.
The Laplacian on $\mathbb{H}^2$ (upper half plane model) is

$$(-y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -4(Im(z))^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

The Laplacian on the Disk Model

$$-\left(1-|z|^2\right)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=-4\left(1-|z|^2\right)^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}.$$

For general Riemannian metrics, there is a more general formula for the Laplacian. The Laplacian is used in several areas of physics.
Example: *Heat equation:* $u(x,y,t)$= temperature at time t at position (x,y) satisfies

$$\frac{\partial u}{\partial t}=-\Delta u.$$

Wave equation: $u(x,y,t)$= position of point on wave when vibrating at time t satisfies

$$\frac{\partial^2 u}{\partial t^2}=-\Delta u.$$

Given a polyhedron in $\mathbb{H}^2$, we have various boundary conditions for functions ψ defined on the polyhedron:

1. Dirichlet boundary condition: $\psi(z)=0$ when z is on boundary.
2. Neumann boundary condition: $\frac{\partial}{\partial n}\psi(z)=0$ when $z\in$ boundary. Here $\frac{\partial}{\partial n}$ means the outward normal derivative at points of the boundary.
3. Periodic boundary conditions on a polygon is equivalent to considering functions on $\mathbb{H}^2$ that satisfy a periodicity condition.

5.3 Selberg Conjecture and Fundamental Gap Conjectures

1. Selberg Conjecture: Consider the group
$\Gamma(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\Big|\, a,b,c,d\in\mathbb{Z}, ad-bc=1, a\equiv d\equiv 1, b\equiv c\equiv 0 \text{ mod N}\right\}$. $\Gamma(N)$ acts on $\mathbb{H}$ by $\begin{pmatrix} a & b\\ c & d\end{pmatrix} z=\frac{az+b}{cz+d}$. Let $X(N)$ be the space of bounded functions f on $\Gamma(N)\backslash\mathbb{H}$. Equivalently, they are the bounded, $\Gamma(N)$- periodic functions on $\mathbb{H}$ We define $\lambda_n(X(N))$ being the n^{th} smallest eigenvalue for the Laplacian on $X(N)$. Then, there is a lower bound for the first non-zero eigenvalue $\lambda_1(X(N))$ such that for $N\geq 1$

$$\lambda_1(X(N))\geq\frac{1}{4}.$$

2. Fundamental Gap Conjecture: Consider the Laplacian on a bounded convex domain Ω in $\mathbb{R}^n$ with Dirichlet boundary conditions. Given the eigenvalues listed in increasing order $0<\lambda_1(\Omega)<\lambda_2(\Omega)\leq\lambda_3(\Omega)\leq ...\to\infty$. Then the difference between the first two eigenvalues satisfies

$$\lambda_2-\lambda_1\geq\frac{3\pi^2}{d^2}$$

with d being the diameter of the convex domain Ω.

5.4 Methods for estimating the first two eigenvalues

The Spectral theorem tells us that eigenvalues are real and discrete (i.e. no accumulation points) ([6]). The inner product on functions on a domain G with Dirichlet boundary conditions ($f(x,y)=0$ if $(x,y)\in$ boundary) is

$$\langle f,g\rangle = \int_G f(x,y)\overline{g(x,y)}\frac{dxdy}{y^2}.$$

On $\mathbb{H}$, the Laplacian is

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right).$$

We have

$$\begin{aligned}\langle \Delta f,g\rangle &= \langle f,\Delta g\rangle\\ &= \langle \nabla f,\nabla g\rangle.\end{aligned}$$

$\forall f,g$ in the domain, where $\nabla f = -y^2\dfrac{\partial f}{\partial x} - y^2\dfrac{\partial f}{\partial y}$. We also have $\langle \partial_x,\partial_x\rangle = \langle \partial_y,\partial_y\rangle = \dfrac{1}{y^2}$

Proof: According to ([6]), the formula for the Laplcian Δ on Riemannian manifold with $(g_{ij}(x))$ metric

$$\Delta = \frac{-1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_i f\right)$$

with $\sqrt{g}=det(g_{ij}(x))$. We have (g^{ij}) is the inverse of (g_{ij}). Also

$$grad\ f = \nabla f = \sum g^{ij}(\partial_j f)\partial_i,$$

$$div\ v = \nabla\cdot v = \frac{-1}{\sqrt{g}}\partial_i\left(\sqrt{g}v_i\right).$$

In $\mathbb{H}$

$$\begin{aligned}g_{ij} &= \begin{pmatrix}\frac{1}{y^2} & 0\\ 0 & \frac{1}{y^2}\end{pmatrix},\\ g^{ij} &= \begin{pmatrix}y^2 & 0\\ 0 & y^2\end{pmatrix}.\end{aligned}$$

Green's identities formulas for Riemannian manifold

$$\begin{aligned}&\int_M (f\Delta g - g\Delta f)\\ =\ \ & -\int_{\partial_M} f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu}\end{aligned}$$

where $\dfrac{\partial}{\partial \nu}$ is the normal derivative. Under Dirichlet boundary conditions

$$f\Big|_{\partial_M} = 0,\ \ g\Big|_{\partial_M} = 0.$$

Under Neumann boundary conditions

$$\left.\frac{\partial f}{\partial \nu}\right|_{\partial_M} = 0, \left.\frac{\partial g}{\partial \nu}\right|_{\partial_M} = 0.$$

The Laplacian is symmetric and is an elliptic differential operator of order 2. If f is an eigenfunction with eigenvalue λ,

$$\langle \Delta f, f\rangle = \langle \lambda f, f\rangle = \lambda\langle f, f\rangle.$$

we also have $\langle \nabla f, \nabla f\rangle \geq 0$ and $\langle f, f\rangle \geq 0$. So $\lambda \geq 0$.

5.5 Approximation of Hyperbolic Laplacian Eigenvalues

We will now investigate the first eigenvalue of the Dirichlet Laplacian on hyperbolic triangles. We construct the formula for building a hyperbolic equilateral triangle. Let $\triangle$ ABC be a hyperbolic equilateral with $\angle A = \angle B = \angle C = \alpha$ and sides $a = b = c$ be the opposite to the angle at A,B,C. Using the hyperbolic Law of Cosines II, we have

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cosh(a).$$

so,

$$\begin{aligned} cosh(a) &= \frac{\cos\alpha + \cos^2\alpha}{\sin^2\alpha} \\ a &= \text{arcCosh}\left(\frac{\cos\alpha + \cos^2\alpha}{\sin^2\alpha}\right). \end{aligned}$$

We construct an isomorphic image of ABC in the complex coordinate system. Let $B = (0, 1)$, let y_0 be A's coordinate on the y-axis. We have $y(t) = t$ with t from 1 to y_0 is the function of the y-axis ranging from A to B.

$$AB = a = \int_1^{y_0} \frac{1}{y(t)}dt = \ln(y_0).$$

Let $A = y_0 = e^a$. To determine the coordinate of C, we have B is a point on the Euclidean semi-circle in $\mathbb{H}$ with the center $O_1 = (\cot\alpha, 0)$ and A is a point on the Euclidean semi-circle in $\mathbb{H}$ with the center $O_2 = (-\exp(a)\cot\alpha, 0)$. We denote r_1 and r_2 as the length of the radius of the two circles (O_1) and (O_2) respectively. We have

$$\begin{aligned} r_1 &= \exp(a)\csc(\alpha) \\ r_2 &= \csc(\alpha). \end{aligned}$$

Solving the system two equation we get the coordinates of C.

$$\begin{aligned} (x_C - \cot(\alpha))^2 + y_C^2 &= r_1^2 \\ (x_C + e^a\cot(\alpha))^2 + y_C^2 &= r_2^2. \end{aligned}$$

Thus,

$$x_C = \frac{r_1^2 - r_2^2 + (e^a \cot(\alpha))^2 + (\cot(\alpha))^2}{2(-\exp(a)\cot(\alpha) - \cot(\alpha))}.$$
$$y_C = \sqrt{r_1^2 - (x_C - \cot(\alpha))^2}.$$

Let z_1 and z_2 be the parametrization of the two circles (O_1) and (O_2), we have

$$z_1 = \cot\alpha + r_1(-\cos\theta + i\sin\theta)$$

with $\alpha < \theta < \arctan\left(\dfrac{y_C}{\cot(\alpha) - x_C}\right)$.

$$z_2 = -e^a \cot(\alpha) + r_2(\cos(\theta) + i\sin(\theta))$$

with $\arctan\left(\dfrac{y_C}{e^a\cot(\alpha) + x_C}\right) < \theta < \alpha$.

For any linear transformation L, the eigenvalues of L are the critical values of

$$f \to \frac{\langle Lv, v\rangle}{\langle v, v\rangle}.$$

In particular, the smallest critical value is λ_1

The Laplacian from space of all functions

$$f \to \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle}.$$

λ_1, the first eigenvalue, is the minimum critical value of $\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle}$. λ_2, the second eigenvalue is the next critical value.

Using Green's theorem and the Rayleigh Quotient, we have the eigenvalues λ_1 of the Laplacian is the minimum value of $R(f)$ among all f satisfying the boundary conditions.

$$R(f) = \frac{\int\int \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \right) dxdy}{\int\int \left(\frac{1}{y^2} f^2\right) dxdy}$$

We start with hyperbolic triangle $\triangle ABC$ with $A = (x_1, y_1) = (0, e)$, $B = (x_2, y_2) = (0, 1)$, and $C = (x_0, y_0)$. By changing the coordinates of C, or x_0, and y_0 and using Dirichlet boundary conditions, we can estimate the change in the first two eigenvalues and their difference.

We have the radii of the circles that pass through A, C and B, C respectively are

$$r_2 = \sqrt{e^2 + \frac{1}{4}\Big(\frac{e^2 - x_0^2 - y_0^2}{-x_0}\Big)},$$

$$r_3 = \sqrt{1 + \frac{1}{4}\Big(\frac{1 - x_0^2 - y_0^2}{-x_0}\Big)}.$$

The center of the circles that pass through A, C and B, C respectively are

$$c_2 = \frac{1}{2}\frac{e^2 - x_0^2 - y_0^2}{-x_0},$$

$$c_3 = \frac{1}{2}\frac{1 - x_0^2 - y_0^2}{-x_0}.$$

The formulas for AC and BC respectively are:

$$s(x) = \sqrt{r_2^2 - (x - c_2)^2},$$

$$t(x) = \sqrt{r_3^2 - (x - c_3)^2}.$$

The transformation (u, v) turns hyperbolic $\triangle ABC$ to hyperbolic $\triangle ABC'$. $C' = (x_3, y_3)$ with $x_3 = x_0 + \Delta x$ and $y_3 = y_0 + \Delta y$. The formulas for AC' and BC' respectively are:

$$h(x) = \sqrt{r_{2N}^2 - (x - c_{2N})^2},$$

$$g(x) = \sqrt{r_{3N}^2 - (x - c_{3N})^2},$$

where r_{2N} and r_{3N} are the radii of the circles that pass through A, C' and B, C' respectively. The numbers c_{2N} and c_{3N} are the x-coordinates of the centers of the circles that pass through A, C and B, C. We have

$$r_{2N} = \sqrt{e^2 + \frac{1}{4}\Big(\frac{e^2 - x_3^2 - y_3^2}{-x_3}\Big)},$$

$$r_{3N} = \sqrt{1 + \frac{1}{4}\Big(\frac{1 - x_3^2 - y_3^2}{-x_3}\Big)},$$

$$c_{2N} = \frac{1}{2}\frac{e^2 - x_3^2 - y_3^2}{-x_3},$$

$$c_{3N} = \frac{1}{2}\frac{1 - x_3^2 - y_3^2}{-x_3}.$$

Let $(x, y) \mapsto (u, v)$ be the transformations that change $\triangle ABC$ to $\triangle ABC'$. We have

$$u = \frac{xx_0}{x_3},$$

$$v = t\left(\frac{xx_0}{x_3}\right) + \left(\frac{y - g(x)}{h(x) - g(x)}\right)\left(s\left(\frac{xx_0}{x_3}\right) - \left(\frac{xx_0}{x_3}\right)\right).$$

By changing the coordinates, we have the Rayleigh quotient:

$$R(f) = \frac{\int_0^{x_3}\int_{g(x)}^{h(x)} \left(\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right)(x,y) \left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right) - \left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right) dydx}{\int_0^{x_3}\int_{g(x)}^{h(x)} \frac{1}{v^2(x,y)} f^2(u(x,y), v(x,y) \left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right) - t\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right) dydx}.$$

Let $C(x_0+\Delta x, y_0+\Delta y)$ be the third vertex of the hyperbolic triange whose other vertices are $(0,1)$ and $(0,e)$. Then

$$x_3 = x_0 + \Delta x,$$

$$y_3 = y_0 + \Delta y.$$

We use Taylor series to estimate the change in the Rayleigh quotient with respect to Δx and Δy in order to obtain a differential equation for the level curve of λ_1 as a function of (x_0, y_0). We have

$$s\left(\frac{xx_0}{x_3}\right) = s(x) + \Delta x \frac{x - c_2}{\sqrt{r_2^2 - (x-c_2)^2}} \frac{-xx_0}{x_0^2},$$

$$t\left(\frac{xx_0}{x_3}\right) = t(x) + \Delta x \frac{x - c_3}{\sqrt{r_3^2 - (x-c_3)^2}} \frac{-xx_0}{x_0^2},$$

$$r_{2N} = \sqrt{e^2 + \frac{1}{4}\Big(\frac{e^2 - x_0^2 - y_0^2}{-x_0}\Big)} + \Delta x \frac{\frac{e^2+x_0^2-y_0^2}{x_0^2}}{8\sqrt{e^2 + \frac{1}{4}\left(\frac{e^2-x_0^2-y_0^2}{-x_0}\right)}} + \Delta y \frac{y_0}{4x_0\sqrt{e^2 + \frac{1}{4}\left(\frac{e^2-x_0^2-y_0^2}{-x_0}\right)}},$$

$$r_{3N} = \sqrt{1 + \frac{1}{4}\Big(\frac{1 - x_0^2 - y_0^2}{-x_0}\Big)} + \Delta x \frac{\frac{1+x_0^2-y_0^2}{x_0^2}}{8\sqrt{1 + \frac{1}{4}\left(\frac{1-x_0^2-y_0^2}{-x_0}\right)}} + \Delta y \frac{y_0}{4x_0\sqrt{1 + \frac{1}{4}\left(\frac{1-x_0^2-y_0^2}{-x_0}\right)}},$$

$$c_{2N} = \frac{1}{2}\frac{e^2 - x_0^2 - y_0^2}{-x0} + \Delta x \left(\frac{e^2 + x_0^2 - y0^2}{2x_0^2}\right) + \Delta y \left(\frac{y_0}{x_0}\right),$$

$$c_{3N} = \frac{1}{2}\frac{1 - x_0^2 - y_0^2}{-x0} + \Delta x \left(\frac{1 + x_0^2 - y0^2}{2x_0^2}\right) + \Delta y \left(\frac{y_0}{x_0}\right),$$

$$\begin{aligned}
h(x) &= \sqrt{r_{2N}^2(x_0,y_0) - (x - c_{2N}(x_0,y_0))^2} + \Delta x \frac{r_{2N}\frac{\partial r_{2N}}{\partial \Delta x} + (x - c_{2N})\frac{\partial c_{2N}}{\partial \Delta x}}{\sqrt{r_{2N}^2(x_0,y_0) - (x - c_{2N}(x_0,y_0))^2}} \\
&\quad + \Delta y \frac{r_{2N}\frac{\partial r_{2N}}{\partial \Delta y} + (x - c_{2N})\frac{\partial c_{2N}}{\partial \Delta y}}{\sqrt{r_{2N}^2(x_0,y_0) - (x - c_{2N}(x_0,y_0))^2}}, \\
g(x) &= \sqrt{r_{3N}^2(x_0,y_0) - (x - c_{3N}(x_0,y_0))^2} + \Delta x \frac{r_{3N}\frac{\partial r_{3N}}{\partial \Delta x} + (x - c_{3N})\frac{\partial c_{3N}}{\partial \Delta x}}{\sqrt{r_{3N}^2(x_0,y_0) - (x - c_{3N}(x_0,y_0))^2}} \\
&\quad + \Delta y \frac{r_{3N}\frac{\partial r_{3N}}{\partial \Delta y} + (x - c_{3N})\frac{\partial c_{3N}}{\partial \Delta y}}{\sqrt{r_{3N}^2(x_0,y_0) - (x - c_{3N}(x_0,y_0))^2}},
\end{aligned}$$

Let

$$
\begin{aligned}
\alpha_s &= \frac{x-c_2}{\sqrt{r_2^2-(x-c_2)^2}}\frac{-xx_0}{x_0^2}, \\
\alpha_t &= \frac{x-c_3}{\sqrt{r_3^2-(x-c_3)^2}}\frac{-xx_0}{x_0^2}, \\
\alpha_{r_{2N}} &= \frac{\frac{e^2+x_0^2-y_0^2}{x_0^2}}{8\sqrt{e^2+\frac{1}{4}\left(\frac{e^2-x_0^2-y_0^2}{-x_0}\right)}}, \\
\beta_{r_{2N}} &= \frac{y_0}{4x_0\sqrt{e^2+\frac{1}{4}\left(\frac{e^2-x_0^2-y_0^2}{-x_0}\right)}}, \\
\alpha_{r_{3N}} &= \frac{\frac{1+x_0^2-y_0^2}{x_0^2}}{8\sqrt{1+\frac{1}{4}\left(\frac{1-x_0^2-y_0^2}{-x_0}\right)}}, \\
\beta_{r_{3N}} &= \frac{y_0}{4x_0\sqrt{1+\frac{1}{4}\left(\frac{1-x_0^2-y_0^2}{-x_0}\right)}}, \\
\alpha_{c_{2N}} &= \left(\frac{e^2+x_0^2-y_0^2}{2x_0^2}\right), \\
\beta_{c_{2N}} &= \left(\frac{y_0}{x_0}\right), \\
\alpha_{c_{3N}} &= \left(\frac{1+x_0^2-y_0^2}{2x_0^2}\right), \\
\beta_{c_{3N}} &= \left(\frac{y_0}{x_0}\right), \\
\alpha_h &= \frac{r_{2N}\frac{\partial r_{2N}}{\partial \Delta x}+(x-c_{2N})\frac{\partial c_{2N}}{\partial \Delta x}}{\sqrt{r_{2N}^2(x_0,y_0)-(x-c_{2N}(x_0,y_0))^2}}, \\
\beta_h &= \frac{r_{2N}\frac{\partial r_{2N}}{\partial \Delta y}+(x-c_{2N})\frac{\partial c_{2N}}{\partial \Delta y}}{\sqrt{r_{2N}^2(x_0,y_0)-(x-c_{2N}(x_0,y_0))^2}}, \\
\alpha_g &= \frac{r_{3N}\frac{\partial r_{3N}}{\partial \Delta x}+(x-c_{3N})\frac{\partial c_{3N}}{\partial \Delta x}}{\sqrt{r_{3N}^2(x_0,y_0)-(x-c_{3N}(x_0,y_0))^2}}, \\
\beta_g &= \frac{r_{3N}\frac{\partial r_{3N}}{\partial \Delta y}+(x-c_{3N})\frac{\partial c_{3N}}{\partial \Delta y}}{\sqrt{r_{3N}^2(x_0,y_0)-(x-c_{3N}(x_0,y_0))^2}}.
\end{aligned}
$$

We have

$$\begin{aligned}
\frac{y-g(x)}{h(x)-g(x)} &= \frac{y-g(x_0,y_0)-\frac{\Delta x}{2}\alpha_g-\frac{\Delta y}{2}\beta_g}{h(x_0,y_0)-g(x_0,y_0)+\frac{\Delta x}{2}(\alpha_h-\alpha g)+\frac{\Delta y}{2}(\beta_h-\beta g)} \\
&= \left(y-g(x_0,y_0)-\frac{\Delta x}{2}\alpha_g-\frac{\Delta u}{2}\beta_g\right)\left(\frac{1}{h(x_0,y_0)-g(x_0,y_0)}\right. \\
&\quad \left.+\frac{1}{(h(x_0,y_0)-g(x_0,y_0))^2}\left(\frac{\Delta x}{2}(\alpha_h-\alpha_g)+\frac{\Delta y}{2}(\beta_h-\beta_g)\right)\right) \\
&= \frac{y-g(x_0,y_0)}{h(x_0,y_0)-g(x_0,y_0)}-\frac{\Delta x\alpha_g}{2(h(x_0,y_0)-g(x_0,y_0))}-\frac{\Delta y\beta_g}{2(h(x_0,y_0)-g(x_0,y_0))} \\
&\quad +\frac{\Delta x(y-g(x_0,y_0))}{2(h(x_0,y_0)-g(x_0,y_0))^2}(\alpha_h-\alpha_g)+\frac{\Delta y(y-g(x_0,y_0))}{2(h(x_0,y_0)-g(x_0,y_0))^2}(\beta_h-\beta_g).
\end{aligned}$$

Let

$$h_0 = h(x_0,y_0) = s(x),$$
$$g_0 = g(x_0,y_0) = t(x).$$

So

$$\begin{aligned}
&\frac{y-g(x)}{h(x)-g(x)}\left(s\left(\frac{xx_0}{x_3}\right)-t\left(\frac{xx_0}{x_3}\right)\right) \\
=\ &\frac{y-g(x)}{h(x)-g(x)}\left(s(x)-t(x)-\Delta x s'(x)\frac{xx_0}{x_0^2}+\Delta x t'(x)\frac{xx_0}{x_0^2}\right) \\
=\ &\frac{y-g_0}{h_0-g_0}(s(x)-t(x))+\frac{y-g_0}{h_0-g_0}\Delta x\left(\alpha_s-\alpha_t\right)+\Delta x(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\alpha_g\right. \\
&\left.+\frac{y-g_0}{2(h_0-g_0)^2}(\alpha_h-\alpha_g)\right)+\Delta y(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\beta_g+\frac{y-g_0}{2(h_0-g_0)^2}(\beta_h-\beta_g)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
v(x) &= t(x)+\Delta x\alpha_t+\frac{y-g_0}{h_0-g_0}(s(x)-t(x))+\frac{y-g_0}{h_0-g_0}\Delta x\left(\alpha_s+\alpha_t\right)+\Delta x(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\alpha_g\right. \\
&\quad \left.+\frac{y-g_0}{2(h_0-g_0)^2}(\alpha_h-\alpha_g)\right)+\Delta y(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\beta_g+\frac{y-g_0}{2(h_0-g_0)^2}(\beta_h-\beta_g)\right) \\
&= t(x)+\frac{y-g_0}{h_0-g_0}(s(x)-t(x))+\Delta x\left(\alpha_t+\frac{y-g_0}{h_0-g_0}(\alpha_s+\alpha_t)+(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\alpha_g\right.\right. \\
&\quad \left.\left.+\frac{y-g_0}{2(h_0-g_0)^2}(\alpha_h-\alpha_g)\right)\right)+\Delta y(s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\beta_g+\frac{y-g_0}{2(h_0-g_0)^2}(\beta_h-\beta_g)\right).
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_v &= \alpha_t + \frac{y-g_0}{h_0-g_0}(\alpha_s+\alpha_t) + (s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\alpha_g + \frac{y-g_0}{2(h_0-g_0)^2}(\alpha_h-\alpha_g)\right),\\
\beta_v &= (s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\beta_g + \frac{y-g_0}{2(h_0-g_0)^2}(\beta_h-\beta_g)\right),\\
\alpha_u &= \frac{-x}{x_0},\\
\beta_u &= 0.
\end{aligned}$$

Using Taylor series for approximation, we also have

$$\frac{x_0}{x_3} = \frac{x_0}{x_0+\Delta x} = 1 - \frac{\Delta x}{x_0^2}.$$

$$\begin{aligned}
&\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right) - t\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\\
&= \left(1-\frac{\Delta x}{x_0^2}\right)\left(\frac{s(x)-t(x)+\Delta x(\alpha_s-\alpha_t)}{h_0-g_0+\Delta x(\alpha_h-\alpha_g)+\Delta y(\beta_h-\beta_g)}\right)\\
&= \left(1-\frac{\Delta x}{x_0^2}\right)\left(1-\frac{s(x)-t(x)+\Delta x(\alpha_s-\alpha_t)}{(h_0-g_0)^2}\left(\Delta x(\alpha_h-\alpha_g)+\Delta y(\beta_h-\beta_g)\right)\right)\\
&= \left(1-\frac{\Delta x}{x_0^2}\right)\left(1-\Delta x\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\alpha_h-\alpha_g) - \Delta y\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\beta_h-\beta_g)\right)\\
&= 1-\Delta x\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\alpha_h-\alpha_g) - \Delta y\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\beta_h-\beta_g) - \Delta x\frac{s(x)-t(x)}{x_0^2(h_0-g_0)}\\
&= 1-\Delta x\left(\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\alpha_h-\alpha_g) + \frac{s(x)-t(x)}{x_0^2(h_0-g_0)}\right) - \Delta y\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\beta_h-\beta_g)\\
&= 1-\Delta x\left(\left(\frac{1}{s(x)-t(x)}\right)(\alpha_h-\alpha_g) + \frac{1}{x_0^2}\right) - \Delta y\left(\frac{1}{s(x)-t(x)}\right)(\beta_h-\beta_g).
\end{aligned}$$

$$\frac{1}{v^2(x,y)} = \frac{1}{v(x_0,y_0)^2} - \Delta x\frac{2}{v(x_0,y_0)^3} - \Delta y\frac{2}{v(x_0,y_0)^3}$$

Let $u_0 = u(x_0, y_0)$ (when $\Delta x = 0$),$v_0 = v(x_0, y_0)$, and $f_0 = f(u_0, v_0)$. We have

$$
\begin{aligned}
& \frac{1}{v^2(x,y)} f^2(u(x,y), v(x,y) \left(\frac{x_0}{x_3} \left(\frac{s\left(\frac{xx_0}{x_3}\right) + t\left(\frac{xx_0}{x_3}\right)}{h(x) - g(x)} \right) \right) \\
= \; & \left(\frac{1}{v_0^2} - \Delta x \frac{2}{v_0^3} - \Delta y \frac{2}{v_0^3} \right) \left(f_0^{@} + 2\Delta x f_0 \left(\frac{\partial f}{\partial u} \alpha_u + \frac{\partial f}{\partial v} \alpha_v \right) + 2\Delta y f_0 \left(\frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial v} \beta_v \right) \right) \Bigg(1 \\
& - \Delta x \left(\left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\alpha_h - \alpha_g) + \frac{s(x) - t(x)}{x_0^2 (h_0 - g_0)} \right) - \Delta y \left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\beta_h - \beta_g) \Bigg) \\
= \; & \left(\frac{f_0^2}{v_0^2} - \Delta x \frac{2 f_0}{v_0^3} - \Delta y \frac{2 f_0}{v_0^3} + \frac{2\Delta x f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \alpha_u + \frac{\partial f}{\partial v} \alpha_v \right) + \frac{2 \Delta y f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial v} \beta_v \right) \right) \Bigg(1 \\
& - \Delta x \left(\left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\alpha_h - \alpha_g) + \frac{s(x) - t(x)}{x_0^2 (h_0 - g_0)} \right) - \Delta y \left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\beta_h - \beta_g) \Bigg) \\
= \; & \frac{f_0^2 (s(x) - t(x))}{v_0^2 (h_0 - g_0)} - \Delta x \frac{f_0^2}{v_0^2} \left(\left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\alpha_h - \alpha_g) + \frac{s(x) - t(x)}{x_0^2 (h_0 - g_0)} \right) \\
& - \Delta y \frac{f_0^2}{v_0^2} \left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\beta_h - \beta_g) + \Delta x \frac{(s(x) - t(x))}{(h_0 - g_0)} \left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \alpha_u + \frac{\partial f}{\partial v} \alpha_v \right) - \frac{2 f_0}{v_0^3} \right) \\
& + \Delta y \frac{(s(x) - t(x))}{(h_0 - g_0)} \left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial v} \beta_v \right) - \frac{2 f_0}{v_0^3} \right) \\
= \; & \frac{f_0^2}{v_0^2} + \Delta x \Bigg(\frac{(s(x) - t(x))}{(h_0 - g_0)} \left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \alpha_u + \frac{\partial f}{\partial v} \alpha_v \right) - \frac{2 f_0}{v_0^3} \right) - \frac{f_0^2}{v_0^2} \Bigg(\left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\alpha_h - \alpha_g) \\
& + \frac{s(x) - t(x)}{x_0^2 (h_0 - g_0)} \Bigg) \Bigg) + \Delta y \Bigg(\frac{(s(x) - t(x))}{(h_0 - g_0)} \left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial v} \beta_v \right) - \frac{2 f_0}{v_0^3} \right) \\
& - \frac{f_0^2}{v_0^2} \left(\frac{s(x) - t(x)}{(h_0 - g_0)^2} \right) (\beta_h - \beta_g) \Bigg) \\
= \; & \frac{f_0^2}{v_0^2} + \Delta x \Bigg(\left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \alpha_u + \frac{\partial f}{\partial v} \alpha_v \right) - \frac{2 f_0}{v_0^3} \right) - \frac{f_0^2}{v_0^2} \left(\left(\frac{1}{s(x) - t(x)} \right) (\alpha_h - \alpha_g) + \frac{1}{x_0^2 (s(x) - t(x))} \right) \Bigg) \\
& + \Delta y \Bigg(\left(\frac{2 f_0}{v_0^2} \left(\frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial v} \beta_v \right) - \frac{2 f_0}{v_0^3} \right) - \frac{f_0^2}{v_0^2} \left(\frac{1}{s(x) - t(x)} \right) (\beta_h - \beta_g) \Bigg).
\end{aligned}
$$

We also have

$$\left(\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right)(x,y)\left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right)+t\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right)$$

$$\begin{aligned}
= & \left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2+\Delta x\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta x}\right)+\Delta x\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta x}\right)\right. \\
& \left.+\Delta y\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta y}\right)+\Delta y\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta y}\right)\right)\left(1\right. \\
& \left.-\Delta x\left(\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\alpha_h-\alpha_g)+\frac{s(x)-t(x)}{x_0^2(h_0-g_0)}\right)-\Delta y\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\beta_h-\beta_g)\right) \\
= & \left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)-\Delta x\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\alpha_h-\alpha_g)+\frac{s(x)-t(x)}{x_0^2(h_0-g_0)}\right) \\
& -\Delta y\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\frac{s(x)-t(x)}{(h_0-g_0)^2}\right)(\beta_h-\beta_g)+\Delta x\left(\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta x}\right)\right. \\
& \left.+\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta x}\right)\right)+\Delta y\left(\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta y}\right)\right. \\
& \left.+\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta y}\right)\right) \\
= & \left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)+\Delta x\left[\left(\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta x}\right)+\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta x}\right)\right)\right. \\
& \left.-\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\left(\frac{1}{s(x)-t(x)}\right)(\alpha_h-\alpha_g)-\frac{1}{x_0^2}\right)\right] \\
& +\Delta y\left[\left(\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial \Delta y}\right)+\frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial \Delta y}\right)\right)\right. \\
& \left.-\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\frac{1}{s(x)-t(x)}\right)(\beta_h-\beta_g)\right]
\end{aligned}$$

with

$$
\begin{aligned}
\frac{\partial u}{\partial \Delta x} &= \frac{-x x_0}{x_0^2}, \\
\frac{\partial u}{\partial \Delta y} &= 0, \\
\frac{\partial v}{\partial \Delta x} &= \alpha_t + \frac{y-g_0}{h_0-g_0}(\alpha_s+\alpha_t) + (s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\alpha_g + \frac{y-g_0}{2(h_0-g_0)^2}(\alpha_h-\alpha_g)\right), \\
\frac{\partial v}{\partial \Delta y} &= (s(x)-t(x))\left(\frac{-1}{2(h_0-g_0)}\beta_g + \frac{y-g_0}{2(h_0-g_0)^2}(\beta_h-\beta_g)\right).
\end{aligned}
$$

Let

$$
\begin{aligned}
N_x &= \left[\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta x} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial \Delta x}\right) + \frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta x} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial \Delta x}\right)\right] \\
&\quad - \left(\left(\frac{\partial f}{\partial u_0}\right)^2 + \left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\left(\frac{1}{s(x)-t(x)}\right)(\alpha_h-\alpha_g) - \frac{1}{x_0^2}\right), \\
N_y &= \left(\frac{2\partial f}{\partial u}\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial \Delta y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial \Delta y}\right) + \frac{2\partial f}{\partial v}\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial \Delta y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial \Delta y}\right)\right) \\
&\quad - \left(\left(\frac{\partial f}{\partial u_0}\right)^2 + \left(\frac{\partial f}{\partial v_0}\right)^2\right)\left(\frac{1}{(s(x)-t(x))}\right)(\beta_h-\beta_g), \\
D_x &= \left(\frac{2f_0}{v_0^2}\left(\frac{\partial f}{\partial u}\alpha_u + \frac{\partial f}{\partial v}\alpha_v\right) - \frac{2f_0}{v_0^3}\right) - \frac{f_0^2}{v_0^2}\left(\left(\frac{1}{s(x)-t(x)}\right)(\alpha_h-\alpha_g) + \frac{1}{x_0^2(s(x)-t(x))}\right), \\
D_y &= \left(\frac{2f_0}{v_0^2}\left(\frac{\partial f}{\partial u}\beta_u + \frac{\partial f}{\partial v}\beta_v\right) - \frac{2f_0}{v_0^3}\right) - \frac{f_0^2}{v_0^2}\left(\frac{1}{s(x)-t(x)}\right)(\beta_h-\beta_g).
\end{aligned}
$$

We have

$$
\begin{aligned}
& \int_0^{x_3}\int_{g(x)}^{h(x)} \left(\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right)(x,y)\left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right) - \left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right) dydx \\
= & \int_0^{x_0+\Delta x}\int_{t(x)+\Delta x\alpha_h+\Delta y\beta_h}^{s(x)+\Delta x\alpha_g+\Delta y\beta_g} \left(\left(\frac{\partial f}{\partial u_0}\right)^2 + \left(\frac{\partial f}{\partial v_0}\right)^2\right) + \Delta x N_x + \Delta y N_y dydx \\
= & \int_0^{x_0}\int_{t(x)}^{s(x)} \left(\left(\frac{\partial f}{\partial u_0}\right)^2 + \left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx + \Delta x\left(\int_0^{x_0}\int_{t(x)}^{s(x)} N_x dydx\right) \\
& +\Delta y\left(\int_0^{x_0}\int_{t(x)}^{s(x)} N_y dydx\right) + \Delta x\int_{t(x)}^{s(x)} f(x_0,y)dy + \alpha_g\Delta x\int_0^{x_0} f(x,s(x))dx \\
& -\alpha_h\Delta x\int_0^{x_0} f(x,t(x))dx + \beta_g\Delta y\int_0^{x_0} f(x,s(x))dx - \beta_h\Delta y\int_0^{x_0} f(x,t(x))dx \\
= & \int_0^{x_0}\int_{t(x)}^{s(x)} \left(\left(\frac{\partial f}{\partial u_0}\right)^2 + \left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx + \Delta x\Bigg(\int_0^{x_0}\int_{t(x)}^{s(x)} N_x dydx \\
& +\int_{t(x)}^{s(x)} f(x_0,y)dy + \alpha_g\int_0^{x_0} f(x,s(x))dx - \alpha_h\int_0^{x_0} f(x,t(x))dx\Bigg) \\
& +\Delta y\Bigg(\int_0^{x_0}\int_{t(x)}^{s(x)} N_y dydx + \beta_g\int_0^{x_0} f(x,s(x))dx - \beta_h\int_0^{x_0} f(x,t(x))dx\Bigg).
\end{aligned}
$$

Also,

$$
\begin{aligned}
& \int_0^{x_3}\int_{g(x)}^{h(x)} \frac{1}{v^2(x,y)} f^2(u(x,y),v(x,y)\left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right)+t\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right) dydx \\
= & \int_0^{x_0+\Delta x}\int_{t(x)+\Delta x\alpha_h+\Delta y\beta_h}^{s(x)+\Delta x\alpha_g+\Delta y\beta_g} \left(\frac{f_0^2}{v_0^2}+\Delta x D_x+\Delta y D_y\right) dydx \\
= & \int_0^{x_0}\int_{t(x)}^{s(x)} \frac{f_0^2}{v_0^2} dydx + \Delta x\left(\int_0^{x_0}\int_{t(x)}^{s(x)} D_x dydx\right) + \Delta y\left(\int_0^{x_0}\int_{t(x)}^{s(x)} D_y dydx\right) \\
& +\Delta x \int_{t(x)}^{s(x)} f(x_0,y)dy + \alpha_g \Delta x \int_0^{x_0} f(x,s(x))dx \\
& -\alpha_h \Delta x \int_0^{x_0} f(x,t(x))dx + \beta_g \Delta y \int_0^{x_0} f(x,s(x))dx - \beta_h \Delta y \int_0^{x_0} f(x,t(x))dx \\
= & \int_0^{x_0}\int_{t(x)}^{s(x)} \frac{f_0^2}{v_0^2} dydx + \Delta x\Bigg(\int_0^{x_0}\int_{t(x)}^{s(x)} D_x dydx + \int_{t(x)}^{s(x)} f(x_0,y)dy \\
& +\alpha_g \int_0^{x_0} f(x,s(x))dx - \alpha_h \int_0^{x_0} f(x,t(x))dx\Bigg) + \Delta y\Bigg(\int_0^{x_0}\int_{t(x)}^{s(x)} D_y dydx \\
& +\beta_g \int_0^{x_0} f(x,s(x))dx - \beta_h \int_0^{x_0} f(x,t(x))dx\Bigg).
\end{aligned}
$$

Let

$$
\begin{aligned}
P_x &= \int_0^{x_0}\int_{t(x)}^{s(x)} N_x dydx + \int_{t(x)}^{s(x)} f(x_0,y)dy + \alpha_g \int_0^{x_0} f(x,s(x))dx - \alpha_h \int_0^{x_0} f(x,t(x))dx, \\
P_y &= \int_0^{x_0}\int_{t(x)}^{s(x)} N_y dydx + \beta_g \int_0^{x_0} f(x,s(x))dx - \beta_h \int_0^{x_0} f(x,t(x))dx, \\
Q_x &= \int_0^{x_0}\int_{t(x)}^{s(x)} D_x dydx + \int_{t(x)}^{s(x)} f(x_0,y)dy + \alpha_g \int_0^{x_0} f(x,s(x))dx - \alpha_h \int_0^{x_0} f(x,t(x))dx, \\
Q_y &= \int_0^{x_0}\int_{t(x)}^{s(x)} D_y dydx + \beta_g \int_0^{x_0} f(x,s(x))dx - \beta_h \int_0^{x_0} f(x,t(x))dx.
\end{aligned}
$$

We have

$$
\begin{aligned}
R(f) &= \frac{\int_0^{x_3}\int_{g(x)}^{h(x)}\left(\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right)(x,y)\left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right)-\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right)dydx}{\int_0^{x_3}\int_{g(x)}^{h(x)}\frac{1}{v^2(x,y)}f^2(u(x,y),v(x,y)\left(\frac{x_0}{x_3}\left(\frac{s\left(\frac{xx_0}{x_3}\right)-t\left(\frac{xx_0}{x_3}\right)}{h(x)-g(x)}\right)\right)dydx} \\
&= \frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx+\Delta xP_x+\Delta yP_y}{\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx+\Delta xQ_x+\Delta yQ_y} \\
&= \frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx+\Delta xP_x+\Delta yP_y}{\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx} \\
&\quad -\Delta xQ_x\frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx+\Delta xP_x+\Delta yP_y}{\left(\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx\right)^2} \\
&\quad -\Delta yQ_y\frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx+\Delta xP_x+\Delta yP_y}{\left(\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx\right)^2} \\
&= \frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx}{\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx} \\
&\quad +\Delta x\left(\frac{P_x}{\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx}-Q_x\frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx}{\left(\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx\right)^2}\right) \\
&\quad +\Delta y\left(\frac{P_y}{\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx}-Q_y\frac{\int_0^{x_0}\int_{t(x)}^{s(x)}\left(\left(\frac{\partial f}{\partial u_0}\right)^2+\left(\frac{\partial f}{\partial v_0}\right)^2\right)(x,y)dydx}{\left(\int_0^{x_0}\int_{t(x)}^{s(x)}\frac{f_0^2}{v_0^2}dydx\right)^2}\right).
\end{aligned}
$$

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