# LEAFWISE MORSE-NOVIKOV COHOMOLOGICAL INVARIANTS OF FOLIATIONS

by

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# Contents

# 1 Introduction

Consider an $n$-dimensional smooth manifold $M$; denote by $\Omega^k(M)$ the collection of all degree $k$ differential forms on $M$ and by $H^k(M)$ the corresponding de Rham cohomology group. Let $\omega$ be a closed 1-form that is not necessarily exact. We consider the twisted operator $d_\omega : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by $d_\omega = d + \omega\wedge$, where $d$ is the usual exterior derivative. It turns out that $(d_\omega)^2 = 0$. The differential cochain complex $(\Omega^*(M), d_\omega)$ is called the Morse-Novikov complex of the manifold $M$. The cohomology groups $H^k_\omega(M)$ of this cochain complex are called the Morse-Novikov or Lichnerowicz cohomology groups of $M$ and have been utilized by many researchers. Morse-Novikov cohomology was first studied by A. Lichnerowicz in [24], and used in the context of Poisson geometry. The idea of Lichnerowicz has been exploited to study many properties of manifolds. In [26] and [27] S.P.Novikov proved a generalization of the Morse inequalities by comparing the ranks of these cohomology groups with combinatorial invariants derived from the zeros of the form $\omega$. Pazhintov [33] gave an analytic proof of the real part of Novikov's inequalities. E. Witten used the Morse-Novikov cohomology for exact $\omega$ in his famous discovery [45] of what is now known as *Witten deformation*. In this case Morse-Novikov cohomology is isomorphic to de Rham cohomology (see Corollary 2.4). M. Shubin and S. P. Novikov applied the deformation method to a rigorous treatment of eigenvalue limits of Witten Laplacians for more general 1-forms and vector fields in [28] and [38]. Many other researchers have extended and generalized this work, such as Braverman and Farber [6] in cases of nonisolated zeros of 1-forms and vector fields. See [14] for a good reference on these related topics. Alexandra Otiman studied Morse-Novikov cohomology for particular classes of closed 1-forms in [31]. I. Vaisman studied locally conformal symplectic manifolds in [41], and L. Ornea, and M. Verbitsky studied Morse-Novikov cohomology of locally conformally Kähler manifolds in [30]. The closed 1-form used in the work of Vaisman, Ornea, and Verbitsky is called the *Lee form.* In [8], X. Chen showed that if a Riemannian manifold $M$ has almost non-negative sectional curvature and nonzero first de Rham cohomology group, then all the Morse-Novikov cohomology groups of $M$ vanish irrespective of the choice of the closed non-exact 1-form $\omega$. In [25], L. Meng established an analogue of the Leray-Hirsch Theorem for de Rham cohomology and a blowup formula fo Dolbeault-Morse-Novikov cohomology on complex manifolds. Morse-Novikov cohomology theory has also been used to study locally conformal symplectic manifolds (see [42], [41], and [43]).

In our study we work with Morse-Novikov cohomology applied in the foliation setting; the kernel of a $d_\omega$-closed form is involutive and hence gives rise to a foliation of the manifold. In the presence of a metric, if the $d_\omega$-closed form is a volume form of transverse distribution of a foliation, it turns out that the closed 1-form $\omega$ in $d + \omega\wedge$ is the mean curvature form of the transverse distribution. Later on, we restrict the 1-form to be closed along leaves of a foliation. In the first section of our study, we review basic properties of Morse-Novikov cohomology of a manifold — for example, homotopy invariance (Proposition 2.17 and Corollary 2.18), Hodge decomposition (Theorem 2.24), and Poincaré duality (Theorem 2.25). In fact, we give a proof of Poincaré duality for Morse-Novikov cohomology (originally shown in Proposition 3.5 of [32]), using the Hodge star operator and the Hodge Theorem 2.24 for Morse-Novikov cohomology. We compute the Morse-Novikov cohomology groups of $\mathbb{S}^n$, $\mathbb{R}P^n$, in Examples 2.6, and 2.8. It turns out that they are independent of the 1-form $\omega$. We also compute the Morse-Novikov cohomology of $\mathbb{T}^2$ in Example 2.10 for a particular 1-form $\omega$, and this computation can be adapted to an arbitrary 1-form.

There are mainly two types of Morse-Novikov cohomology associated to foliations: basic (see [34]) and leafwise. Liviu Ornea and Vladimir Slesar studied basic Morse-Novikov cohomology in [29]. K. Richardson and G. Habib used basic Morse-Novikov cohomology to prove that the basic signature and the Álvarez class of a Riemannian foliation are homotopy invariants [17]. They have also used a modified differential as in Morse-Novikov cohomology to define a twisted basic cohomology for Riemannian foliations that satisfies Poincaré duality [16]. J. A. Álvarez Lopez, Y. Kordyukov, and E. Leichtnam studied leafwise Hodge decomposition on Riemannian foliations with bounded geometry and extended the Morse-Novikov differential complex [3].

Using $d + \omega\wedge$ as the differential for a closed 1-form $\omega$, we study leafwise Morse-Novikov cohomology groups whose isomorphism classes turn out to be smooth invariants of the foliation. In the cases where $\omega$ is truly a closed 1-form on the manifold, one further obtains Morse-Novikov cohomology groups from the foliation. In Section 3 we study the basic properties of leafwise Morse-Novikov cohomology groups, including the homotopy axiom in Proposition 3.13, and foliated homotopy invariance in Corollary 3.14. With the additional assumption that the foliation is Riemannian, we give a proof of the Hodge decomposition in Corollary 3.20 and Poincaré duality in Corollary 3.21. We extend these results to more general settings of forms of $p, q$ type: homotopy axioms for general leafwise Morse-Novikov cohomology in Proposition 3.24, Hodge decomposition for general leafwise Morse-Novikov cohomology in Theorem 3.25, Poincaré duality for general leafwise Morse-Novikov cohomology in Proposition 3.27. The assumption that the foliation is Riemannian is required to obtain Hodge theory and Poincaré duality; for general smooth foliations, those results are false, even for the case when $\omega = 0$.

Ordinary foliations arise from distributions, subbundles of the tangent bundle of a manifold that are involutive and thus generate integral submanifolds through each point of the manifold. Generalized foliations allow leaves that are not necessarily of the same dimension at each point. They may arise from Pfaffian systems (see [21] for details). In Section 4, we study an interesting connection between these distributions and Morse-Novikov cohomology, which classifies the types of foliations. Given a foliation $(M, \mathcal{F})$ on a smooth manifold $M$ of codimension $q$, there exists a $q$-form $\nu$ such that the tangent bundle to the foliation $T\mathcal{F}$ may be defined as the set of vectors $v$ such that $v \lrcorner \nu = 0$, where $v \lrcorner$ denotes interior product. Let $\nu$ be an arbitrary $q$-form on a manifold that satisfies the differential equation of the form $(d + \omega\wedge)\nu = 0$ for some closed differential 1-form $\omega$. Then each Morse-Novikov cohomology class $[\nu] \in H^q_\omega(M)$ defines an involutive cosmooth distribution and hence a smooth foliation whose tangent bundle is the maximal smooth distribution contained in $\ker \nu$; see Proposition 4.22, and Corollary 4.23. If $\dim(M) \leq 3$ and if $\ker \nu$ has locally constant rank, it is involutive if and only if there exists a 1-form $\omega$ such that $d\nu = \omega \wedge \nu$. In other words, every cosmooth involutive distribution of locally constant dimension is associated with a particular set of leafwise Morse-Novikov cohomology groups; see Corollary 4.34. In general, it is not true that $(d + \omega\wedge)$-cohomologous $q$-forms determine the same foliations; see Example 4.42 and Example 4.43. But $\ker \nu$ may be involutive even if it does not satisfy the differential equation $(d + \omega\wedge)\nu = 0$ for any closed 1-form $\omega$; see Example 4.39.

The $q$-form $\nu$ may be taken to be the volume form of the normal bundle $N\mathcal{F}$ in the presence of a Riemannian metric $g$, and in general such a form may be found even when the foliation is not regular, i.e. when the leaf dimension is not constant. A volume form $\nu$ satisfies a differential equation of the form $(d + \omega\wedge)\nu = 0$ for some closed 1-form $\omega$ if and only if $\ker(\nu)$ is involutive and thus defines a foliation. It turns out that $\omega$ is a leafwise closed form; therefore $\nu$ yields leafwise Morse-Novikov cohomology groups; see Theorem 4.10. In the presence of metrics, this form $\omega$ may be derived from the mean curvature of the orthogonal distribution associated to the foliation; see Corollary 4.11, and mean curvature is leafwise cohomologous for different metrics; therefore the isomorphism classes of leafwise Morse-Novikov cohomology groups associated to $[\nu]$ are independent of the metric. In other words, a transversely oriented foliation of a Riemannian manifold uniquely determines leafwise Morse-Novikov cohomology groups whose isomorphism classes are independent of the choice of the metric; see Lemma 4.13 and Corollary 4.12. The leafwise Morse-Novikov cohomology groups are also invariant under diffeomorphism; see Corollary 4.16. Let $\nu$ be a transverse volume form of a foliation $(M, \mathcal{F})$ on a Riemannian manifold, and let $\kappa$ be the mean curvature of the normal distribution $\mathcal{NF}$. The foliation is minimal and the normal bundle $\mathcal{NF}$ is involutive if and only if $\nu$ is $(d + \kappa\wedge)$-harmonic; see Proposition 4.36.

We further explore some special geometric situations in the latter part of Section 4. Given a smooth distribution with characteristic $p$-form $\chi$, from Rummler's formula (see [37]) we have $d\chi = -\kappa \wedge \chi + \phi_0$, where $\kappa$ is the mean curvature 1-form of the distribution that is of type $(0, 1)$, and $\phi_0$ is a $(2, p-1)$ type form. In many interesting cases, such as when the foliation is Riemannian and has a metric with basic mean curvature, then we have $d\kappa = 0$. Because of this, it turns out that $\phi_0$ is $d + \kappa\wedge$-closed, so that $\ker \phi_0$ defines a generalized foliation that is an invariant of the distribution; see Example 4.27.

A smooth singular foliation of a given Riemannian manifold $(M, g, \mathcal{F})$ is a partition of M into smooth, connected, injectively immersed submanifolds, called leaves of the foliation, if there are possibly an arbitrary (infinite) family of smooth vector fields $X_1, X_2, \cdots$ on $M$ that spans the tangent bundle $T\mathcal{F}$ at all points of $M$. See Examples 4.3, and 4.4. The Morse-Novikov cohomology groups of these smooth singular foliations turn out to be foliated homotopy invariants; see Corollary 3.14, Proposition 4.47, and Proposition 4.48. It is important to emphasize that these invariants are new even for regular foliations, and luckily the construction applies even to the singular foliation setting; see Proposition 4.20. In the case of orbits of compact Lie group actions, several homotopy invariants are known, but none of these known invariants generalize easily to the singular foliation setting.

Let $(M, \mathcal{F})$ be a transversely oriented $q$-dimensional foliation with transverse volume form $\nu$. By Theorem 4.10, there exists an 1-form $\omega$ such that $d\nu = -\omega \wedge \nu$. This 1-form is leafwise cohomologous to the mean curvature 1-form of the transverse distribution of the foliation. It was observed by Godbillon and Vey (see [15]) that the form $\omega \wedge (d\omega)^q$ is closed and its de Rham cohomology class depends only on the foliation $(M, \mathcal{F})$. This cohomology class is called the Godbillon-Vey invariant of $(M, \mathcal{F})$. In the codimension one case, the Godbillon-Vey class measures some type of exponential growth of the leaves of the foliation (see Theorem 3.1 in [13], and [7]). The Godbillon-Vey class vanishes often. If $\nu$ is an invariant transverse volume form of $(M, \mathcal{F})$, then $\omega$ is zero. This is the case for Riemannian foliations or singular Riemannian foliations; thus the Godbillon-Vey class is trivial and the Morse-Novikov classes are actually de Rham cohomology classes (see Example 4.28). When there exists a submanifold of dimension $q$ that is transverse to all the leaves (as is the case for many taut foliations), this class is always nonzero since the integral of the transverse volume form over the submanifold is nonzero. In some cases the associated leafwise Morse-Novikov cohomology group may not be trivial while the Godbillon-Vey class is trivial, but in other cases both classes can be trivial — for example, nontaut Riemannian foliations. Example 4.38 gives a nice example of a non-Riemannian foliation where the mean curvature 1-form $\omega$ is not exact and where the corresponding Morse-Novikov cohomology groups are also nontrivial while the Godbillon-Vey class is trivial. In this sense leafwise Morse-Novikov cohomology groups is a finer invariants than the Godbillon-Vey class.

# 2 Morse-Novikov cohomology and corresponding de Rham Laplacian

Consider an $n$-dimensional smooth manifold $M$; denote by $\Omega^k(M)$ the collection of all degree $k$ differential forms on $M$ and by $H^k(M)$ the corresponding de Rham cohomology group. Let $\omega$ be a closed 1-form which is not necessarily exact. We consider the twisted operator $d_\omega : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by $d_\omega = d + \omega\wedge$, where $d$ is the usual exterior derivative. Since $d \circ d = d^2 = 0$, $\omega \wedge \omega = 0$,and $d(\omega \wedge \alpha) = d\omega \wedge \alpha - \omega \wedge d\alpha$ for any $k$-form $\alpha$, it follows that $d_\omega \circ d_\omega = (d_\omega)^2 = 0$. The differential cochain complex $(\Omega^*(M), d_\omega)$ is called the Morse-Novikov complex of the manifold $M$. Let $d_k^\omega$ be the restriction of $d_\omega$ to $\Omega^k(M)$. The cohomology groups $H^k_\omega(M) = \frac{ker(d_k^\omega)}{im(d_{k-1}^\omega)}$ of this cochain complex are called the Morse-Novikov cohomology groups of $M$. We review some standard results in Morse-Novikov cohomology, which can be found in [24]. This cohomology theory is also called Lichnerowicz cohomology.

**Example 2.1.** For the circle $S^1$, consider $d_\omega = d + \omega\wedge$, where $[\omega] = \omega = d\theta \in \Omega^1(S^1)$. If $d_\omega f = 0$ for some function $f \in \Omega^0(S^1)$ then

$$\begin{aligned} df + f d\theta &= 0 \\ \Rightarrow f &= ce^{-\theta}, \end{aligned}$$

where c is a constant. Notice $f = ce^{-\theta}$ is not periodic unless $c = 0$. Therefore no nonzero function on $S^1$ is $d_\omega$-closed. In other words $ker(d_\omega)$ is empty. Hence $H^0_\omega(S^1) = 0$. Clearly any 1-form on $S^1$ is $d_\omega$-closed. If $g(\theta)d\theta \in \Omega^1(S^1)$ and we wish to solve $d_\omega f =$

$g(\theta)d\theta$. Expanding $f(\theta) = \sum a_m e^{im\theta}$ and $g(\theta) = \sum b_n e^{in\theta}$ in Fourier series, we obtain $f(\theta) = \sum \frac{b_n}{in+1} e^{in\theta}$, and $g(\theta)d(\theta)$ is $d_\omega$ exact. If $g$ is smooth i.e. rapidly decreasing coefficients, so is $f$. Hence $H^1_\omega(S^1) = 0$.

Note that we could deduce the vanishing of the Morse-Novikov cohomology from the main theorem of [25], since $S^1$ has almost nonnegative sectional curvature.

**Proposition 2.2.** *If $\omega$ and $\theta = \omega + dg$ are cohomologous in $H^1(M)$, then for each $k$ the Morse-Novikov cohomology groups $H^k_\omega(M)$ and $H^k_\theta(M)$ are isomorphic. i.e. the map given by $[\alpha] \mapsto [e^{-g}\alpha]$ is an isomorphism.*

$$H^k_\omega(M) \cong H^k_\theta(M).$$

**Remark 2.3.** Therefore, to calculate all possible $H^k_\omega(M)$, it suffices to restrict to one representative in each de Rham cohomology class in $H^1(M)$. So for instance, if a particular metric is chosen, since every closed form is cohomologous to a harmonic form, one could always pick the harmonic representative [35].

*Proof.* If $\omega$ and $\theta$ are cohomologous, then there exists $g \in \Omega^0(M)$, such that $\theta - \omega = dg$. Define the mapping $\phi : H^k_\omega(M) \to H^k_\theta(M)$ by $\phi([\alpha]) = [e^{-g}\alpha]$. One can check that $\phi$ is well-defined and is a group homomorphism, since

$$\begin{aligned} (d + \theta\wedge)\left(e^{-g}\alpha\right) &= (d + \omega \wedge + dg\wedge)\left(e^{-g}\alpha\right) \\ &= e^{-g}\left(d + \omega\wedge\right)(\alpha), \end{aligned}$$

for all $\alpha \in \Omega^k(M)$.

Suppose $\alpha, \beta \in \Omega^k(M)$ are cohomologous. Then there exists $\nu \in \Omega^{k-1}(M)$ such that $\alpha - \beta = (d + \omega\wedge)\nu$. We have

$$\begin{aligned} \phi([\alpha - \beta]) &= [e^{-g}(d + \omega\wedge)\nu] \\ \Rightarrow \left[(d + \theta\wedge)\left(e^{-g}\nu\right)\right] &= [0]. \end{aligned}$$

Similarly if, $\phi([\alpha]) = 0$, Then $[\alpha] = 0 \in H^k_\omega(M)$, and $\phi$ is injective.

If $[\alpha] \in H^k_\theta(M)$ then we find similarly that $[e^g\alpha] \in H^k_\omega(M)$, so that $\phi$ is surjective. □

**Corollary 2.4.** *If $\omega$ is an exact 1-form, then for each $k$ the Morse-Novikov cohomology group $H^k_\omega(M)$ and the de Rham cohomology group $H^k(M)$ are isomorphic.*

$$H^k_\omega(M) \cong H^k(M).$$

**Corollary 2.5.** *If the first de Rham cohomology group $H^1(M)$ vanishes then for every closed 1-form $\omega$ and for each $k$ the Morse-Novikov cohomology groups satisfy $H^k_\omega(M) = H^k(M)$.*

**Example 2.6.** For $n \geq 2$ de Rham cohomology group of the sphere $H^1(S^n)$ is 0, so the Morse-Novikov cohomology $H^k_\omega(S^n) = H^k(S^n)$ for each $k$. Similarly $H^k_\omega(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ for $k > 1$ and all closed 1-forms $\omega$ in $\mathbb{R}^n$.

**Corollary 2.7.** *If the fundamental group $\pi_1(M)$ of a manifold $M$ is finite, then its de Rham and Morse-Novikov cohomology groups are isomorphic.*

*Proof.* The abelianization of the fundamental group $\pi_1(M)$ is isomorphic to the homology group $H_1(M, \mathbb{Z})$. By the universal coefficient theorem of singular cohomology with integer coefficients, the singular cohomology $H^1(M,\mathbb{R})$ of $M$ is isomorphic to $\mathrm{Hom}\left(H_1\left(M,\mathbb{Z}\right),\mathbb{R}\right)$. Since a homomorphism from a finite group into the group of the set of the integers is the zero homomorphism, $H^1(M,\mathbb{R})=0$. By the de Rham theorem $H^1(M,\mathbb{R})=H^1(M)\cong 0$. □

**Example 2.8.** The fundamental group $\pi_1(\mathbb{R}P^n)$ of the real projective space $\mathbb{R}P^n$ is $\mathbb{Z}_2$. So $H^k_\omega(\mathbb{R}P^n)\cong H^k(\mathbb{R}P^n)$ for all k and any closed 1-form $\omega$.

**Lemma 2.9.** *For any smooth manifold $M$, the Morse-Novikov cohomology $H^0_\omega\left(M\right)=\{0\}$ if and only if $\omega$ is not exact.*

*Proof.* Suppose first that $H^0_\omega\left(M\right)\neq\{0\}$ for a closed 1-form $\omega$ on $M$, then there is a nonzero function $f\in C^\infty\left(M\right)$, such that

$$\begin{aligned} d_\omega f &= 0 \\ \Rightarrow df+f\omega &= 0 \\ \Rightarrow d\left(\log(\frac{1}{f})\right) &= \omega, \end{aligned}$$

which implies $\omega$ is exact. Conversely, suppose that $\omega$ is exact. There exists a function $g\in C^\infty\left(M\right)$ such that $dg=\omega$. Then

$$d_\omega\left(e^{-g}\right) = -e^{-g}dg+e^{-g}dg=0,$$

which shows $H^0_\omega\left(M\right)\neq\{0\}$. □

**Example 2.10.** We compute the Morse-Novikov cohomology groups of the torus $T^2 = \{(x, y) \in \mathbb{R}^2\}/2\pi\mathbb{Z}^2$. Suppose the closed 1-form $\omega$ is not exact. Since the de Rham cohomology ring over $\mathbb{R}$ of torus is generated by the differential forms $\{1, dx, dy, dx\wedge dy\}$, the first cohomology group is $H^1(T^2) = \{[adx + bdy]\,|a, b \in \mathbb{R}\}$. We compute the $H^k_\omega(T^2)$ for $\omega = dx$. The computation is analogous for an arbitrary $\omega = adx + bdy$.

If $f \in \Omega^0(T^2)$ such that $d_\omega f = 0$ Then

$$\begin{aligned} df + fdx &= 0 \\ f &= ce^{-x} \end{aligned}$$

But $f(x, y) = ce^{-x}$ is not a periodic function unless $c = 0$. So there is no function on $T^2$ which is $d_\omega$ closed. Hence $H^0_\omega(T^2) = 0$.

To compute $H^0_\omega(T^2)$ consider the functions

$$f(x, y) = \sum_{m,n\in\mathbb{Z}} f_{mn}e^{i(mx+ny)},$$

$$a(x, y) = \sum_{m,n\in\mathbb{Z}} a_{mn}e^{i(mx+ny)},$$

and

$$b(x, y) = \sum_{m,n\in\mathbb{Z}} b_{mn}e^{i(mx+ny)}$$

in $\Omega^0(T^2)$ in their Fourier expansions. If $\tau = adx + bdy \in \Omega^1(T^2)$ and $d_\omega \tau = 0$ then

$$\begin{aligned} -\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} + b &= 0 \\ -ina_{mn} + imb_{mn} + b_{mn} &= 0 \\ b_{mn} &= \frac{ina_{mn}}{im+1}. \end{aligned}$$

In particular $b_{m0} = 0$ for all $m$.

The equation $d_w f = \tau = adx + bdy$ has a solution $f$ if

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}} (imf_{mn} + f_{mn} - a_{mn})e^{i(mx+ny)}dx + \sum_{m,n\in\mathbb{Z}} (inf_{mn} - b_{mn})e^{i(mx+ny)}dy &= 0, \text{ or} \\ imf_{mn} + f_{mn} - a_{mn} &= 0, \text{ and} \\ inf_{mn} - b_{mn} &= 0 \end{aligned}$$

for $m, n \in \mathbb{Z}$. So we have $f_{mn} = \frac{a_{mn}}{im+1} = \frac{b_{mn}}{in}$ for $n \neq 0$. Hence $\tau = adx + bdy$ is $d_\omega$ exact. Therefore $H^1_\omega(T^2) = 0$.

To compute $H^2_\omega(T^2)$, suppose $\mu = \nu dx \wedge dy \in \Omega^2(T^2)$ is an arbitrary 2-form for some function $\nu$. Clearly $d_\omega \mu = 0$. If $\mu$ is $d_\omega$-exact, then there exists an 1-form $\tau = \alpha dx + \beta dy$ such that $d_\omega \tau = \mu$, for some functions $\alpha$ and $\beta$. We have $d_\omega \tau = \mu$ if and only if

$$\begin{aligned} (-\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} + \beta - \nu)dx \wedge dy &= 0 \\ \Leftrightarrow -\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} &= \nu - \beta \\ \Leftrightarrow d\tau &= \gamma dx \wedge dy, \end{aligned}$$

where $\gamma = \nu - \beta$. By Stokes' theorem, the equation $d\tau = \gamma dx \wedge dy$ will have a global solution on $T^2$ if $\nu - \beta$ integrates to 0 on $T^2$, which may be achieved by choosing $\beta = \frac{\int_{T^2} \nu}{\text{vol}(T^2)}$. Therefore $H^2_\omega(T^2) = 0$.

Again we could deduce the vanishing of the Morse-Novikov cohomology from the main theorem of [25], since the flat torus has almost nonnegative sectional curvature.

**Definition 2.11.** Let $M$ and $N$ be smooth manifolds and $I = [0, 1]$. Two smooth maps $f, g : M \to N$ are smoothly homotopic if there is a smooth map $F : M \times I \to N$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

**Example 2.12.** let $f, g : M \to \mathbb{R}^n$ be two smooth maps. Then $F(x, t) : M \times \mathbb{R} \to \mathbb{R}^n$ defined by $F(x, t) = (1 - t)f(x) + tg(x)$ is a smooth homotopy from $f$ to $g$.

**Definition 2.13.** A map $f : M \to N$ is a smooth homotopy equivalence if there exists a map $g : N \to M$ such that $f \circ g$ is homotopic to identity map of $N$ and $g \circ f$ is homotopic to the identity map of $M$. We say that $M$ and $N$ are homotopy equivalent, or that $M$ and $N$ have the same homotopy type.

Notice that homotopy is an equivalence on the set of all smooth maps from $M$ to $N$. Smooth homotopy equivalence is an equivalence relation on the set of all smooth manifolds. Clearly diffeomorphic manifolds have same homotopy type.

**Example 2.14.** The punctured Euclidean space $\mathbb{R}^{n+1} - \{0\}$ and the $n$-dimensional sphere $S^n$ have the same homotopy type. Let $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ be the inclusion map and let $r : \mathbb{R}^{n+1} - \{0\} \to S^n$ be defined by $r(x) = \frac{x}{\|x\|}$. Then $r \circ i$ is the identity map of $S^n$ and $i \circ r$ is homotopic to the identity of $\mathbb{R}^{n+1} - \{0\}$ with the straight line homotopy defined in the previous example. Therefore $r$ and $i$ are homotopy inverses to each other, and $\mathbb{R}^{n+1} - \{0\}$ and $S^n$ have the same homotopy type.

It is well known that the differential $d$ commutes with the pullback by a map, but $d_\omega$ does not commutes with pullback. We have the following proposition.

**Proposition 2.15.** *Let $F : M \to N$ be a smooth map of manifolds and $\omega \in \Omega^1(N)$ be a closed 1-form. If $\tau \in \Omega^k(N)$, then $F^* d_\omega \tau = d_{F^*\omega} F^* \tau$.*

*Proof.* Let $\tau$ be a smooth $k$-form on $N$. Then

$$\begin{aligned} d_{F^*\omega} F^* \tau &= d\left(F^*\tau\right) + F^*\omega \wedge F^*\tau \\ &= F^*\left(d\tau\right) + F^*\left(\omega \wedge \tau\right) \\ &= F^*\left(d\tau + \omega \wedge \tau\right) \\ &= F^* d_\omega \tau. \end{aligned}$$

□

Let $f : M \to N$ be a smooth map. If $\omega \in \Omega^k(N)$ is a closed form, then $f^*\omega \in \Omega^k(M)$ is closed and if $\omega \in \Omega^k(N)$ is an exact form, then $f^*\omega \in \Omega^k(M)$ is exact.

**Proposition 2.16.** *For a smooth map $f : M \mapsto N$, the pullback of forms $f^* : \Omega^k(N) \to \Omega^k(M)$ induces a linear map in Morse-Novikov cohomology $f^* : H^k_\omega(N) \to H^k_{f^*\omega}(M)$, defined by $f^*\left([\tau]\right) = [f^*\tau]$.*

*Proof.* We must must prove that the linear map $f^*$ maps closed forms to closed forms and exact forms to exact forms. For any $[\tau] \in H^k_\omega(N)$,

$$\begin{aligned} d_{f^*\omega}(f^*\tau) &= f^* d_\omega \tau \\ &= 0. \end{aligned}$$

Thus $d_\omega$ closed forms on $N$ are mapped to $d_{f^*\omega}$ closed forms on $M$. Similarly, $d_\omega$ exact forms are mapped to $d_{f^*\omega}$ exact forms. □

Now we can state the homotopy invariance of Morse-Novikov cohomology.

**Proposition 2.17.** *(Homotopy axiom for the Morse-Novikov cohomology). Let $f : M \to N$ and $g : M \to N$ be homotopic maps, and $\omega$ be a closed 1-form on $N$. Then there exists a positive function $h : M \to \mathbb{R}$ such that*

$$f^* = hg^* : H^k_\omega(N) \to H^k_{f^*\omega}(M) \text{ for all } k.$$

*Proof.* Since $f$ and $g$ are homotopic maps, by the homotopy axiom of de Rham cohomology, they induce the same map in de Rham cohomology. Therefore, for any closed 1-form $\omega \in \Omega^1(N)$, the pullback forms $f^*\omega$ , $g^*\omega \in H^1(M)$ are cohomologous. There exists a function $\nu : M \to \mathbb{R}$ such that $g^*\omega - f^*\omega = d\nu$. We define $h = e^\nu$. Then from proof of Proposition 2.2, for any $d_\omega$-closed form $\alpha$ on $N$, $[hg^*\alpha] = [f^*\alpha] \in H^k_{f^*\omega}(M)$. □

**Corollary 2.18.** *If $f : M \to N$ is a homotopy equivalence and $\omega$ is a closed 1-form, then the Morse-Novikov cohomology groups $H^*_\omega(N)$ and $H^*_{f^*\omega}(M)$ are isomorphic.*

$$H^k_\omega(N) \cong H^k_{f^*\omega}(M), \text{ for all } k.$$

*Proof.* There exists a map $g : N \to M$ such that $f \circ g$ is homotopic to the identity map $\mathbb{I}_N$ of $N$ and $g \circ f$ is homotopic to the identity map $\mathbb{I}_M$ of $M$ . We have linear maps

$$H^*_\omega(N) \xrightarrow{f^*} H^*_{f^*\omega}(M) \xrightarrow{g^*} H^*_{g^*f^*\omega}(N).$$

By the homotopy axiom of Morse-Novikov cohomology, there exists a positive function $h : N \to \mathbb{R}$ such that $g^*f^*\omega = \omega + d(ln(h))$ then we have

$$\mathbb{I}_N = hg^*f^* = h\,(f \circ g)^* : H^*_\omega(N) \to H^*_\omega(N).$$

And similarly, for some positive function $\bar{h} : M \to \mathbb{R}$ such that $f^*g^*\omega = \omega + d(ln(\bar{h}))$ then we have

$$\mathbb{I}_M = \bar{h}f^*g^* = h\,(g \circ f)^* : H^*_\omega(M) \to H^*_\omega(M).$$

Since multiplication by a positive function is an isomorphism of Morse-Novikov cohomology, $f^*$ and $g^*$ are isomorphisms. □

**Corollary 2.19.** *Suppose $S$ is a submanifold of a manifold $M$ and $F$ is a deformation retraction from $M$ to $S$. Let $r : M \to S$ be the retraction $r(x) = F(x, 1)$. Then $r$ induces* *an isomorphism in Morse-Novikov cohomology*

$$r^* : H^*_\omega(S) \to H^*_{r^*\omega}(M).$$

Since the differential $d_\omega$ commutes with the restriction map to open subsets, in the same way as for de Rham cohomology [5], we can construct a Mayer-Vietoris exact cohomology sequence for the Morse-Novikov cohomology. Suppose $U$, $V$ are two open subsets of the manifold $M$ such that $M = U \cup V$. Then for a closed 1-form $\omega$, one can verify the following short exact sequence of cochain complexes.

$$0 \to (\Omega^*(M), d_\omega) \xrightarrow{\alpha} (\Omega^*(U) \oplus \Omega^*(V), d_\omega|_U \oplus d_\omega|_V) \xrightarrow{\beta} (\Omega^*(U \cap V), d_\omega|_{U\cap V}) \to 0,$$

where $\alpha$ is restriction homomorphism induced from the inclusion map $i : U \to M$ and $\beta$ is difference homomorphism induced from the inclusion map $j : U \cap V \to M$,

$$\alpha(\tau) \mapsto (i^*_U \tau, i^*_V \tau) \text{ and } \beta(\tau, \eta) = j^*|_V \eta - j^*|_U \tau.$$

It is known from homological algebra that a short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology [5]. If $\rho_U, \rho_V$ is a partition of unity corresponding to the open cover $U, V$ of $M$, we have the following Mayer-Vietoris sequence of Morse-Novikov cohomology.

$$\cdots \to H^k_\omega(M) \xrightarrow{\alpha_*} H^k_{\omega|U}(U) \oplus H^k_{\omega|V}(V) \xrightarrow{\beta_*} H^*_{\omega|(U\cap V)} \xrightarrow{\delta^*} H^{k+1}_\omega(M) \to \cdots,$$

where

$$\begin{aligned}\alpha_*([\tau]) &= ([\tau|_U], [\tau|_V]), \\ \beta_*([\theta], [\eta]) &= [\theta|_{U\cup V}] - [\eta|_{U\cup V}],\end{aligned}$$

and the connecting homomorphism is defined by $\delta^*[\tau] = [-d_\omega(\rho_V \tau)]$ on $U$ and $\delta^*[\tau] = [-d_\omega(\rho_U \tau)]$ on $V$.

We now review some well-known facts (see, e.g [35]). Let $(M, g)$ be a closed compact oriented Riemannian manifold of dimension $n$. At every point $p \in M$, we have an inner product $g_p$ on the tangent space $T_pM$, and therefore also an inner product on the cotangent space $T_p^*M$ determined by the inverse matrix of the matrix of $g_p$. This inner product is extended in a natural way to differential forms. So each vector bundle $\Lambda^k T^*M$ carries a metric that allows us to define an inner product on the space of smooth $k$-forms on $M$ by the following formula

$$\langle \alpha, \beta \rangle = \int_M g(\alpha, \beta)\text{vol}.$$

Let $\alpha \in \Omega^k(M)$ be a $k$-form. Define the linear Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ such that for all $\beta \in \Omega^k(M)$

$$\alpha \wedge *\beta = g(\alpha, \beta)\text{vol}.$$

So the inner product defined above can be expressed by the even simpler formula

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta.$$

It turns out that $**\alpha = (-1)^{k(n+k)}\alpha$ for $\alpha \in \Omega^k(M)$ and that $\beta \wedge *\alpha = \alpha \wedge *\beta$ for all $\alpha, \beta \in \Omega^k(M)$.

The codifferential $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ in the exterior algebra may be expressed in terms of the Hodge $*$ operator; for $\beta \in \Omega^k(M)$,

$$d^*\beta = (-1)^{nk+n+1} * (d * \beta).$$

**Lemma 2.20.** *(See, for example, [35]) On a closed compact Riemannian manifold, $d^*$ is the formal adjoint of $d$ with respect to the global inner product defined above.*

It follows that $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is an isomorphism. Since $*$ commutes with $\Delta = d^*d + dd^*$, $*$ is the Poincaré duality isomorphism of de Rham cohomology of a compact oriented manifold,

$$H^k(M) \cong H^{n-k}(M) \text{ for every } 0 \leq k \leq n.$$

The interior product in the exterior algebra is defined in terms of the Hodge $*$ operator; for $\beta \in \Omega^k(M)$ and $\omega \in \Omega^1(M)$ is a covector, the interior product $\omega \lrcorner : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined as

$$\omega \lrcorner \beta = (-1)^{nk+n} * (\omega \wedge *\beta).$$

**Lemma 2.21.** *The adjoint of $\omega\wedge$ with respect to the inner product defined above is $\omega\lrcorner$.*

*Proof.* Let $\beta \in \Omega^k(M)$ and $\gamma \in \Omega^{k-1}(M)$, then

$$\begin{aligned}(\gamma, \omega\lrcorner\beta)\,\mathrm{vol} &= (-1)^{nk+n}\,\gamma \wedge * * (\omega \wedge *\beta) \\ &= (-1)^{nk+n+(n-k+1)(-k+1)}\,\gamma \wedge \omega \wedge *\beta \\ &= (-1)^{k+1}(-1)^{k-1}\,\omega \wedge \gamma \wedge *\beta\end{aligned}$$

so that $(\gamma, \omega\lrcorner\beta)\,\mathrm{vol} = (\omega \wedge \gamma, \beta)\,\mathrm{vol}$. □

Laplace and Dirac type operators [35], [44] are examples of elliptic operators. We first define the *principal symbol* of a differential or pseudodifferential operator. If $\pi : E \to M$ and $\pi' : F \to M$ are two vector bundles and $P : \Gamma(E) \to \Gamma(F)$ is a differential operator of order $k$ acting on sections, then in local coordinates of a local trivialization of the vector bundles $P$ can be written as

$$P = \sum_{|\alpha|=k} s_\alpha(x)\frac{\partial^k}{\partial x^\alpha} + \text{lower order terms},$$

where the summation is over all possible multi-indices $\alpha = (\alpha_1, \cdots, \alpha_k)$ of length $|\alpha| = k$ and each $s_\alpha(x) \in \mathrm{Hom}(E_x, F_x)$ is a linear transformation. If $\xi = \sum \xi_j dx^j \in T_x^*(M)$ is a non-zero covector at $x$, we define the *principal symbol* of $P$ to be

$$\sigma(P)(\xi) = i^k \sum_{|\alpha|=k} s_\alpha(x)\xi^\alpha \in Hom(E_x, F_x),$$

where $\xi^\alpha = \xi_{\alpha_1}. \cdots .\xi_{\alpha_n}$. It turns out that the principal symbol is invariant under coordinate transformations. One coordinate-free definition of $\sigma(P)_x : T^*_x M \to \mathrm{Hom}(E_x, F_x)$ can be given as follows. For any $\xi \in T^*_x M$ choose a locally defined function $f$ such that $df_x = \xi$. Then we define the operator

$$\sigma_m(P)(\xi) = \lim_{t\to\infty} \frac{1}{t^m}(e^{-itf} P e^{itf}),$$

where $(e^{-itf} P e^{itf})(u) = e^{-itf}(P(e^{itf}u))$. Then the order $k$ of the operator and symbol are defined to be $k = \sup\{m : \sigma_m(P)(\xi)\} < \infty$ and $\sigma(P)(\xi) = \sigma_k(P)(\xi)$. It follows that if $P$ and $Q$ are two differential operators such that the composition $PQ$ is defined, then

$$\sigma(PQ)(\xi) = \sigma(P)(\xi)\sigma(Q)(\xi).$$

**Definition 2.22.** An elliptic differential operator $P$ on $M$ is defined to be an operator such that its principal symbol $\sigma(P)(\xi)$ is invertible for all nonzero covectors $\xi \in T^*M$.

**Example 2.23.** The symbol of the Dirac operator $D = \sum c(e_j)\nabla_{e_j}$ is

$$\sigma(D)(\xi) = i\sum c(e_j)\xi_j = i\sum c(\xi^j e_j) = ic(\xi^\sharp).$$

The symbol of the Dirac Laplacian $D^2$ is

$$\sigma(D^2)(\xi) = \sigma(D)(\xi)\sigma(D)(\xi) = (ic(\xi^\sharp))^2 = \|\xi^\sharp\|^2,$$

where $\xi^\sharp$ is the corresponding vector of the covector $\xi$ induced by the metric on $M$.

The last equality is a consequence of the definition of Clifford multiplication; see [22]. Therefore for non-zero $\xi$, both these symbols are invertible, and hence $D$ and $D^2$ are elliptic differential operators.

An operator $P$ is *strongly elliptic* if there exists $c > 0$ such that

$$\sigma(P)(\xi) \geq c|\xi|^2$$

for all non-zero $\xi \in T^*M$. The Laplacian $\Delta$ of $\mathbb{R}^n$ and $D^2$ on Clifford bundle are strongly elliptic. For more about elliptic differential operators on manifolds see [35], [44], [22].

Let $M$ be a closed, compact, and oriented Riemannian manifold. We consider the de Rham operator for the differential

$$d_w : \Omega^{e/o}(M) \to \Omega^{o/e}(M),$$

where $\Omega^e(M)$ and $\Omega^o(M)$ denote the bundle of differential forms of even degree and odd degree respectively. We choose a Riemannian metric $g$ on $M$; this induces a volume form on $M$ and Hermitian inner products on all the spaces $\Omega^k(M)$. Since $d_\omega$ is a linear differential operator and the bundle in question carries a Hermitian metric induced from the Hermitian inner product, there exists an unique adjoint of $d_\omega$, denoted by $d_\omega^*$. Combining $d_\omega$ and $d_\omega^*$ we obtain a deformed differential operator

$$D_\omega = d_\omega + d_\omega^* : \Omega^{e/o}(M) \to \Omega^{o/e}(M).$$

For each $k$, we define the *Laplace operator* $\Delta_\omega : \Omega^k(M) \to \Omega^k(M)$ by the formula $\Delta_\omega = (d_\omega + d_\omega^*)^2 = d_\omega d_\omega^* + d_\omega^* d_\omega$. A form $\tau \in \Omega^k(M)$ is called $\omega$-harmonic if $\Delta_\omega \tau = 0$. We denote $\mathcal{H}_\omega^k(M) = \ker \Delta_\omega$, the space of all $\omega$-harmonic forms of degree $k$. Notice that $\Delta_\omega$ is a second order, formally self adjoint, linear differential operator on $\Omega^k(M)$. Because $d_\omega$ and $d_\omega^*$ square to zero,

$$(\Delta_\omega \alpha, \beta) = (d_\omega \alpha, d_\omega \beta) + (d_\omega^* \alpha, d_\omega^* \beta) = (\alpha, \Delta_\omega \beta)\,.$$

Since the principal symbols of $d_\omega + d_\omega^*$, and $\Delta_\omega$ are the same as that of $d + d^*$ and $\Delta$, the operators$d_\omega + d_\omega^*$ and $\Delta_\omega$ are elliptic operators. The following sequence

$$\Gamma\left(M, \Lambda^0(M)\right) \stackrel{d_\omega}{\to} \Gamma\left(M, \Lambda^1(M)\right) \stackrel{d_\omega}{\to} \cdots \stackrel{d_\omega}{\to} \Gamma\left(M, \Lambda^n(M)\right)$$

is an *elliptic complex*, since the associated symbol sequence

$$0 \to \pi^*\Gamma\left(M, \Lambda^0(M)\right) \stackrel{\sigma(d_\omega)}{\to} \cdots \stackrel{\sigma(d_\omega)}{\to} \pi^*\Gamma\left(M, \Lambda^n(M)\right) \to 0$$

is exact, where $\Gamma\left(M, \Lambda^k(M)\right) = \Omega^k(M)$ is the set of smooth sections of the bundle $\pi : \Lambda^k(M) \to M$, and $\sigma(d_\omega)$ is the principal symbol of $d_\omega$. See Chapter IV, Example 2.5 of [44]. We may therefore apply the theorem concerning an elliptic differential complex of vector bundles (see Chapter IV, Theorem 5.2 of [44]) to conclude that $\mathcal{H}_\omega^k(M) = \ker \Delta_\omega$ is finite dimensional, and we have the following orthogonal decomposition of $\Omega^k(M)$:

$$\Omega^k(M) = \mathcal{H}_\omega^k \oplus \operatorname{im}(\Delta_\omega G)\,,$$

where $G : \Omega^k(M) \to \Omega^k(M)$ is a Green's operator. Now we can state and prove the Hodge theorem for the Morse-Novikov cohomology.

**Theorem 2.24.** *Let $(M, g)$ be a closed compact and oriented Riemannian manifold. Then $\mathcal{H}^k_\omega(M) \cong H^k_\omega(M)$. In other words, every Morse-Novikov cohomology class has a unique $\omega$-harmonic representative.*

*Proof.* Let $\alpha \in \mathcal{H}^k_\omega(M)$, which is smooth by elliptic regularity. Then we have

$$
\begin{aligned}
(\Delta_\omega \alpha, \alpha) &= 0 \\
\Rightarrow (d_\omega \alpha, d_\omega \alpha) + (d^*_\omega \alpha, d^*_\omega \alpha) &= 0 \\
\Rightarrow \|d_\omega \alpha\|^2 + \|d^*_\omega \alpha\|^2 &= 0.
\end{aligned}
$$

This implies that $\alpha$ is $\omega$-harmonic if and only if $d_\omega \alpha = 0$ and $d^*_\omega \alpha = 0$. These $\omega$-harmonic forms are closed and therefore define classes in Morse-Novikov cohomology. We have a map $\mathcal{I} : \mathcal{H}^k_\omega(M) \to H^k_\omega(M)$ defined by $\mathcal{I}(\alpha) = [\alpha]$. We show that this map is a bijection.

Suppose $\alpha \in \mathcal{H}^k_\omega$ is $d_\omega$ exact, say $\alpha = d_\omega \tau$ for some $\tau \in \Omega^{k-1}(M)$. Then

$$
\|\alpha\|^2 = (\alpha, \alpha) = (\alpha, d_\omega \tau) = (d^*_\omega \alpha, \tau) = 0,
$$

and therefore $\alpha = 0$. To prove the surjectivity, let $\alpha \in \Omega^k(M)$ such that $d_\omega \alpha = 0$. Then by the decomposition $\Omega^k(M) = \mathcal{H}^k_\omega \oplus \operatorname{im}(\Delta_\omega G)$, for some $\tau \in \mathcal{H}^k_\omega(M)$ and $\beta \in \Omega^k(M)$, we have

$$\alpha = \tau + \Delta_\omega G\beta = \tau + d_\omega d_\omega^* G\beta + d_\omega^* d_\omega G\beta.$$

Applying $d_\omega$ on both sides of this equation, it follows that $d_\omega d_\omega^* d_\omega G\beta = 0$, and therefore

$$\|d_\omega^* d_\omega G\beta\|^2 = (d_\omega^* d_\omega G\beta, d_\omega^* d_\omega G\beta) = (d_\omega G\beta, d_\omega d_\omega^* d_\omega G\beta)$$

proving that $d_\omega^* d_\omega G\beta = 0$. Hence we have $\alpha = \tau + d_\omega d_\omega^* G\beta$; therefore $[\alpha] = [\tau]$. □

Now we give a proof of Poincaré duality for Morse-Novikov cohomology (see [32, Proposition 3.5]), using the Hodge star operator and the Hodge theorem for Morse-Novikov cohomology.

**Theorem 2.25.** *If $M$ is a closed compact oriented manifold of dimension $n$ and $\omega$ is a closed 1-form, then the Hodge star operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ induces the isomorphism*

$$H_\omega^k(M) \cong H_{-\omega}^{n-k}(M).$$

*Proof.* From $(\omega \lrcorner) = (-1)^{nk+n} * (\omega\wedge) *$, $*^2 = (-1)^{k(n-k)}$, and $d^* = (-1)^{n(k+1)+1} * d*$ on $\Omega^k(M)$, we have the following identities for operators acting on $\Omega^k(M)$. For any $\beta \in \Omega^k(M)$

$$\begin{aligned}(\omega \lrcorner) * \beta &= (-1)^{n(n-k)+n} * (\omega\wedge) *^2 \beta \\ &= (-1)^{n^2+nk+n} (-1)^{k(n-k)} * (\omega\wedge) \beta,\end{aligned}$$

so that $(\omega\lrcorner) * = (-1)^k * (\omega\wedge)$ on $\Omega^k(M)$. Also,

$$* (\omega\lrcorner)\beta = (-1)^{nk+n} *^2 (\omega\wedge) * \beta$$

$$= (-1)^{nk+n} (-1)^{(n-k+1)(n-(n-k+1))} (\omega\wedge) * \beta,$$

so that $* (\omega\lrcorner) = (-1)^{k+1} (\omega\wedge) *$ on $\Omega^k(M)$. Next

$$d^* * \beta = (-1)^{n(n-k+1)+1} * d *^2 \beta$$

$$= (-1)^{n(n-k+1)+1} (-1)^{k(n-k)} * d\beta,$$

so that $d^* * = (-1)^{k+1} * d$ on $\Omega^k(M)$. Finally

$$*d^*\beta = (-1)^{n(k+1)+1} *^2 d * \beta$$

$$= (-1)^{n(k+1)+1} (-1)^{(n-k+1)(n-(n-k+1))} d * \beta,$$

so that $*d^* = (-1)^k d*$ on $\Omega^k(M)$. From these equations we have

$$(d^* + \omega\lrcorner) * = (-1)^{k+1} * (d - \omega\wedge)$$

$$(d + \omega\wedge) * = (-1)^k * (d^* - \omega\lrcorner)$$

on $\Omega^k(M)$. As before $d^*$ is the $L^2$ adjoint of $d$, and $\lrcorner$ represents interior product. It turns out that the $L^2$ adjoint of $d_\omega = d + \omega\wedge$ is $d_\omega^* = d^* + \omega\lrcorner$ and the Laplacian is

$\Delta_\omega = (d_\omega + d_\omega^*)^2 = d_\omega d_\omega^* + d_\omega^* d_\omega = (d + \omega\wedge)(d^* + \omega\lrcorner) + (d^* + \omega\lrcorner)(d + \omega\wedge)$. If $\beta \in \Omega^k(M)$, then by the formulas above we have for all $\beta \in \Omega^k(M)$,

$$
\begin{aligned}
*\Delta_\omega\beta &= *(d + \omega\wedge)(d^* + \omega\lrcorner)\beta + *(d^* + \omega\lrcorner)(d + \omega\wedge)\beta \\
&= (-1)^{k-1}(d^* - \omega\lrcorner) * (d^* + \omega\lrcorner)\beta + (-1)^k(d - \omega\wedge) * (d + \omega\wedge)\beta \\
&= (-1)^{k-1}(-1)^k(d^* - \omega\lrcorner)(d - \omega\wedge) * \beta + (-1)^k(-1)^{k+1}(d - \omega\wedge)(d^* - \omega\lrcorner) * \beta \\
&= -\left((d^* - \omega\lrcorner)(d - \omega\wedge) + (d - \omega\wedge)(d^* - \omega\lrcorner)\right) * \beta \\
&= -\Delta_{-\omega} * \beta.
\end{aligned}
$$

Thus the operator $*$ maps $\omega$-harmonic forms to $(-\omega)$-harmonic forms, so from the Hodge theorem for the Morse-Novikov cohomology $*$ induces the required isomorphism. □

For relative Morse-Novikov cohomology see [20].

# 3 Leafwise de Rham cohomology and leafwise Morse-Novikov cohomology

Suppose we are given a smooth foliation $(M, \mathcal{F})$ on a manifold $M$ without any metric. The leafwise tangent bundle $T\mathcal{F}$ is the restriction of the tangent bundle $TM$ of the manifold to the leaves, therefore the dual bundle $T^*\mathcal{F}$ is well-defined. Hence $\Lambda T^*\mathcal{F}$ is well-defined. The conormal bundle $N^*\mathcal{F}$ is defined as the set of all linear functionals that map each vector in $T\mathcal{F}$ to zero, and this bundle can be canonically identified with a subbundle of $T^*M$. On the other hand, the normal bundle $Q = TM/T\mathcal{F}$ may not be uniquely identified with a subbundle of $TM$ without the presence of a metric. The sections of the dual bundle $T^*\mathcal{F}$ are not differential forms on the manifold because they are defined only on the sections of the tangent bundle $T\mathcal{F}$. However, if we choose a metric on the manifold then the normal bundle $N\mathcal{F}$ is defined and we can identify the dual bundle $T^*\mathcal{F}$ with the set of covectors that kill $N\mathcal{F}$. The value of an element of $N^*\mathcal{F}$ and the corresponding element of $T^*M$ return the same value when applied to a vector in $T\mathcal{F}$. Now we can decompose any covector in $T^*M$ into $T^*\mathcal{F}$ and $N^*\mathcal{F}$ components and by using this decomposition we can decompose all differential forms using

$$\Lambda^{i,j}(M, \mathcal{F}) = \Lambda^i N^*\mathcal{F} \wedge \Lambda^j T^*\mathcal{F}.$$

Let $\Omega^{i,j}(M, \mathcal{F}) = \Gamma\Lambda^{i,j}(M, \mathcal{F})$. The exterior derivative can then be decomposed as $d = d_{0,1} + d_{1,0} + d_{2,-1}$ with

$$d_{i,j}\omega \in \Omega^{r+i,s+j}(M, \mathcal{F})$$

for all $\omega \in \Omega^{r,s}(M,\mathcal{F})$. Then it is easy to see that since $d^2 = 0$ we also have $d_{0,1}^2 = 0$.

The elements of $\Omega^{0,k}(M,\mathcal{F})$ are called leafwise $k$-forms. Let $\Gamma(T\mathcal{F})$ be the set of smooth sections of $T\mathcal{F}$. If $X, Y \in \Gamma(T\mathcal{F})$, then by the Frobenius theorem $[X,Y] \in \Gamma(T\mathcal{F})$. The leafwise exterior differential operator $d_{0,1}^k = d_{\mathcal{F}}^k : \Omega^{0,k}(M,\mathcal{F}) \to \Omega^{0,k+1}(M,\mathcal{F})$ may also be defined by

$$d_{\mathcal{F}}^k \omega(X_0, \cdots, X_{k+1}) = \sum_{0 \le i \le k} (-1)^i [X_i \omega\left(X_0, \cdots, \hat{X}_i, \cdots, X_k\right)] \\ + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k\right)$$

for $X_0, \cdots, X_{k+1} \in \Gamma(T\mathcal{F})$. The differential operator $d_{\mathcal{F}}$ is the restriction of the usual differential on differential forms on the leaves of $\mathcal{F}$. Similar to the usual exterior differential, the leafwise differential satisfies $d_{\mathcal{F}}^{k+1} \circ d_{\mathcal{F}}^k = 0$. For $k \ge 0$, the $k^{\text{th}}$ leafwise cohomology group $H^k(M, d_{\mathcal{F}})$ is the $k^{\text{th}}$ cohomology group

$$H^k(M, d_{\mathcal{F}}) = \frac{\ker d_{\mathcal{F}}^k}{\operatorname{im} d_{\mathcal{F}}^{k-1}}$$

of the cochain complex $(\Omega^{0,*}(M,\mathcal{F}), d_{\mathcal{F}})$.

**Example 3.1.** $H^0(M, d_{\mathcal{F}})$ is the space of the smooth functions that are constant on each leaf. Hence, if the foliation $\mathcal{F}$ has a dense leaf, then $H^0(M, d_{\mathcal{F}}) \cong \mathbb{R}$. One such example is a foliation of dimension 1 of the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ determined by the vector field $X = \partial_x + m\partial_y$ for any irrational number $m$.

**Example 3.2.** Suppose $X = \partial_x + m\partial_y$ is the vector field generating a one-dimensional foliation on $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. If we choose the slope $m$ to be a Liouville irrational number, it turns out that $H^1(M, d_{\mathcal{F}})$ is infinite dimensional, and if we choose $m$ to be a Diophantine irrational number then $H^1(M, d_{\mathcal{F}}) = \mathbb{R}$ (see [18] and [36]).

For the purpose of having Laplacian and Hodge decompositions, we need to consider reduced leafwise cohomology

$$\bar{H}^k(M, d_{\mathcal{F}}) = \frac{\ker d_{\mathcal{F}}^k}{\overline{\operatorname{im} d_{\mathcal{F}}^{k-1}}}.$$

Here the closure $\overline{\operatorname{im} d_{\mathcal{F}}^{k-1}}$ is taken with respect to the Frechét topology on $\Omega^{0,k}(M, \mathcal{F})$. The cup product induced from exterior product of forms makes $\bar{H}^*(M, d_{\mathcal{F}})$ into a graded commutative algebra over $C^\infty(M)$.

**Example 3.3.** For the foliation of the flat torus in Example 3.1, $\bar{H}^0(M, d_{\mathcal{F}}) = \mathbb{R}$, $\bar{H}^1(M, d_{\mathcal{F}}) = \mathbb{R}$ and $\bar{H}^k(M, d_{\mathcal{F}}) = 0$, for $k \geq 2$.

Let $f : M \to N$ be a smooth map of the foliated manifold which maps leaves into leaves. Then the pullback maps

$$f^* : \Gamma\left(\Lambda^k T^* \mathcal{F}_N\right) \to \Gamma\left(\Lambda^k T^* \mathcal{F}_M\right)$$

are defined for all $k$. They commute with $d_{\mathcal{F}}$ and respect the exterior product; therefore they induce a continuous map of the reduced cohomology ring.

$$f^* : \bar{H}^k(N, d_{\mathcal{F}_0}) \to \bar{H}^k(M, d_{\mathcal{F}_1}).$$

Such maps are called foliated maps. Two smooth foliated maps $f, g : (N, \mathcal{F}_0) \to (M, \mathcal{F}_1)$

between two foliated manifolds are leafwise homotopic if there is a map $F : N \times [0,1] \to M$ such that if $F_t$ denotes the restriction $F_t = F|_{N\times t} : N \to M$ for $t \in [0,1]$, then $F|_0 = f, F|_1 = g$, $F_t(\mathcal{F}_0) \in \mathcal{F}_1$ for all $t \in [0,1]$, and for every $x \in N$ the points $F_{t_1}(x), F_{t_2}(x)$ lie in the same leaf of $\mathcal{F}_1$ for all $t_1, t_2 \in [0,1]$. Thus, a leafwise homotopy consists of leaf-preserving maps. Denote the identity maps of $(N, \mathcal{F}_0)$, $(M, \mathcal{F}_1)$ by $l_{\mathcal{F}_0}$, $l_{\mathcal{F}_1}$ respectively. A leafwise map $f : (N, \mathcal{F}_0) \to (M, \mathcal{F}_1)$ is a leafwise homotopy equivalence if there exists a leafwise map $g : (M, \mathcal{F}_1) \to (N, \mathcal{F}_0)$ with $f \circ g$ is leafwise homotopic to $l_{\mathcal{F}_0}$ and $g \circ f$ is leafwise homotopic to $l_{\mathcal{F}_1}$.

**Proposition 3.4.** *[19, Theorem I, 3.2]) If the maps $f, g : (N, \mathcal{F}_0) \to (M, \mathcal{F}_1)$ between two foliated manifolds are leafwise homotopic, then $f^* = g^* : H^k(M, d_{\mathcal{F}_1}) \to H^k(N, d_{\mathcal{F}_0})$. That is, leafwise homotopic maps induce the same map on leafwise cohomology groups.*

**Corollary 3.5.** *If a map $f : (N, \mathcal{F}_0) \to (M, \mathcal{F}_1)$ is a smooth foliated homotopy equivalence, then $f^*$ induces an isomorphism between $H^k(N, d_{\mathcal{F}_0})$ and $H^k(M, d_{\mathcal{F}_1})$*

For more structure, we consider Riemannian foliations, characterized by the existence of a bundle-like metric $g$ such that a geodesic of the metric $g$ is orthogonal to all leaves that it meets whenever it is orthogonal to one of them.

**Example 3.6.** Any foliation of codimension one given by a closed nonzero 1-form is an example of a Riemannian foliation.

Let $M$ be an oriented manifold endowed with a foliation $\mathcal{F}$ of dimension $p$. The graded Frechét space $\Omega^{0,*}(M, \mathcal{F})$ can be endowed with the natural metric

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_{\mathcal{F}} \,\mathrm{vol}.$$

In this formula $\langle , \rangle_{\mathcal{F}}$ is the Riemannian metric on $\Lambda T^{*}\mathcal{F}$ induced from the Riemannian metric $g$ on $M$, and vol is the volume form associated to the metric $g$. We denote the formal adjoint of the leafwise differential $d_{\mathcal{F}}$ with respect to this inner product by $\delta_{\mathcal{F}}$; then the corresponding Laplacian is

$$\Delta_{\mathcal{F}} = \delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}.$$

Since $\mathcal{F}$ is Riemannian, the restriction of $\delta_{\mathcal{F}}$ to any leaf is the codifferential of the leaf with respect to the induced metric [2, Lemma 3.2], i.e.

$$\left(\delta_{\mathcal{F}}\alpha\right)|_{F} = \delta_{\mathcal{F}}\left(\alpha\right)|_{F} \text{ for all } \alpha \in \Omega^{0,k}\left(M, \mathcal{F}\right),$$

where $F$ denotes a leaf of the foliation. Now we assume that the tangent bundle $T\mathcal{F}$ is orientable. The choice of an orientation determines a volume form $\chi_{\mathcal{F}} \in \Omega^{0,p}\left(M, \mathcal{F}\right)$. Now we can define leafwise Hodge star-operator

$$*_{\mathcal{F}} : \Lambda^{0,k}T^{*}\mathcal{F} \to \Lambda^{0,p-k}T^{*}\mathcal{F} \text{ for each } k \text{ and } x \in M,$$

and it is determined by the relation

$$\alpha \wedge *_{\mathcal{F}}\beta = \langle \alpha, \beta \rangle \chi_{\mathcal{F}}, \text{ for } \alpha, \beta \in \Lambda^{0,k}T_{x}^{*}\mathcal{F}.$$

This fibrewise star-operator determines the leafwise star-operator

$$*_{\mathcal{F}} : \Omega^{0,k}(M,\mathcal{F}) \to \Omega^{0,p-k}(M,\mathcal{F}) \text{ for each } k.$$

Now we state some important properties of leafwise cohomology. Suppose $M$ is compact, and $\mathcal{F}$ is a $p$-dimensional oriented Riemannian foliation of $M$ with a bundle-like metric $g$.

**Proposition 3.7.** *[10, Theorem 0.2] The map $\phi : \ker \Delta^k_{\mathcal{F}} \to \bar{H}^k(M, d_{\mathcal{F}})$ defined by $\phi(\omega) = \omega \bmod \overline{\operatorname{im} d^k_{\mathcal{F}}}$ is a topological isomorphism of Frechét spaces. This isomorphism, in general, does not hold for non-Riemannian foliations.*

Under the same assumptions, the next deep result is due to Álvarez López and Kordyukov.

**Theorem 3.8.** *[2, Corollary C] The Hodge star-operator induces an isomorphism*

$$*_{\mathcal{F}} : \ker \Delta^k_{\mathcal{F}} \to \ker \Delta^{p-k}_{\mathcal{F}}.$$

*Moreover $*_{\mathcal{F}}$ commutes with $\Delta^k_{\mathcal{F}}$ up to a sign. From the previous proposition we have the following isomorphism*

$$*_{\mathcal{F}} : \bar{H}^k(M, d_{\mathcal{F}}) \to \bar{H}^{p-k}(M, d_{\mathcal{F}}).$$

Let $\omega$ be a leafwise closed 1-form, which is not necessarily exact. We consider the twisted operator $d^{\omega}_{\mathcal{F}} : \Omega^{0,k}(M,\mathcal{F}) \to \Omega^{0,k+1}(M,\mathcal{F})$ defined by $d^{\omega}_{\mathcal{F}} = d_{\mathcal{F}} + \omega\wedge$, where $d_{\mathcal{F}}$ is the exterior derivative along the leaf. Since $d_{\mathcal{F}} \circ d_{\mathcal{F}} = d^2_{\mathcal{F}} = 0$, $\omega \wedge \omega = 0$, and

$d_{\mathcal{F}}(\omega \wedge \alpha) = -\omega \wedge d_{\mathcal{F}}\alpha$ for any $k$-form $\alpha$, it follows that $(d^{\omega}_{\mathcal{F}})^2 = 0$. The differential cochain complex $(\Omega^{0,*}(M,\mathcal{F}), d^{\omega}_{\mathcal{F}})$ is called the leafwise Morse-Novikov complex of the foliated manifold $(M,\mathcal{F})$. Let $d^{\omega,k}_{\mathcal{F}}$ be the restriction of $d^{\omega}_{\mathcal{F}}$ to $\Omega^{0,k}(M,\mathcal{F})$. The cohomology groups

$$H^k_{\omega}(M, d_{\mathcal{F}}) = \frac{\ker\left(d^{\omega,k}_{\mathcal{F}}\right)}{\operatorname{im}\left(d^{\omega,k-1}_{\mathcal{F}}\right)}$$

of this cochain complex are called the leafwise Morse-Novikov cohomology groups of $(M,\mathcal{F})$. For the purpose of obtaining Hodge decomposition, we need to consider the reduced leafwise Morse-Novikov cohomology

$$\bar{H}^k_{\omega}(M, d_{\mathcal{F}}) = \frac{\ker\left(d^{\omega,k}_{\mathcal{F}}\right)}{\overline{\operatorname{im}\left(d^{\omega,k-1}_{\mathcal{F}}\right)}}.$$

Here the closure $\overline{\operatorname{im}\left(d^{\omega,k-1}_{\mathcal{F}}\right)}$ is taken with respect to the Frechét topology on $\Omega^{0,k}(M,\mathcal{F})$. The cup product induced from exterior product of forms makes $\bar{H}^*_{\omega}(M, d_{\mathcal{F}})$ into a graded commutative algebra over $C^{\infty}(M)$.

**Proposition 3.9.** *If $\omega$ and $\theta = \omega + d_{\mathcal{F}}g$ are cohomologous in $H^1(M, d_{\mathcal{F}})$, then for each $k$, the leafwise Morse-Novikov cohomology groups $H^k_{\omega}(M, d_{\mathcal{F}})$ and $H^k_{\theta}(M, d_{\mathcal{F}})$ are isomorphic. That is, the map $\Phi : H^k_{\omega}(M, d_{\mathcal{F}}) \to H^k_{\theta}(M, d_{\mathcal{F}})$ given by $\Phi([\alpha]) = [e^{-g}\alpha]$ is an isomorphism.*

*Proof.* The proof of Proposition 2.2 translates almost verbatim for the proof of this proposition. We need only replace the differential $d$ by $d_{\mathcal{F}}$. □

**Corollary 3.10.** *If $\omega$ is a $d_{\mathcal{F}}$ exact 1-form, then for each $k$ the leafwise Morse-Novikov cohomology group $H^k_\omega(M, d_{\mathcal{F}})$ and the leafwise de Rham cohomology group $H^k(M, d_{\mathcal{F}})$ are isomorphic.*

$$H^k_\omega(M, d_{\mathcal{F}}) \cong H^k(M, d_{\mathcal{F}}).$$

**Corollary 3.11.** *If the first leafwise de Rham cohomology group $H^1(M, d_{\mathcal{F}})$ equals 0, then for every $d_{\mathcal{F}}$ closed 1-form $\omega$ and for each $k$ the leafwise Morse-Novikov cohomology groups satisfy $H^k_\omega(M, d_{\mathcal{F}}) = H^k(M, d_{\mathcal{F}})$.*

**Lemma 3.12.** *For any smooth foliation $(M, \mathcal{F})$ the leafwise Morse-Novikov cohomology $H^0_\omega(M, d_{\mathcal{F}}) = \{0\}$ if and only if $\omega$ is not $d_{\mathcal{F}}$ exact.*

*Proof.* Similar to the proof of Lemma 2.9. □

**Proposition 3.13.** *(Homotopy axiom for the leafwise Morse-Novikov cohomology). Let $f : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)$ and $g : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)$ be foliated homotopic maps, and let $\omega$ be a leafwise closed $1 - form$ on $(N, \mathcal{F}_1)$. Then there exists a positive function $h : (M, \mathcal{F}_0) \to \mathbb{R}$ such that for all $k$*

$$f^* = hg^* : H^k_\omega(N, d_{\mathcal{F}_1}) \to H^k_{f^*\omega}(M, d_{\mathcal{F}_0}).$$

*Proof.* Since $f$ and $g$ are foliated homotopic maps, by the homotopy axiom of leafwise de Rham cohomology (Proposition 3.4), they induce the same map in leafwise de Rham cohomology. Therefore, for any leafwise closed $1-$ form $\omega \in \Omega^1(N, \mathcal{F}_1)$, the pullback forms $f^*\omega$ , $g^*\omega \in H^1_\omega(M, d_{\mathcal{F}_0})$ are cohomologous. There exists a function $\nu : (M, \mathcal{F}_0) \to \mathbb{R}$ such that $g^*\omega - f^*\omega = d_{\mathcal{F}}\nu$. We define $h = e^\nu$. Then from the proof of Proposition 3.9, for any $d^\omega_{\mathcal{F}}$ closed form $\alpha$ on $(N, \mathcal{F}_1)$, $[hg^*\alpha] = [f^*\alpha] \in H^k_{f^*\omega}(M, d_{\mathcal{F}_0})$. □

**Corollary 3.14.** *If $f : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)$ is a foliated homotopy equivalence and $\omega$ is a leafwise closed $1-$form, then the leafwise Morse-Novikov cohomology groups $H^*_\omega(M, d_{\mathcal{F}_0})$ and $H^*_{f^*\omega}(N, d_{\mathcal{F}_1})$ are isomorphic; i.e.*

$$H^k_\omega(M, d_{\mathcal{F}_0}) \cong H^k_{f^*\omega}(N, d_{\mathcal{F}_1}), \text{ for all } k.$$

*Proof.* Similar to the proof of Corollary 2.18, replacing $d$ with $d_{\mathcal{F}}$. □

In the following, let $\mathcal{F}$ be a Riemannian foliation of a manifold $M$ endowed with a bundle-like metric $g_M$, $\dim(M) = n$, $\dim(\mathcal{F}) = p$, and $\text{codim}(\mathcal{F}) = q$. Let $T\mathcal{F}$ be the tangent bundle and $Q = TM/T\mathcal{F}$ be the normal bundle. We may canonically identify the $Q$ with $T\mathcal{F}^\perp$. Then we have $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$. This direct sum decomposition of the tangent bundle $TM$ induces a bigrading of the algebra $\Omega(M)$ of smooth differential forms:

$$\Omega^{u,v}(M, \mathcal{F}) = \Gamma(M, \Lambda^v T\mathcal{F}^* \otimes \Lambda^u T\mathcal{F}^{\perp *}).$$

We choose a tangential and a transversal orientation for $\mathcal{F}$ on any open subset $\mathcal{U} \subset M$. We obtain the Hodge star operator $*_{\mathcal{F}}$ on $T\mathcal{F}^*$ and $*_\perp$ on $T\mathcal{F}^{\perp *}$ to $\mathcal{U}$ such that $*_\perp(1) \wedge *_{\mathcal{F}}(1)$ is a positive volume form on $\mathcal{U} \subset M$.

**Lemma 3.15.** *(Lemma 3.2 in [2]) The Hodge star operator on $T^*M = T\mathcal{F}^{\perp *} \otimes T\mathcal{F}^*$ on $\mathcal{U}$ satisfies*

$$* = (-1)^{(q-u)v} *_{\perp} \otimes *_{\mathcal{F}} : \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \to \Lambda^{q-u} T\mathcal{F}^{\perp *} \otimes \Lambda^{p-v} T\mathcal{F}^*.$$

**Lemma 3.16.** $*^2 = (-1)^{(u+v)(p+q+1)}$ *on* $\Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^*$.

*Proof.* We have $*_{\perp}^2 = (-1)^{u(q+1)}$ and $*_{\mathcal{F}}^2 = (-1)^{v(p+1)}$ on $\Omega^{u,v}(M, \mathcal{F})$.

$$\begin{aligned} *^2 &= (-1)^{(q-u)v}(-1)^{u(p-v)} *_{\perp}^2 \otimes *_{\mathcal{F}}^2 : \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \to \\ &\qquad \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \\ &= (-1)^{(q-u)v}(-1)^{u(p-v)}(-1)^{u(q+1)}(-1)^{v(p+1)} id_{\perp} \otimes id_{\mathcal{F}} : \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \to \\ &\qquad \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \\ &= (-1)^{(u+v)(p+q+1)} id_{\perp} \otimes id_{\mathcal{F}} : \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^* \to \\ &\qquad \Lambda^u T\mathcal{F}^{\perp *} \otimes \Lambda^v T\mathcal{F}^*. \end{aligned}$$

□

**Lemma 3.17.** *(Formula 17 in [2]) The adjoint $\delta_{\mathcal{F}} = d_{\mathcal{F}}^*$ of $d_{\mathcal{F}}$ is given by*

$$\delta_{\mathcal{F}} \beta = d_{\mathcal{F}}^* \beta = (-1)^{pk+p+1} *_{\mathcal{F}} d_{\mathcal{F}} *_{\mathcal{F}} \beta,$$

*for any $\beta \in \Omega^{0,k}(M, \mathcal{F})$.*

*Proof.* The standard proof that $d^* = (-1)^{nk+n+1} * d *$ on $n$-manifolds applies on a foliated manifold in a local neighborhood. □

**Lemma 3.18.** *$\omega\lrcorner = (-1)^{pk+p} *_{\mathcal{F}} (\omega\wedge) *_{\mathcal{F}}$ for all $\omega \in \Omega^{0,k}(M,\mathcal{F})$.*

*Proof.* The proof of Lemma 2.22 carries over, replacing $\Omega^k(M)$ with $\Omega^{0,k}(M,\mathcal{F})$. □

By Lemmas above and the identity $*_{\mathcal{F}}^2 = (-1)^{k(p-k)}$, we have on $\Omega^{0,k}(M,\mathcal{F})$

$$(\omega\lrcorner) *_{\mathcal{F}} = (-1)^k *_{\mathcal{F}} (\omega\wedge)$$

$$*_{\mathcal{F}} (\omega\lrcorner) = (-1)^{k+1} (\omega\wedge) *_{\mathcal{F}}$$

$$*_{\mathcal{F}} d_{\mathcal{F}}^* = (-1)^k d_{\mathcal{F}} *_{\mathcal{F}}$$

$$d_{\mathcal{F}}^* *_{\mathcal{F}} = (-1)^{k+1} *_{\mathcal{F}} d_{\mathcal{F}}.$$

The adjoint of the leafwise differential $(d_{\mathcal{F}} + \omega\wedge)$ is $(d_{\mathcal{F}}^* + \omega\lrcorner)$. We denote the Laplacian corresponding to the differential $d_{\mathcal{F}} + \omega\wedge$ by $\Delta_\omega^{\mathcal{F}}$. Then

$$\Delta_\omega^{\mathcal{F}} = (d_{\mathcal{F}} + \omega\wedge)(d_{\mathcal{F}}^* + \omega\lrcorner) + (d_{\mathcal{F}}^* + \omega\lrcorner)(d_{\mathcal{F}} + \omega\wedge).$$

**Proposition 3.19.** *If $M$ is a closed compact oriented manifold of dimension $n = p + q$ and $\omega$ is a leafwise closed 1-form, then for the Hodge star operator $*_{\mathcal{F}}$, we have*

$$*_{\mathcal{F}} \Delta_\omega^{\mathcal{F}} = \Delta_{-\omega}^{\mathcal{F}} *_{\mathcal{F}}.$$

*Proof.* The proof of Theorem 2.25 carries over by replacing $*$ with $*_{\mathcal{F}}$ and $d$ with $d_{\mathcal{F}}$, for all $\beta \in \Omega^{0,k}(M, \mathcal{F})$:

$$\begin{aligned}
*_{\mathcal{F}}\Delta_{\omega}^{\mathcal{F}}\beta &= *_{\mathcal{F}}(d_{\mathcal{F}} + \omega\wedge)(d_{\mathcal{F}}^{*} + \omega\lrcorner)\beta + *_{\mathcal{F}}(d_{\mathcal{F}}^{*} + \omega\lrcorner)(d_{\mathcal{F}} + \omega\wedge)\beta \\
&= (-1)^{k}(d_{\mathcal{F}}^{*} - \omega\lrcorner) *_{\mathcal{F}} (d_{\mathcal{F}}^{*} + \omega\lrcorner)\beta + (-1)^{k+1}(d_{\mathcal{F}} - \omega\wedge) *_{\mathcal{F}} (d_{\mathcal{F}} + \omega\wedge)\beta \\
&= (-1)^{k}(-1)^{k}(d_{\mathcal{F}}^{*} - \omega\lrcorner)(d_{\mathcal{F}} - \omega\wedge) *_{\mathcal{F}} \beta + (-1)^{k+1}(-1)^{k+1}(d_{\mathcal{F}} - \omega\wedge)(d_{\mathcal{F}}^{*} - \omega\lrcorner) *_{\mathcal{F}} \beta \\
&= ((d_{\mathcal{F}}^{*} - \omega\lrcorner)(d_{\mathcal{F}} - \omega\wedge) + (d_{\mathcal{F}} - \omega\wedge)(d_{\mathcal{F}}^{*} - \omega\lrcorner)) *_{\mathcal{F}} \beta \\
&= \Delta_{-\omega}^{\mathcal{F}} *_{\mathcal{F}} \beta.
\end{aligned}$$

Thus the operator $*_{\mathcal{F}}$ maps $\Delta_{\omega}^{\mathcal{F}}$-harmonic forms to $\Delta_{-\omega}^{\mathcal{F}}$-harmonic forms. □

**Corollary 3.20.** *If we restrict the Laplacian $\Delta_{\omega}^{\mathcal{F}}$ on $\Omega^{0,v}(M, \mathcal{F})$, then* $\ker \Delta_{\omega}^{\mathcal{F}}$ *is finite dimensional, and every reduced leafwise Morse-Novikov cohomology class has a $\Delta_{\omega}^{\mathcal{F}}$ harmonic representative.*

*Proof.* Notice that the operator $\Delta_{\omega}^{\mathcal{F}}$ is defined on all forms in $\Omega^{u,v}(M, \mathcal{F})$, but it is elliptic when restricted on the forms $\Omega^{0,v}(M, \mathcal{F})$ along the leaves of the foliation (Section 1 in [2]). Using this ellipticity we can conclude that $\mathcal{H}_{\omega}^{k}(M, d_{\mathcal{F}}) = \ker \Delta_{\omega}^{\mathcal{F}} \subset \Omega^{0,k}(M, \mathcal{F})$ is isomorphic to $\bar{H}_{\omega}^{k}(M, d_{\mathcal{F}})$, and

$$\mathcal{H}_{\omega}^{k}(M, d_{\mathcal{F}}) \cong \bar{H}_{\omega}^{k}(M, d_{\mathcal{F}}).$$

□

**Corollary 3.21.** $\bar{H}^{k}_{\omega}(M, d_{\mathcal{F}}) \cong \bar{H}^{p-k}_{-\omega}(M, d_{\mathcal{F}})$.

*Proof.* Since the operator $*_{\mathcal{F}}$ maps $\Delta^{\mathcal{F}}_{\omega}$-harmonic forms to $\Delta^{\mathcal{F}}_{-\omega}$-harmonic forms, it induces the isomorphism

$$\mathcal{H}^{k}_{\omega}(M, d_{\mathcal{F}}) \cong \mathcal{H}^{p-k}_{-\omega}(M, d_{\mathcal{F}}).$$

□

We now extend leafwise Morse-Novikov cohomology to forms of general $u, v$ type. Let $\omega$ be a leafwise closed 1-form, which is not necessarily exact. We consider the twisted operator $d^{\omega}_{\mathcal{F}} : \Omega^{u,v}(M, \mathcal{F}) \to \Omega^{u,v+1}(M, \mathcal{F})$ defined by $d^{\omega}_{\mathcal{F}} = d_{\mathcal{F}} + \omega\wedge$, where $d_{\mathcal{F}}$ is the exterior derivative along the leaf.

**Proposition 3.22.** $(d_{\mathcal{F}} + \omega\wedge)^2 = 0$. *Therefore $d_{\mathcal{F}} + \omega\wedge$ is a differential of the sections of $\Omega^{u,v}(M, \mathcal{F})$.*

*Proof.* Observe that for any section $\alpha \wedge \beta \in \Omega^{u,v}(M, \mathcal{F})$, we have

$$\begin{aligned} (d_{\mathcal{F}} + \omega\wedge)^2 (\alpha \wedge \beta) &= (d_{\mathcal{F}} + \omega\wedge)\left((-1)^u \alpha \wedge d_{\mathcal{F}}\beta + \omega \wedge (\alpha \wedge \beta)\right) \\ = d_{\mathcal{F}}\left((-1)^u \alpha \wedge d_{\mathcal{F}}\beta\right) + (-1)^{u+1}\omega \wedge (\alpha \wedge d_{\mathcal{F}}\beta) &+ (-1)^u \left(\omega \wedge (\alpha \wedge d_{\mathcal{F}}\beta)\right) + \omega \wedge (\omega \wedge (\alpha \wedge \beta)) \\ &= (-1)^{u+1}\left(\omega \wedge (\alpha \wedge d_{\mathcal{F}}\beta) - \omega \wedge (\alpha \wedge d_{\mathcal{F}}\beta)\right) = 0. \end{aligned}$$

□

We call the differential cochain complex $(\Omega^{*,*}(M, \mathcal{F}), d^{\omega}_{\mathcal{F}})$ the general leafwise Morse-

Novikov complex of the foliated manifold $(M, \mathcal{F})$. The cohomology groups

$$H_{\omega}^{*,*}(M, d_{\mathcal{F}}) = \frac{\ker(d_{\mathcal{F}}^{\omega})}{\operatorname{im}(d_{\mathcal{F}}^{\omega})}$$

of this cochain complex are called the general leafwise Morse-Novikov cohomology groups of $(M, \mathcal{F})$. For the purpose of having Laplacian and Hodge decomposition, we need to consider reduced general leafwise Morse-Novikov cohomology

$$\bar{H}_{\omega}^{*,*}(M, d_{\mathcal{F}}) = \frac{\ker(d_{\mathcal{F}}^{\omega})}{\overline{\operatorname{im}(d_{\mathcal{F}}^{\omega})}}.$$

Here the closure $\overline{\operatorname{im}(d_{\mathcal{F}}^{\omega})}$ is taken with respect to the Frechét topology on $\Omega^{*,*}(M, \mathcal{F})$.

**Proposition 3.23.** *If $\omega$ and $\theta = \omega + d_{\mathcal{F}} g$ are cohomologous in $H^1(M, d_{\mathcal{F}})$, then for each $\ell, k$, the general leafwise Morse-Novikov cohomology groups $H_{\omega}^{\ell,k}(M, d_{\mathcal{F}})$ and $H_{\theta}^{\ell,k}(M, d_{\mathcal{F}})$ are isomorphic via the isomorphism $[\alpha] \mapsto [e^{-g}\alpha]$.*

*Proof.* Similar to the proof of Proposition 3.9. □

**Proposition 3.24.** *(Homotopy axiom for the general leafwise Morse-Novikov cohomology). Let $f : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)$ and $g : (M, \mathcal{F}_0) \to (N, \mathcal{F}_1)$ be foliated homotopic maps, and $\omega$ be a leafwise closed $1-$form on $(N, \mathcal{F}_1)$. Then there exists a positive function $h : (M, \mathcal{F}_0) \to \mathbb{R}$ such that, for all $\ell, k$*

$$f^* = hg^* : H_{\omega}^{\ell,k}(N, d_{\mathcal{F}_1}) \to H_{f^*\omega}^{\ell,k}(M, d_{\mathcal{F}_0}).$$

*Proof.* Similar to the proof of Proposition 3.13. □

Let $P = d_{\mathcal{F}} + \omega\wedge$, then its formal adjoint $P^*$ is $\delta_{\mathcal{F}} - \omega\lrcorner$. Let $D = P + P^*$ be the corresponding Dirac operator. Then the corresponding Laplacian $\Delta_{\mathcal{F}}^{\omega} = (P + P^*)^2 = PP^* + P^*P$ is a nonnegative, self-adjoint second order differential operator on the smooth sections on $\Omega^{p,q}(M, \mathcal{F})$. For each integer $k \geq 0$, let $H_k(\Delta_{\mathcal{F}}^{\omega})$ be the Hilbert space completion of the space $\Omega^{u,v}(M, \mathcal{F})$ with respect to the scalar product

$$\langle \alpha, \beta \rangle = \sum_{j=0}^{j=k} \left\langle (\Delta_{\mathcal{F}}^{\omega})^j \alpha, \beta \right\rangle$$

for $\alpha, \beta \in \Omega^{u,v}(M, \mathcal{F})$. For the corresponding norm $\|.\|_k$, we have

$$k \leq k' \Rightarrow \|\alpha\|_k \leq \|\alpha\|_{k'} \text{ for all } \alpha \in \Omega^{u,v}(M, \mathcal{F}).$$

Thus we obtain the chain of continuous inclusions

$$H = H_0(\Delta_{\mathcal{F}}^{\omega}) \supset H_1(\Delta_{\mathcal{F}}^{\omega}) \supset H_2(\Delta_{\mathcal{F}}^{\omega}) \supset \cdots \supset H_\infty(\Delta_{\mathcal{F}}^{\omega}),$$

where

$$H_\infty(\Delta_{\mathcal{F}}^{\omega}) = \bigcap_{k\geq 0} H_k(\Delta_{\mathcal{F}}^{\omega})$$

equipped with the Frechét topology.

**Theorem 3.25.** *Let $(M, \mathcal{F})$ be a smooth foliation of a closed Riemannian manifold. The Laplacian $\Delta_{\mathcal{F}}^{\omega}$ gives rise to an orthogonal direct sum decomposition*

$$H_{\infty}\left(\Delta_{\mathcal{F}}^{\omega}\right) \cong \ker \overline{\Delta_{\mathcal{F},\infty}^{\omega}} \oplus \overline{\operatorname{im}\overline{\Delta_{\mathcal{F},\infty}^{\omega}}} \cong \ker \overline{\Delta_{\mathcal{F},\infty}^{\omega}} \oplus \overline{\operatorname{im}\overline{P_{\infty}}} \oplus \overline{\operatorname{im}\overline{P_{\infty}^{*}}},$$

*where $\overline{\Delta_{\mathcal{F},\infty}^{\omega}}$, $\overline{P_{\infty}}$, and $\overline{P_{\infty}^{*}}$ are canonical continuous extensions of the corresponding differential operators.*

*Proof.* The complexification of the Dirac operator $D = P + P^{*}$ satisfies the hypothesis of Chernoff's Lemma 2.1 in [9]. This can be verified from Corollary 1.4 of [9]. Then with the ideas explained in Section 2 of [4], we have the real Hilbert spaces $H_{k}\left(\Delta_{\mathcal{F}}^{\omega}\right)$ and $H_{\infty}\left(\Delta_{\mathcal{F}}^{\omega}\right)$. We can extend the operators $D$ and $\Delta_{\mathcal{F},\infty}^{\omega}$

$$\overline{D_{\infty}}, \overline{\Delta_{\mathcal{F},\infty}^{\omega}} : H_{\infty}\left(\Delta_{\mathcal{F}}^{\omega}\right) \to H_{\infty}\left(\Delta_{\mathcal{F}}^{\omega}\right),$$

yielding the orthogonal decompositions

$$H_{\infty}\left(\Delta_{\mathcal{F}}^{\omega}\right) \cong \ker \overline{\Delta_{\mathcal{F},\infty}^{\omega}} \oplus \overline{\operatorname{im}\overline{\Delta_{\mathcal{F},\infty}^{\omega}}} \cong \ker \overline{D_{\infty}} \oplus \overline{\operatorname{im}\overline{D_{\infty}}}.$$

Notice the spaces $\Omega^{p,q}(M, \mathcal{F})$ are orthogonal to each other with respect to the inner product $\langle , \rangle_{k}$ defined above, for each $k \geq 0$. Therefore, it follows that $\overline{D_{\infty}}$ can be decomposed as the sum of the continuous operators

$$\overline{P_{\infty}}, \overline{P_{\infty}^{*}} : H_{\infty} \to H_{\infty},$$

which are extensions of $P_\infty$ and $P_\infty^*$ respectively. Since $\mathrm{im}P$ and $\mathrm{im}P^*$ are $\langle , \rangle_k$-orthogonal for each $k \geq 0$, we obtain the following orthogonal decomposition:

$$H_\infty\left(\Delta_{\mathcal{F}}^{\omega}\right) \cong \ker \overline{\Delta_{\mathcal{F},\infty}^{\omega}} \oplus \overline{\mathrm{im}\overline{P_\infty}} \oplus \overline{\mathrm{im}\overline{P_\infty^*}}.$$

□

**Corollary 3.26.** *Every reduced general leafwise Morse-Novikov cohomology class has a $\Delta^\omega$-harmonic representative.*

For any $\alpha \wedge \beta \in \Omega^{u,v}(M, \mathcal{F})$, from formula 17 in [4] we have $\delta_{\mathcal{F}} = (-1)^{n(u+v)+n+1} * d_{\mathcal{F}} *$. By using the identities

$$\omega \lrcorner = (-1)^{n(u+v)+n} * \omega \wedge *$$

$$\text{and } *^2 = (-1)^{(u+v)(n+1)},$$

it can be shown that

$$(\omega \lrcorner) * = (-1)^{u+v} * (\omega \wedge)\,.$$

$$* (\omega \lrcorner) = (-1)^{u+v+1} (\omega \wedge) *\,.$$

$$*\delta_{\mathcal{F}} = (-1)^{u+v}\, d_{\mathcal{F}} *\,.$$

$$\delta_{\mathcal{F}} * = (-1)^{u+v+1} * d_{\mathcal{F}}.$$

**Proposition 3.27.** *If $M$ is a closed compact oriented manifold of dimension $n = p + q$ and $\omega$ is a leafwise closed 1-form, then for the Hodge star operator $*$, we have*

$$*\Delta_{\mathcal{F}}^{\omega} = \Delta_{\mathcal{F}}^{-\omega} * .$$

*Proof.* Similar to Proposition 3.19 and Theorem 2.25, using the formulas above. Thus the operator $*$ maps $\Delta_{\mathcal{F}}^{\omega}$-harmonic forms to $\Delta_{\mathcal{F}}^{-\omega}$-harmonic forms. □

# 4 Generalized distributions and foliations

Let $M$ be a smooth manifold. A choice of $k$-dimensional linear subspace $D_x \subset T_xM$ at each point $x \in M$ is called a $k$-dimensional tangent distribution or just a distribution. A distribution is smooth if $D = \coprod_{x\in M} D_x \subset TM$ is a smooth subbundle of the tangent bundle $TM$. An immersed submanifold $N \subset M$ is called an integral manifold of $D$ if $T_xN = D_x$ at each point $x \in N$. If $V$ is any nowhere vanishing vector field on $M$, span$(V)$ is an example of 1-dimensional distribution, and the image of any integral curve of $V$ is an integral manifold of $V$. In $\mathbb{R}^n$, the vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ span a smooth $k$-dimensional distribution. The $k$-dimensional affine subspaces parallel to $\mathbb{R}^k$ are integral manifolds. It is not necessary that every smooth distribution has integral manifolds. For example, let $D$ be a distribution on $\mathbb{R}^3$ spanned by the vector fields $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y}$; it turns out that $D$ has no integral manifolds. It is said that $D$ is *involutive* if given any two smooth sections $X$ and $Y$ of $D$ defined on an open subset of $M$, their Lie bracket $[X, Y]$ is also a smooth section of $D$. It is called integrable if each point of $M$ is contained in some integral manifold of $D$. It turns out that every integrable distribution is involutive. Given a $k$-dimensional distribution $D \subset TM$, a coordinate chart $(U, \phi)$ on $M$ is called flat for $D$, if $\phi(U)$ is a product of connected open sets $V \times W \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and at points of $U$, the first $k$ coordinates vector fields $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^k}$ span $D$. A distribution $D \subset TM$ is called *completely integrable* if there exists a flat chart for $D$ in a neighborhood of every point of $M$. Every completely integrable distribution is integrable and therefore involutive. The famous Frobenius theorem states that every involutive distribution is completely integrable. Another way to say this is that the distribution is the tangent bundle of foliation of $M$. A $k$-dimensional foliation on an $n$-dimensional manifold $M$ is a collection of a disjoint, connected, immersed $k$-dimensional submanifolds of $M$ whose union is $M$, such that in a neighborhood $U$ of each point $x \in M$, there exists a flat chart as defined above. Each of the submanifolds is called a leaf of the foliation. For an example, if $\tau$ is a fixed real number, the image of the map

$$t \mapsto \alpha_\theta(t) = \left(e^{it}, e^{i(\tau t+\theta)}\right)$$

as $\theta$ ranges over $\mathbb{R}$ forms a 1-dimensional foliation of a torus $T^2$. If $\tau$ is rational, the leaves are circles, and if it is irrational each leaf is dense. Foliations are in one to one correspondence with involutive distributions. In other words, let $\mathcal{F}$ be a smooth foliation on a manifold $M$, then the collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on $M$. If the dimension of the subspace $D_p$ is a constant function of $p \in M$, it is called a regular distribution. For regular distributions, the global Frobenius theorem implies the converse. For details on distributions and the Frobenius theorem, see [23]. If the dimension of a distribution is not constant, it is called a *generalized distribution*. Sussman [40] and Stefan [39] extended the Frobenius theorem to smooth generalized distributions. In the theory of differential forms, the intersection of kernels of differential 1-forms $\ker \omega_1 \cap \cdots \cap \ker \omega_k$ defines a generalized distribution called a cosmooth distribution [12]. The system of equations $\omega_i(X) = \cdots = \omega_k(X) = 0$ for An integral manifold of this system is a submanifold whose tangent space at every point $p \in M$ is annihilated by each $\omega_i$. A maximal integral manifold is a submanifold $i : N \hookrightarrow M$ such that the kernel of the restriction map on forms $i^* : \Omega^1_p(M) \to \Omega^1_p(N)$

is spanned by the kernels of the $\omega_i$ at every point $p \in N$. A Pfaffian system is said to be completely integrable if $M$ admits a foliation by maximal integral manifolds. The necessary and sufficient conditions for complete integrability of a regular Pfaffian system are given by the Frobenius theorem [23]. An equivalent version states that if the ideal $\mathcal{I}$ generated by $\omega_i, \cdots, \omega_k$ is differentially closed, in other words $d\mathcal{I} \subset \mathcal{I}$, then the system admits a foliation by maximal integral manifolds.

**Definition 4.1.** A *smooth singular foliation* of a given manifold $(M, g, \mathcal{F})$ is a partition of M into smooth, connected, injectively immersed submanifolds, called leaves of the foliation, if there are possibly an arbitrary (infinite) family of smooth vector fields $X_1, X_2, \cdots$ on $M$ that spans the leaves at all points:

$$\forall p \in M, T_p\mathcal{F}_p = \operatorname{span}\left\{X_1(p), X_2(p), \cdots\right\}$$

where $\mathcal{F}_p$ denotes the leaf through the point $p \in M$.

**Remark 4.2.** Since smooth distributions are finitely generated [12], without loss of generality, we may assume the family of vector fields in the definition of the smooth singular foliation is finite.

**Example 4.3.** Let $\mathcal{F}$ be the partition of $\mathbb{R}^2$ by concentric circles around the origin. The leaves of this foliation consists of circles, except at the origin where the leaf is a point. Obviously all leaves are smooth, connected, and injectively immersed. Since this foliation is is spanned by the vector field $X = x\partial y - y\partial x$, it is a singular foliation.

**Example 4.4.** Let $\mathcal{F}$ be the partition of $\mathbb{R}^2$ defined by $L(x,y) = (x,y)$ when $y \geq 0$, and $L(x,y) = \mathbb{R} \times y$ when $y < 0$. The leaves of this foliation are either horizontal lines or points. Therefore they are smooth, connected, and injectively immersed. Notice that the foliation is spanned everywhere by the vector field $X = \phi(x,y)\,\partial_x$, where $\phi(x,y) = 0$ whenever $y \geq 0$. and $\phi(x,y) > 0$ for $y < 0$.

We want to study distributions (subbundles of the tangent bundle) given by the kernels of differential forms on a Riemannian manifold $M$ of dimension $n$. Given any differential $q$-form $\nu$, the kernel of the differential form is the distribution defined at a point $x \in M$ by

$$(\ker \nu)|_x = \{X \in T_xM : X \lrcorner \nu = 0\}.$$

If $M$ is a Riemannian manifold of dimension $n$ and $e_1, \cdots, e_p, b_1, \cdots, b_q$ such that $p + q = n$ is an adapted local orthonormal basis of $T_xM$ for the distribution defined as $\mathrm{span}\{e_1, \cdots, e_p\}$ then $\omega = b^1 \wedge \cdots \wedge b^q$ is called its *transverse volume form*, which is defined up to a sign that depends on transverse orientation. For example, consider the form $\nu = dx^1 \wedge dx^2 \wedge dx^3$. It turns out $\ker(\nu)$ defines a 2-dimensional distribution on $\mathbb{R}^5$. In general the dimension of the kernel of a $q$-differential form at a point where it is not identically zero may be less than $n - q$. Consider the differential form $\nu = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ in $\mathbb{R}^5$ and the vector field $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4 + X^5\partial_5$. Then $X \lrcorner \nu = 0$ if and only if $X^j = 0$ for $1 \leq j \leq 4$. Hence $\ker(\nu) = \mathrm{span}\{\partial_5\}$, and $\dim(\ker(\nu)) = 1$. Notice that if $\nu$ is the transverse volume form of a distribution $D$, then $D = \ker(\nu)$. A distribution that arises as the kernel of a differential form is a cosmooth distribution. A cosmooth distribution may have different dimensions at different points of the manifold, as shown in the example above.

**Proposition 4.5.** *The rank of the kernel of a $q$-form $\nu$ on a manifold $M$ of dimension $n$ at a point $x_0$ where it is not zero is at most $n-q$. That is,*

$$\dim(\ker(\nu)) \leq n - q.$$

*Proof.* Let $\nu$ be an arbitrary $q$-form and $\text{rank}(\ker(\nu)) = p$ at point $x_0 \in M$. Let $v_1, \cdots, v_p$ be a basis of $\ker(\nu) \subseteq T_{x_0}M$. We can form an adapted coordinate chart in a neighborhood of $x_0$ such that $v_1, \cdots, v_p, v_{p+1}, \cdots, v_n$ is a basis of the tangent space $T_{x_0}M$ and $v_j|_{x_0} = \partial_j$ for $1 \leq i \leq p$. Since they are not required to be orthonormal, we may choose $v_i = \partial_i$ for $i > p$. In the corresponding dual basis $v_1^*, \cdots, v_p^*, v_{p+1}^*, \cdots, v_n^*$ of $T_{x_0}^*M$, the form $\nu$ can be written as $\nu = \sum_I c_I v_{i_1}^* \wedge \cdots \wedge v_{i_q}^*$. Notice that if $i \notin \{1, \cdots, p\}$ then $\partial_i \lrcorner \nu|_{x_0} = 0$ and if $i \in \{1, \cdots, p\}$ then we have

$$\partial_i \lrcorner \left( \sum_I c_I v_{i_1}^* \wedge \cdots \wedge v_{i_q}^* \right) \quad = \quad 0.$$

Now if $i \notin I = \{i_1, \cdots, i_p\}$ then $\partial_i \lrcorner v_{i_1}^* \wedge \cdots \wedge v_{i_q}^* = 0$ at $x_0$, and if $i \in I$ then we have $\partial_i \lrcorner v_{i_1}^* \wedge \cdots \wedge v_{i_q}^* \neq 0$ is a nonzero $(q-1)$-form. Then from the equation $\sum_I c_I \partial_i \lrcorner v_{i_1}^* \wedge \cdots \wedge v_{i_q}^* = 0$ and linear independence of the forms $v_{i_1}^* \wedge \cdots \wedge v_{i_q}^*$, we have $C_I = 0$ if $i \in I$. Therefore $C_I \neq 0$ only if $i_1, \cdots, i_q \in \{p+1, \cdots, n\}$, and this implies $q \leq (n-p)$. Hence $p \leq (n-q)$. □

**Corollary 4.6.** *Let $\nu = \nu_q + \cdots + \nu_0$, where $\nu_i \in \Omega^i(M)$ for $i = 0, \cdots, q$, be a differential form that has mixed degree terms and $X \in \Gamma(TM)$. If $\nu_q$ is nonzero at a point, then $X \lrcorner \nu = 0$ if and only if $X \lrcorner \nu_i = 0$ for $i = 0, \cdots, q$. Therefore* $\dim(\ker \nu) \leq n - q$.

**Proposition 4.7.** *Let $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$ be a smooth two form on a manifold $M$ of dimension $n$. At any point, the space* $\ker(\omega)$ *is even dimensional when $n$ is even and, it is odd dimensional when $n$ is odd.*

*Proof.* Let $X = \sum_{i=1}^n X^i \partial_i \in T_x M$. We have $X \lrcorner \omega = 0$ if and only if

$$\left(\sum_{k=1}^n X^k \partial_k\right) \lrcorner \left(\sum_{i<j} \omega_{ij} dx^i \wedge dx^j\right) = 0,$$

if and only if

$$\begin{aligned} \sum_{k<i} \omega_{ki} X^k dx^i - \sum_{i<k} \omega_{ik} X^k dx^i &= 0 \\ \sum_{k<i} \omega_{ki} X^k - \sum_{i<k} \omega_{ik} X^k &= 0, \end{aligned}$$

which is equivalent to the homogeneous system of linear equations $AX = 0$, where the matrix $A = (a_{ij})$ is a square matrix of size $n \times n$ with entries $a_{ij} = \omega_{ij}$, and $a_{ji} = -\omega_{ij}$ for $i \leq j$. Notice that the dimension of the $\ker(\omega)$ is the nullity of the coefficient matrix $A$, and $A$ is a skew symmetric matrix. Since the rank of a skew symmetric matrix is always even, the nullity of the matrix $A$ is even when $n$ is even and the nullity of the matrix $A$ is odd when $n$ is odd. □

**Theorem 4.8.** *(Generalized Rummler's Formula [37]) Let $\nu \in \Omega^{q,0}$ be a $q$-form on a Riemannian manifold of dimension $n$ which at each point is a multiple of a transverse* *volume form of the cosmooth distribution* $D = \ker(\nu)$*, with* $1 \leq q < n$*. Then there exists a* 1*-form* $\omega \in \Omega^{1,0}$ *and a* $(q+1)$*-form* $\phi_0 \in \Omega^{2,q-1}$ *such that*

$$d\nu = -\omega \wedge \nu + \phi_0;$$

*that is,*

$$(d + \omega\wedge)\nu = \phi_0.$$

*Proof.* Let the rank of $D$ be $p < n$ on a Riemannian manifold $M$ near a point $x_0 \in M$ and $e_1, \cdots, e_p$ be an orthonormal frame of $D$ at a point $x_0 \in M$ and $b_1, \cdots, b_q$ be an orthonormal frame of the normal distribution $D^\perp$ of $D$ at the point $x_0 \in M$. Then $e_1, \cdots, e_p, b_1, \cdots, b_q$ is called an adapted orthonormal frame of $T_{x_0}M$ such that $p + q = n = \dim(M)$. Let $e^1, \cdots, e^p, b^1, \cdots, b^q$ be the dual frame corresponding to the adapted frame $e_1, \cdots, e_p, b_1, \cdots, b_q$ of $T_{x_0}M$. By extending this frame on a neighborhood of the point $x_0 \in M$, we have $\nu = fb^1 \wedge \cdots \wedge b^q$, of $D^\perp$ for some nonzero function $f$ near $x_0$. Taking the differential of $\nu$, we have

$$d\nu = \sum_{i=1}^{i=p} e_i(f)e^i \wedge b^1 \wedge \cdots \wedge b^j \wedge \cdots \wedge b^q + \sum_{j=1}^{q} (-1)^{(j+1)} fb^1 \wedge \cdots \wedge db^j \wedge \cdots \wedge b^q.$$

We can write the 2-form $db^j$ as

$$db^j = \sum c^j_{kl} e^k \wedge b^l + \sum_{\alpha<\beta} r^j_{\alpha\beta} b^\alpha \wedge b^\beta + \sum s^j_{uv} e^u \wedge e^v,$$

for some functions $c^j_{kl}$, $r^j_{\alpha\beta}$, $s^j_{\alpha\beta}$, and $1 \leq k, u, v \leq p$ and $1 \leq l, \alpha, \beta \leq q$. Substituting $db^j$

in $d\nu$, we have

$$d\nu = \left( \sum_{k=1}^{k=p} \left( e_k(f) + \sum_{j=1}^{j=q} f c^j_{kj} \right) e^k \right) \wedge b^1 \wedge \cdots \wedge b^q + \phi_0$$
$$= -\omega \wedge \nu + \phi_0.$$

Here $\omega = -\sum_{k=1}^{k=p} \left( e_k(f) + \sum_{j=1}^{j=q} f c^j_{kj} \right) e^k$ is a 1-form and

$$\phi_0 = \sum_{j=1, k<l}^{j=q} (-1)^{(j+1)} f s^j_{kl} b^1 \wedge \cdots \wedge \hat{b^j} \wedge \cdots \wedge b^q \wedge e^k \wedge e^l.$$

□

**Corollary 4.9.** *If $\nu$ is a volume form of degree $(n-1)$, then there exists a 1-form $\omega$ such that $d\nu = -\omega \wedge \nu$.*

**Theorem 4.10.** *Let $\nu$ be a $q$-form on a Riemannian manifold of dimension $n$ which at each point is a multiple of a transverse volume form of the cosmooth distribution $D = \ker(\nu)$, with $1 \leq q < n$. Then there exists a 1-form $\omega$ such that*

$$d\nu = -\omega \wedge \nu,$$

*that is*

$$(d + \omega \wedge)\nu = 0,$$

*if and only if $D$ is involutive, if and only if $D$ is the tangent bundle of a cosmooth foliation whose dimension is $(n-q)$ when $\nu \neq 0$. The form $\omega$ may be chosen to be a leafwise closed 1-form.*

*Proof.* The distribution $D = \ker \nu$ is involutive if and only if for any sections $X_1, X_2 \in \Gamma(D)$, $X_1 \lrcorner \nu = X_2 \lrcorner \nu = [X_1, X_2] \lrcorner \nu = 0$. If $\ker \nu$ is involutive, any vector fields $X_1, \cdots, X_{q+1}$ with $X_1, X_2 \in \ker \nu$, we have

$$d\nu(X_1, X_2, \cdots, X_{q+1}) = \sum (-1)^j X_j \nu(X_1, \cdots, \hat{X}_j, \cdots, X_{q+1}) + \sum (-1)^{i+j} \nu([X_i, X_j] \cdots, \hat{X}_i \cdots \hat{X}_j, \cdots, X_{q+1}).$$

By the generalized Rummler's formula 4.8,

$$\begin{aligned} (\omega \wedge \nu)(X_1, X_2, \cdots, X_{q+1}) + \phi_0(X_1, X_2, \cdots, X_{q+1}) = 0 \\ = \phi_0(X_1, X_2, \cdots, X_{q+1}) = 0, \end{aligned}$$

since $X_3, \cdots, X_{q+1}$ are arbitrary and $X_1, X_2$ are arbitrary sections of $D$. Hence we have $d\nu = -\omega \wedge \nu$ if $D$ is involutive.

Conversely, suppose that $d\nu = -\omega \wedge \nu$ for some 1-form $\omega$ and the given $q$-form $\nu$. For any sections $X_1, X_2 \in \Gamma(D)$, and $X_3, \cdots, X_{q+1} \in \Gamma(TM)$,

$$d\nu(X_1, X_2, \cdots, X_{q+1}) = -(\omega \wedge \nu)(X_1, X_2, \cdots, X_{q+1})$$

implies that

$$\begin{aligned} 0 &= \sum (-1)^{i+j} \nu([X_i, X_j] \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{q+1}) \\ &= \nu([X_1, X_2] \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{q+1}). \end{aligned}$$

Since $X_3, \cdots, X_{q+1}$ are arbitrary vector fields, we get $[X_1, X_2] \in \Gamma(D)$. Therefore $D$ is involutive. From the generalized Rummler's formula, we see that $\omega$ may be chosen to be a leafwise 1-form. Also, since

$$d\,(d + \omega \wedge)\,\nu = d\omega \wedge \nu = d_{\mathcal{F}}\omega \wedge \nu = 0,$$

$\omega$ is a $d_{\mathcal{F}}$ closed form. □

**Corollary 4.11.** *If $\nu = b^1 \wedge \cdots \wedge b^q$ is the transverse volume form of the distribution $D$, then the 1-form $\omega = -\sum c^j_{kj} e^k$ in the previous theorem is the mean curvature form $\kappa$ of the transverse distribution $D^\perp$ defined by*

$$d\nu = -\kappa \wedge \nu.$$

*Proof.* Let $\nabla$ be the Levi-Civita connection of the metric on $TM$. By definition, the coefficients $c^j_{kj}$ are given by

$$\begin{aligned} c^j_{kj} &= db^j(e_k, b_j) \\ &= e_k(b^j(b_j)) - b_j(b^j(e_k)) - b^j([e_k, b_j]) \\ &= -b^j(\nabla_{e_k} b_j - \nabla_{b_j} e_k) \\ &= -b^j(\nabla_{e_k} b_j) + b^j(\nabla_{b_j} e_k) \\ &= -\langle \nabla_{e_k} b_j, b_j \rangle + \langle \nabla_{b_j} e_k, b_j \rangle. \end{aligned}$$

But $e_k\langle b_j, b_j\rangle = 0$, so that

$$\langle \nabla_{e_k} b_j, b_j\rangle = 0,$$

and using the Einstein summation convention,

$$\begin{aligned}\langle \nabla_{b_j} e_k, b_j\rangle &= b_j\langle e_k, b_j\rangle - \langle e_k, \nabla_{b_j} b_j\rangle \\ &= -\kappa(e_k),\end{aligned}$$

where $\kappa$ is the mean curvature 1-form of the distribution $D^{\perp}$. Therefore

$$c_{kj}^{j} = -\kappa(e_k),$$

and

$$\omega = -\sum c_{kj}^{j} e^k = \sum \kappa(e_k) e^k = \kappa.$$

□

**Corollary 4.12.** *With notations as in Corollary 4.11, if $\bar{\nu} = f(x)\nu$ for some positive function $f(x)$, then the mean curvature of the subbundle $D^{\perp}$ is leafwise cohomologous to $\kappa$.*

*Proof.*

$$d\bar{\nu} = df(x) \wedge \nu + f(x) d\nu,$$
$$= (df(x) - f(x)\kappa) \wedge \nu$$
$$= (d(\ln(f(x))) - \kappa) \wedge \bar{\nu}$$
$$= (d_{\mathcal{F}}(\ln(f(x))) - \kappa) \wedge \bar{\nu}.$$

□

For a smooth, codimension $q$, and transversally oriented foliation $(M, g, \mathcal{F})$ on a Riemannian manifold, let $b_1, \cdots, b_p, e_1 \cdots, e_q$ be a local adapted orthonormal frame with corresponding coframe $b^1, \cdots, b^p, e^1 \cdots, e^q$, such that $\nu = e^1 \wedge \cdots \wedge e^q$ be the positive transverse volume form. The form $\nu$ depends only on the transverse orientation and the metric. We have the following useful lemma.

**Lemma 4.13.** *Suppose that a transversally oriented foliation of a Riemannian manifold with metric $g$, and the corresponding transverse volume form is given. For any other metric $\bar{g}$, the corresponding volume form $\bar{\nu}$ satisfies*

$$\bar{\nu} = f\nu,$$

*for some positive function $f$.*

*Proof.* Let $b_1, \cdots, b_p, e_1 \cdots, e_q$ be a local adapted $g$-orthonormal frame such that the transverse volume form is $\nu = e^1 \wedge \cdots \wedge e^q$. Let $\bar{b_1}, \cdots, \bar{b_p}, \bar{e_1} \cdots, \bar{e_q}$ be a local adapted

$\bar{g}$-orthonormal frame. Since at each point $\text{span}\{b_1, \cdots, b_p\} = \text{span}\{\bar{b_1}, \cdots, \bar{b_p}\}$ is the tangent space of the foliation, the coframe satisfies $\bar{e}^j(b_k) = 0$ for all $j, k$, so that we must have $\bar{e}^j = \alpha_{j1}e^1 + \cdots + \alpha_{jq}e^q$ for $j = 1, \cdots, q$, for some functions $\alpha_{jk}$, where the matrix $(\alpha_{jk})$ is invertible and orientation-preserving. Then $\bar{\nu} = \det(\alpha_{jk})\nu$. □

Here we show a particular case of this lemma.

**Example 4.14.** Consider the foliation of the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ by horizontal lines. In standard coordinates, $e^1$ and $e^2$ are the leafwise and transverse volume forms respectively corresponding the orthonormal frame $e_1 = (1, 0), e_2 = (0, 1)$ for the induced metric from standard metric of $\mathbb{R}^2$. Consider another metric by declaring $\bar{e}_1 = e_1$, and $\bar{e}_2 = e_1 + e_2$ to be orthonormal. Since dual of $\bar{e}_2$ satisfies $\bar{e}_2^*(e_1) = 0$ and $\bar{e}_2^*(e_1 + e_2) = 1$, it is simply $e^2$, and since the dual of $\bar{e}_1$ satisfies $\bar{e}_1^*(e_1) = 1$ and $\bar{e}_1^*(e_1 + e_2) = 0$, its easy to see that $\bar{e}_1^* = e^1 - e^2$. The transverse volume form remained the same while the tangential volume form changed.

**Corollary 4.15.** *A transversely oriented foliation of a Riemannian manifold uniquely determines a leafwise Morse-Novikov cohomology class that is independent of the choice of metric.*

*Proof.* It follows from Theorem 4.10, Corollary 4.11, and Lemma 4.13. □

**Corollary 4.16.** *The isomorphism classes of leafwise Morse-Novikov cohomology groups determined by a transversally oriented foliation of a Riemannian manifold are invariant under foliated diffeomorphism.*

*Proof.* Given a foliated diffeomorphism $f : (M, \mathcal{F}_M) \to (N, \mathcal{F}_N)$ pulls back the metric on $N$ to another metric on $M$. The result follows from the previous corollary. □

**Lemma 4.17.** *There is a maximal smooth distribution $S$ contained in each generalized distribution $L$ of a manifold $M$.*

*Proof.* Let $S$ be the distribution defined as $S_p = \{V_p : V \in C^\infty(L)\}$ for all $p \in M$. Notice that $\{0\} \subseteq S_p \subseteq L_p$ at each point $p \in M$. Then $S$ is a smooth distribution, and by construction it is maximal. □

**Lemma 4.18.** *For any smooth distribution $S$ on a manifold $M$, the set of points of $M$ for which the distribution has maximal rank is an open set.*

*Proof.* Consider the function $r : M \to \mathbb{Z}$ defined by $r(x) = \text{rank}(S(x))$. By Drager-Lee-Park-Richardson [12], S is the local span of a finite number of vector fields. Since the rank of the span of a finite number of vector fields is a lower semicontinuous function, $r$ is lower semicontinuous and the set of points $x \in M$ such that the leaves through the points have maximal dimension is an open subset of $M$. □

**Example 4.19.** Let $L$ be the kernel of $\omega = xdx + ydy$ in $\mathbb{R}^2$. It turns out that the maximal smooth distribution $S$ is the span of a single vector field $X = x\partial_y - y\partial_x$. Note that $S = L$ at all points except the origin, and $S$ at the origin is $\{0\}$ and $L$ at the origin is the whole tangent space $\mathbb{R}^2$. So in particular, the $\ker(\omega = xdx + ydy)$ is not a smooth singular distribution, because the rank suddenly jumps up from 1 to 2 at the origin.

**Proposition 4.20.** *Every smooth singular foliation of a Riemannian manifold $(M, g, \mathcal{F})$ uniquely determines an isomorphism class of leafwise Morse-Novikov cohomology groups on the maximal regular part.*

*Proof.* The tangent distribution of the maximal regular part of the singular foliation is involutive and is the kernel of a transverse volume form $\nu$. From Proposition 4.10, we have that $\ker(\nu)$ is involutive if and only if there exists an 1-form $\omega$ such that $(d + \omega\wedge)\,\nu = 0$. Here $\omega$ is the mean curvature 1-form of the transverse distribution. From the generalized Rummler's formula, $\omega$ is a $d_{\mathcal{F}}$ closed form along the leaf of the regular open part of $\mathcal{F}$. The form $\nu$ is in the kernel of the differential $d_{\mathcal{F}} + \omega\wedge$, hence determines a leafwise Morse-Novikov cohomology class of the regular part of the given smooth singular foliation of $M$. If we choose another metric, then by Lemma 4.13, the transverse volume form $\nu$ changes by multiplication by a positive function. Assume $\bar{\nu} = \psi\nu$. Notice $\psi\nu$ has the same involutive kernel on the regular open part of the smooth singular foliation $\mathcal{F}$. Taking the differential of $\bar{\nu}$, we have $d(\psi\nu) = -\bar{\omega} \wedge (\psi\nu)$. From Corollary 4.12, $\bar{\omega} = d(\ln(\psi)) - \omega$. Similar to Proposition 2.2 and Lemma above, the isomorphism classes of Morse-Novikov cohomology groups are independent of the choice of $\psi$. In other words, the isomorphism class is independent of the choice of the metric. □

**Corollary 4.21.** *Suppose $\omega_1 \cdots \omega_k$ are pointwise linearly independent 1-forms, for all $i = 1, \cdots, k$. If the distributions $\ker\omega_1, \cdots, \ker\omega_k$ are integrable, then $\ker\omega_1 \cap \cdots \cap \ker\omega_k$ is also integrable.*

*Proof.* By Proposition 4.10 $\ker\omega_i$ is integrable if and only if $d\omega_i = -\kappa_i \wedge \omega_i$, for all $i = 1, \cdots, k$. [21] By the Frobenius theorem $\ker\omega_1 \cap \cdots \cap \ker\omega_k$ is integrable. That is through each point $x_o \in M$ there exists a maximal connected $(n-k)$-dimensional submanifold $i : N \hookrightarrow M$ such that $i^*\omega_i = 0$ for all $i = 1, \cdots, k$. This submanifold is unique, in the sense that any other such connected submanifold through $x_0$ is a subset of $i(N)$. □

**Proposition 4.22.** *Let $\nu$ be an arbitrary $q$-form on a manifold $M$ of dimension $n$. If $(d+\omega\wedge)\nu = 0$ for some 1-form $\omega$, then the maximal smooth distribution contained in* $\ker(\nu)$ *defines a smooth foliation.*

*Proof.* Assume $(d+\omega\wedge)\nu = 0$ for some 1-form $\omega$. By the Frobenius theorem, it is sufficient to show that $\ker(\nu)$ is involutive. Suppose $X_1, X_2 \in \Gamma(\ker(\nu)) = \{X \in \Gamma(TM) | X \lrcorner \nu = 0\}$ are vector fields on $M$. Consider the arbitrary vector fields $X_3, \cdots, X_{p+1}$ on $M$. Since $(d+\omega\wedge)\nu = 0$ then we have

$$d\nu(X_1, X_2, X_3, \cdots, X_{p+1}) = -(\omega \wedge \nu)(X_1, X_2, X_3, \cdots, X_{p+1})$$

Since $\omega$ is an 1-form, then $(\omega \wedge \nu)(X_1, X_2, X_3, \cdots, X_{p+1}) = 0$. Then

$$\begin{aligned} 0 = d\nu(X_1, X_2, \cdots, X_{q+1}) &= \sum (-1)^{i+1} X_i \nu(X_1, \cdots, \hat{X}_i, \cdots, X_{q+1}) \\ &\quad + \sum (-1)^{i+j} \nu([X_i, X_j] \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{q+1}) \\ &= \nu([X_1, X_2] \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_{q+1}) = 0. \end{aligned}$$

Since $X_3, \cdots, X_{q+1}$ are arbitrary vector fields, we get $[X_1, X_2] \in \ker(\nu)$. Therefore the maximal smooth distribution contained in $\ker(\nu)$ is involutive and thus determines a smooth foliation. □

**Corollary 4.23.** *Every Morse-Novikov cohomology class $[\nu] \in H^q_\omega(M)$ determines a smooth foliation whose tangent bundle is the maximal smooth distribution contained in* $\ker(\nu)$.

Since $H^*_\omega(M)$ is isomorphic to $H^*(M)$ for an exact 1-form $\omega$, the Morse-Novikov cohomology class $[\nu]$ corresponds to the de Rham cohomology class $[f\nu]$, for some positive function $f$. Notice that $\ker(\nu)$ and $\ker(f\nu)$ both define the same foliations of the manifold $M$. Therefore every de Rham cohomology class $[\nu] \in H^q(M)$ determines an involutive smooth foliation determined by $\ker(\nu)$.

Notice that if $(d + \omega\wedge)\alpha = 0$ for any 1-form $\omega$, where $\omega$ is not necessarily a closed, then we have

$$\begin{aligned} d(d+\omega\wedge)\alpha &= 0 \\ \Rightarrow d^2\alpha + d\omega \wedge \alpha - \omega \wedge d\alpha &= 0 \\ \Rightarrow d\omega \wedge \alpha - \omega \wedge (-\omega \wedge \alpha) &= 0 \\ \Rightarrow d\omega \wedge \alpha &= 0. \end{aligned}$$

Therefore $\alpha \in \ker(d\omega\wedge)$. Note that if $\nu$ is a transverse volume form corresponding to a foliation $\mathcal{F}$, we may always choose an 1-form $\omega \in \Omega^{(0,1)}(M, \mathcal{F})$. In that case $\omega$ is the mean curvature form, i.e. uniquely defined. Hence for a cosmooth integrable distribution $D$ defined by the kernel of a $q$-form $\nu$ which is the transverse volume form of the orthogonal subbundle $D^\perp$, we have

$$d\nu = -\kappa \wedge \nu$$

$$\Rightarrow d\kappa \wedge \nu = 0$$

$$\Rightarrow d_{\mathcal{F}}\kappa = 0.$$

**Theorem 4.24.** *Given a foliation $(M, \mathcal{F})$, let $\nu$ be a transverse volume form for one metric, and $\bar{\nu}$ be another transverse volume form for another metric. Then the leafwise Morse-Novikov cohomology groups determined by the foliation are isomorphic.*

*Proof.* From Proposition 4.8 and Corollary 4.12, we have $d\nu = -\kappa \wedge \nu$, and $d\bar{\nu} = -\bar{\kappa} \wedge \bar{\nu}$, where $\kappa$ and $\bar{\kappa}$ are mean curvature 1-forms of the transverse distribution of the foliation for two different metrics. From the discussion above $\kappa$ and $\bar{\kappa}$ are leafwise closed. Since the volume forms of different metrics are different up to multiple of a positive function, from Corollary 4.12, we have $\kappa$ and $\bar{\kappa}$ are leafwise closed and cohomologous. In other words, if $\bar{\nu} = f\nu$ then $\bar{\kappa} = (d_{\mathcal{F}}(\ln(f(x))) - \kappa) \wedge \bar{\nu}$. From Proposition 3.9, we have that the leafwise Morse-Novikov cohomology groups are isomorphic. □

**Remark 4.25.** The theorem above may not hold if the volume form for any metric vanishes at any point on the manifold, e.g, for a singular foliation.

**Theorem 4.26.** *Let $D$ be a cosmooth distribution corresponding to a singular smooth foliation that is the tangent bundle of foliation on its regular part and is all of the tangent space on the singular part. The mean curvature 1-form $\kappa$ of $D^{\perp}$ is $d_{\mathcal{F}}$-closed, and its $d_{\mathcal{F}}$-cohomology class is independent of the choice of metric.*

**Example 4.27.** Given a smooth distribution with characteristic $p$-form (tangent volume form) $\chi$ , Rummler's formula gives us that $d\chi = -\kappa \wedge \chi + \varphi_0$, where $\kappa$ is the mean curvature 1-form of the distribution and $\varphi_0$ is the form of type $(2, p-1)$ that is zero if the normal bundle distribution is involutive. Now we take $d$ of both sides

$$-d\kappa \wedge \chi + \kappa \wedge (d\chi) + d\varphi_0 = 0$$

$$-d\kappa \wedge \chi + \kappa \wedge (-\kappa \wedge \chi + \varphi_0) + d\varphi_0 = 0$$

$$-d\kappa \wedge \chi + \kappa \wedge \varphi_0 + d\varphi_0 = 0.$$

Now in many interesting cases, $d\kappa = 0$. For example, if the distribution is a Riemannian foliation, the bundle-like metric can be chosen so the mean curvature is a basic form, and then $\kappa$ is closed; see [11] and [1]. Then we get $(d + \kappa\wedge)\,\varphi_0 = 0$, so by the usual reasoning, $\ker \varphi_0$ defines a (generalized) foliation and a Morse-Novikov cohomology class. Since $\chi$ is the tangential volume form of the foliation, it is sufficient that $\kappa$ is $d_{1,0}$-closed, where $d_{1,0}$ is the component of the differential that differentiates in the normal direction of the foliation. So in the Riemannian foliation case, this class is a special invariant of the foliation.

In the case of Riemannian flows with basic mean curvature, $\varphi_0$ is a basic 2-form. Therefore we get a codimension 2 foliation that contains the original flow foliation.

If $\chi$ is the leafwise volume form of an 1-dimensional foliation with a $d_{1,0}$-closed mean curvature $\kappa$, then we have $\varphi_0$ is a form of the type $(2,0)$, and $\ker \varphi_0$ determines a cosmooth foliation, typically of codimension 2. Since $\varphi_0$ is a form of the type $(2,0)$, then for any vector field $X$ of the tangent bundle of the foliation, we have $X \lrcorner \varphi_0 = 0$. So the new foliation determined by $\ker \varphi_0$ contains the original foliation. If the manifold is 3-dimensional, then $\varphi_0$ must be a function multiple of the transverse volume form of the foliation.

In higher dimensions, since $\varphi_0$ is a form of type $(2, p-1)$, then $\ker \varphi_0$ is a foliation of dimension $q-1$ or less. If $\dim(\ker \varphi_0) = q-1$, then it can be verified that $q-2$ dimensions of the $\ker \varphi_0$ are in the normal bundle and one dimension is in the leaf direction of the foliation.

**Example 4.28.** The Lie algebra $\mathfrak{Sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ is spanned by the matrices $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We use the standard metric $\langle M, N \rangle = tr(M^t N)$, so that the three vectors are orthogonal with norms $\|a\| = \|b\| = 1$ and $\|c\| = \sqrt{2}$. We define the dual vectors $\alpha = a^*, \beta = b^*,$ and $\gamma = c^*$. The algebra of left invariant vector fields of a Lie group is isomorphic to the Lie algebra. Consider the foliation be determined by the integral curves of the left invariant vector field that corresponds to the tangent vector $c$ at the origin of $SL(2,\mathbb{R})$. For this foliation, $\chi = \frac{1}{\sqrt{2}}\gamma$ is its characteristic form. Its mean curvature is zero, $\varphi_0$ is a constant times the transverse volume form $\alpha \wedge \beta$, and the kernel of $\varphi_0$ is just the original foliation determined by $c$.

**Lemma 4.29.** *Let $\alpha$ be a nonzero 1-form and $\beta$ be a 2-form such that $\beta \wedge \alpha = 0$. Then there exists a 1-form $\tau$ such that $\beta = \alpha \wedge \tau$.*

*Proof.* Let $\alpha, \gamma^1, \cdots, \gamma^{(n-1)}$ be an adapted frame of the cotangent bundle on a neighborhood of a point $x_0 \in M$. We can write

$$\beta = \sum \beta_i \alpha \wedge \gamma^i + \sum_{i<j} \beta_{ij} \gamma^i \wedge \gamma^j$$

for some functions $\beta_i$ and $\beta_{ij}$. Then

$$\beta \wedge \alpha = 0$$

$$\Rightarrow \sum_{i<j} \beta_{ij} \gamma^i \wedge \gamma^j \wedge \alpha = 0.$$

Since $\gamma^i \wedge \gamma^j \wedge \alpha$ are linearly independent, we have $\beta_{ij} = 0$ for all $i, j$. Therefore $\beta = \sum \beta_i \alpha \wedge \gamma^i = \alpha \wedge \left(\sum \beta_i \gamma^i\right) = \alpha \wedge \tau$, where $\tau = \sum \beta_i \gamma^i$. □

**Proposition 4.30.** *Let $\nu$ be a regular 1-form. If $\ker(\nu)$ is involutive, then there exists an 1-form $\omega$ such that $d\nu = -\omega \wedge \nu$.*

*Proof.* If $\nu$ be a nonzero 1-form such that $\ker(\nu)$ is involutive, then by the Frobenius theorem $d\nu \wedge \nu = 0$. Therefore, by Lemma 4.29, we have $d\nu = -\omega \wedge \nu$ for some 1-form $\omega$. Hence every involutive cosmooth distribution determined by the kernel of a 1-form determines a Morse-Novikov cohomology class. □

**Lemma 4.31.** *If $\alpha \in \Omega^k(\mathbb{R}^p)$, $k \geq 1$, is a non-zero differential form, then there exists a vector field $X \in C^\infty(T\mathbb{R}^p)$ such that $X \lrcorner \alpha \neq 0$.*

*Proof.* Let $\alpha \in \Omega^r(\mathbb{R}^p)$. By Proposition 4.5, $\dim(\ker \alpha) \leq p - r$. If $r \neq p$, there exists $X \in C^\infty(T\mathbb{R}^p)$ such that $X \lrcorner \alpha \neq 0$, and if $r = p$, then $\dim(\ker \alpha) \leq 0$, and so $X \lrcorner \alpha \neq 0$ for all $X \in C^\infty(T\mathbb{R}^p)$. □

**Proposition 4.32.** *We consider a foliation $(M, \mathcal{F})$ of dimension $p$, given by the kernel of a $k$-form $\nu \in \Omega^k(M)$. i.e. $T\mathcal{F} = \ker \nu$. In an adapted frame $\{e_1, \cdots, e_p, b_1, \cdots, b_q\}$ of $TM$, $\nu$ can be represented by*

$$\nu = \sum_I \nu_I b^I,$$

*where* $\{e^1, \cdots, e^p, b^1, \cdots, b^q\}$ *is the corresponding dual frame of* $TM$*,* $\nu_I \in C^\infty(M)$*, and* $b^I = b^{I_1} \wedge \cdots \wedge b^{I_k}$*, for all multi-indices* $I = \{1 \leq I_1 < \cdots < I_k \leq q\}$*.*

*Proof.* Any nonzero form $\nu \in \Omega^k(\mathbb{R}^p)$ can be written as $\nu = \sum_I \nu_I b^I$, where $\nu_I$ are differential forms of the form $\nu_I = \sum_J \nu_{IJ} e^J$, where $e^J = e^{J_1} \wedge \cdots \wedge e^{J_s}$ and $b^I = b^{I_1} \wedge \cdots \wedge b^{I_t}$ for all multi-indices $I = \{1 \leq I_1 < \cdots < I_t \leq q\}$ and $J = \{1 \leq J_1 < \cdots < J_s \leq p\}$ such that $s + t = k$. By Lemma 4.29, there always exists a vector field $X = \sum_s X_s e^s$ such that $X \lrcorner \nu_I \neq 0$ if $\nu_I$ has degree greater than or equal to one. Hence $\nu_I$ is a zero form for all $I$. $\square$

The following proposition is a partial converse to Proposition 4.10.

**Proposition 4.33.** *Let* $\nu$ *be an* $(n-1)$*-form. If* $\ker(\omega)$ *is involutive, then there exists an* 1*-form* $\omega$ *such that* $d\nu = -\omega \wedge \nu$*.*

*Proof.* Let $\omega$ be an $(n-1)$-form on a Riemannian manifold of dimension $n$ and $e_1, \cdots, e_p$ be a local orthonormal frame of $\ker(\omega)$ and $e_1, \cdots, e_p, b_1, \cdots, b_q$ be an adapted orthonormal frame of $T_{x_0}M$ such that $p + q = n = \dim(M)$, where $b_1, \cdots, b_q$ is an orthonormal frame of the normal distribution $\ker(\omega)^\perp|_{x_0}$. Let $e^1, \cdots, e^p, b^1, \cdots, b^q$ be the dual frame corresponding to the adapted frame $e_1, \cdots, e_p, b_1, \cdots, b_q$ of $T_{x_0}M$. By extending this frame on a neighborhood $x_0$, we can write

$$\omega = \sum_{j=1}^{p} f_j e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q + \sum_{k=1}^{q} g_k e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q$$

Then

$$
\begin{aligned}
d\omega = \sum_{j=1}^{p} e_j(f_j)e^j \wedge e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} b_k(g_k)b^k e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q \\
+ \sum_{j=1}^{p} f_j \sum_{r=1}^{j-1} (-1)^{r+1} e^1 \wedge \cdots \wedge de^r \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{j=1}^{p} f_j \sum_{r=j+1}^{p} (-1)^{r} e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge de^r \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{j=1}^{p} f_j \sum_{s=1}^{q} (-1)^{p+s} e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge db^s \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} g_k \sum_{l=1}^{p} (-1)^{l+1} e^1 \wedge \cdots \wedge de^l \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} g_k \sum_{m=1}^{k-1} (-1)^{p+m+1} e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge db^m \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} g_k \sum_{m=k+1}^{q} (-1)^{p+m} e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge db^m \cdots \wedge b^q.
\end{aligned}
$$

By substituting

$$
\begin{aligned}
de^r &= \sum c^r_{tu} e^t \wedge b^u + \sum_{t<u} r^r_{tu} b^t \wedge b^u + \sum_{t<u} s^r_{tu} e^t \wedge e^u \\
de^s &= \sum c^s_{tu} e^t \wedge b^u + \sum_{t<u} r^s_{tu} b^t \wedge b^u + \sum_{t<u} s^s_{tu} e^t \wedge e^u \\
db^l &= \sum \bar{c}^l_{tu} e^t \wedge b^u + \sum_{t<u} \bar{r}^l_{tu} b^t \wedge b^u + \sum_{t<u} \bar{s}^l_{tu} e^t \wedge e^u \\
db^m &= \sum \bar{c}^m_{tu} e^t \wedge b^u + \sum_{t<u} \bar{r}^m_{tu} b^t \wedge b^u + \sum_{t<u} \bar{s}^m_{tu} e^t \wedge e^u,
\end{aligned}
$$

we have

$$
\begin{aligned}
d\omega = \sum_{j=1}^{p} e_j(f_j)e^j \wedge e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} b_k(g_k)b^k \wedge e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q \\
+ \sum_{j=1}^{p}(\sum_{r=1}^{r=p} -f_j s^r_{rj} e^j) \wedge e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{j=1}^{p}(\sum_{s=1}^{s=q} +f_j \bar{c}^s_{js} e^j)e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q}(\sum_{l=1}^{l=p} -g_k c^l_{lk} b^k)e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q}(\sum_{m=1}^{m=q} +\bar{g}_k r^m_{km} b^k)e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q
\end{aligned}
$$

If $f_j \neq 0$ and $g_j \neq 0$, combining like terms, we have

$$
\begin{aligned}
d\omega = \sum_{j=1}^{p} -(\sum_{r=1}^{r=p} s^r_{rj} - \sum_{s=1}^{s=q} \bar{c}^s_{js} - e_j(f_j)/f_j)f_j e^j \wedge e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q \\
+ \sum_{k=1}^{q} -(\sum_{m=1}^{m=q} \bar{r}^m_{km} + \sum_{l=1}^{l=p} c^l_{lk} - b_k(g_k)/g_k)g_k b^k \wedge e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q.
\end{aligned}
$$

Let

$$
\begin{aligned}
\alpha_i = (\sum_{r=1}^{r=p} s^r_{ri} - \sum_{s=1}^{s=q} \bar{c}^s_{is} - e_i(f_i)/f_i)e^i \text{ and } \alpha = \sum_{i=1}^{i=p} \alpha_i \\
\beta_i = (-\sum_{m=1}^{m=q} \bar{r}^m_{im} + \sum_{l=1}^{l=p} c^l_{li} - b_i(g_i)/g_i)b^i \text{ and } \beta = \sum_{i=1}^{i=q} \beta_i.
\end{aligned}
$$

We can write

$$d\omega = \sum_{j=1}^{p} -(\sum_{r=1}^{r=p} s^r_{rj} - \sum_{s=1}^{s=q} \bar{c}^s_{js} - e_j(f_j)/f_j) f_j e^j \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q$$

$$+ \sum_{k=1}^{q} -(\sum_{m=1}^{m=q} \bar{r}^m_{km} + \sum_{l=1}^{l=p} c^l_{lk} - e_k(g_k)/g_k) g_k b^k \wedge e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q$$

$$\Rightarrow d\omega = -(\alpha + \beta) \wedge (\sum_{j=1}^{p} f_j e^1 \wedge \cdots \wedge \hat{e^j} \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge b^q$$

$$+ \sum_{k=1}^{q} g_k e^1 \wedge \cdots \wedge e^p \wedge b^1 \wedge \cdots \wedge \hat{b^k} \wedge \cdots \wedge b^q)$$

$$\Rightarrow d\omega = -\tau \wedge \omega \text{ where } \tau = \alpha + \beta.$$

Therefore $\ker(\omega)$ is an involutive distribution, and hence defines a cosmooth foliation of the manifold $M$. □

**Corollary 4.34.** *Suppose* $\dim(M) \leq 3$*, and the* $\ker \nu$ *is locally constant dimensional. Then* $\ker \nu$ *is involutive if and only if there exists a* 1*-form* $\omega$ *such that* $d\nu = -\omega \wedge \nu$*. In other words, every cosmooth involutive distribution of locally constant dimension is associated with a leafwise Morse-Novikov cohomology class.*

*Proof.* This follows from Propositions 4.28, 4.30, and the fact that the kernel of a nonzero top form on a manifold is $\{0\}$. □

An interesting case is to consider when one chooses $\nu$ to be $(d + \omega\wedge)$-harmonic. For instance, if we take $\omega$ to be exact (i.e. the de Rham cohomology case), and if $\nu$ is harmonic then

$$\Delta\nu = (d\delta + \delta d)\,\nu = 0$$

$$\Rightarrow \langle (d\delta + \delta d)\,\nu, \nu\rangle = 0$$

$$\Rightarrow d\nu = 0 \text{ and } \delta\nu = 0.$$

We know the Hodge $*$ operator commutes with the Laplacian $\Delta$. If $\Delta\nu = 0$, then we have

$$\Delta\,(*\nu) = *\,(\Delta\nu) = 0$$

$$\Rightarrow d(*\nu) = 0 \text{ and } \delta * \nu = 0.$$

Therefore $\ker(*\nu)$ is involutive. In the case where $\nu$ is a transverse volume form that means that both the tangent and the normal bundles of the foliations given by $\ker(\nu)$ correspond to foliations with zero mean curvature.

In the case of Morse-Novikov cohomology ($\omega$ is closed but not exact), let $d_\omega\nu = 0$ such that $\nu$ is $d_\omega$-harmonic, where $d_\omega = d + \omega\wedge$. Then we have

$$\Delta_\omega\nu = (d_\omega\delta_\omega + \delta_\omega d_\omega)\,\nu = 0$$

$$\Rightarrow \langle (d_\omega\delta_\omega + \delta_\omega d_\omega)\,\nu, \nu\rangle = 0$$

$$\Rightarrow d_\omega\nu = 0 \text{ and } \delta_\omega\nu = 0.$$

We know from the Poincaré duality of Morse-Novikov cohomology that $*\Delta_\omega = -\Delta_{-\omega}*$

for the Laplacian $\Delta_\omega$ of the differential $d_\omega$. Then we have

$$\Delta_{-\omega}(*\nu) = *(\Delta_\omega \nu) = 0$$

$$\Rightarrow (d - \omega\wedge) * \nu = 0 \text{ and } (\delta - \omega\lrcorner) * \nu = 0.$$

Therefore $\ker(*\nu)$ is involutive.

Suppose $\nu$ is a transverse volume form such that $(d + \kappa\wedge)\nu = 0$, where $\kappa$ is the mean curvature of the transverse distribution of $\ker(\nu)$. If $\nu$ is $(d + \kappa\wedge)$ harmonic, then the leafwise volume form is $(d - \kappa\wedge)$ harmonic, the foliation is minimal, and the normal bundle is involutive.

**Lemma 4.35.** *Let $(M, \mathcal{F})$ be a foliation. Then for any leafwise closed 1-form $\kappa$, the adjoint $(d_{\mathcal{F}} + \kappa\wedge)^*$ of the differential $d_{\mathcal{F}} + \kappa\wedge$ is given by*

$$(d_{\mathcal{F}} + \kappa\wedge)^* = \pm * (d_{\mathcal{F}} - \kappa\wedge) *,$$

*where $*$ is the Hodge star operator on $M$.*

*Proof.* Since $d_{\mathcal{F}}$ is restriction of the differential $d$ along the leaf, the adjoint of $d$ is given by $\delta = (-1)^{nk+n+1} * d*$ [2], and $\kappa \lrcorner = (-1)^{nk+n} * \kappa \wedge *$, we have

$$
\begin{aligned}
(d_{\mathcal{F}} + \kappa\wedge)^* &= (\delta_{\mathcal{F}} + \kappa\lrcorner) \\
\Rightarrow (d_{\mathcal{F}} + \kappa\wedge)^* &= (-1)^{nk+n+1} * d_{\mathcal{F}} * + (-1)^{nk+n} * \kappa \wedge * \\
\Rightarrow (d_{\mathcal{F}} + \kappa\wedge)^* &= (-1)^{nk+n+1} * (d_{\mathcal{F}} - \kappa\wedge) * \\
\Rightarrow (d_{\mathcal{F}} + \kappa\wedge)^* &= \pm * (d_{\mathcal{F}} - \kappa\wedge) * .
\end{aligned}
$$

□

In the following we will use the term $(d + \omega\wedge)$-harmonic for differential forms that satisfy $(d + \omega\wedge)\,\alpha = 0$ and $(d + \omega\wedge)^*\,\alpha = 0$, whether or not $\omega$ is $d$-closed.

**Proposition 4.36.** *Let $\nu \in \Omega^{(q,0)}$ be a transverse volume form of a distribution $D$ of rank $q$, The foliation $(M, \mathcal{F})$ given by the* $\ker(\nu)$ *is minimal and the normal bundle $\mathcal{NF} = D^{\perp}$ is involutive if and only if $\nu$ is $(d + \kappa\wedge)$-harmonic.*

*Proof.* Notice that $(d + \kappa\wedge)\,\nu = 0$ implies $d\kappa \wedge \nu = 0$. Then for the foliation $(M, \mathcal{F})$ given by the kernel of the transverse volume form $\nu$, $d_{\mathcal{F}}\kappa = 0$. If $\nu$ is $(d + \kappa\wedge)$-harmonic, then $(d - \kappa\wedge)(*\nu) = 0$. Therefore $d(*\nu) = 0$, since $\kappa \in \Omega^{0,1}(M)$. Hence $\mathcal{NF} = \ker(*\nu)$ is involutive. From Rummler's formula we know that $d(*\nu) = -\kappa^{\perp} \wedge *\nu + \phi_0 = 0$. Here $\kappa^{\perp} = 0$ and $\phi_0 = 0$, where $*\nu$ is the characteristic form of the foliation and $\kappa^{\perp}$ is its mean curvature 1-form. Therefore $(M, \mathcal{F})$ is minimal.

Conversely, suppose that $\nu \in \Omega^{(q,0)}$ is a volume form such that $(d + \kappa\wedge)\,\nu = 0$ and the foliation $(M, \mathcal{F})$ is minimal and the normal distribution $\mathcal{NF}$ is involutive. Then

$\kappa \in \Omega^{(0,1)}$ has the property that $\left(d + \kappa^{(0,1)}\wedge\right)\nu = 0$. Since $\nu \in \Omega^{(q,0)}$ is a volume form and the foliation $(M, \mathcal{F})$ is minimal, and $\mathcal{NF} = \ker(*\nu)$ is involutive, then $d(*\nu) = 0$. Also $\kappa \wedge *\nu = 0$, so that $(d - \kappa\wedge) * \nu = 0$. Hence $\nu$ is $(d + \kappa\wedge)$ harmonic. □

In the following we consider the case where $\nu$ may or may not be a volume form.

**Proposition 4.37.** *If $\nu \in \Omega^k(M)$ is $(d + \omega\wedge)$-harmonic, and $\ker(\nu)$ and $\ker(*\nu)$ have the maximal rank at $x_0 \in M$, then the tangent bundle $T_{x_0}M$ of the manifold $M$ is the direct sum of the tangent spaces of the foliations $\mathcal{F} = \ker(\nu)$ and $\mathcal{F}' = \ker(*\nu)$. i.e. at $x_0$*

$$T_{x_0}M = \ker(\nu) \oplus \ker(*\nu) = T_{x_0}\mathcal{F} \oplus T_{x_0}\mathcal{F}'.$$

*Proof.* Since $\nu$ is $(d+\omega\wedge)$ harmonic, $(d + \omega\wedge)\nu = 0$ and $(d + \omega\wedge) * \nu = 0$, therefore $\mathcal{F} = \ker(\nu)$ and $\mathcal{F}' = \ker(*\nu)$ are involutive. Since the kernel of $\nu$ and $*\nu$ have maximal rank, $\dim(\ker(\nu)) = n - k$ and $\dim(\ker(*\nu)) = k$. Notice that $\dim(\ker(\nu)) + \dim(\ker(*\nu)) = \dim(M)$. So it will be enough to show that the intersection of $\ker(\nu)$ and $\ker(*\nu)$ contains only the zero vector. Suppose $X \in \ker(\nu) \cap \ker(*\nu)$, then we have

$$X\lrcorner(\nu \wedge *\nu) = X\lrcorner(\nu) \wedge *\nu + (-1)^k \nu \wedge (X\lrcorner * \nu) = 0.$$

Since $\nu \wedge *\nu$ is a positive multiple of the volume form, $X = 0$. □

We would like to discuss an example on a 3-dimensional manifold $M$ from Carrière's work [16] of a distribution determined by the kernel of a transverse volume form corresponding to a nontrivial class $[\omega] \in H^1(M)$ such that there exists a nontrivial Morse-Novikov class $\nu \in H^*_\omega(M)$.

**Example 4.38.** Let $A \in SL(2,\mathbb{Z})$ is a symmetric matrix of trace strictly greater than 2, and $V_1$ and $V_2$ be the eigenvectors of $A$ corresponding to the eigenvalues $\lambda$ and $\frac{1}{\lambda}$ with $\lambda > 1$ and irrational. An example of such a matrix is $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. It is known that map $m \to A(m)$ is a bijection of the flat torus $T^2$. Let the hyperbolic torus $\mathbb{T}^3_A$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation that identifies $(m,t)$ and $(Am, t+1)$. Let $(x,y,t)$ denote the local coordinate in the $V_2$, $V_1$, and $\mathbb{R}$ direction respectively. That is, for $m \in \mathbb{R}^2$, we define $x =< m, V_2 >$ and $y =< m, V_1 >$. We choose the metric to be

$$ds^2 = \lambda^{-2t}dx^2 + \lambda^{2t}dy^2 + dt^2.$$

Let $\nu_{\mathcal{F}} = \lambda^{-t}dx$. Taking the differential, we have

$$\begin{aligned} d\nu_{\mathcal{F}} &= d(\lambda^{-t}dx) \\ = -\log(\lambda)\lambda^{-t}dt \wedge dx &= -\kappa \wedge \nu_{\mathcal{F}} \\ \Rightarrow (d + \kappa\wedge)\nu_{\mathcal{F}} &= 0, \end{aligned}$$

where $\kappa = \log(\lambda)dt$. By the previous result $\ker(\nu_{\mathcal{F}})$ is involutive, and has constant dimension 2, so it determines a smooth foliation of $M$. Notice that $\kappa = \log(\lambda)dt$ is closed, and if $C$ is the circle $\{(0,t) : 0 \leq t \leq 1\}$, then

$$\oint_C \log(\lambda)dt = \int_0^1 \log(\lambda)dt = \log(\lambda).$$

Hence $[\kappa] \in H^1(M)$ is a nontrivial cohomology class. If $\nu_{\mathcal{F}}$ is $(d + \kappa\wedge)$-exact, then there exists some function $f$ on $M$ such that

$$(d+\kappa\wedge)f = \nu_{\mathcal{F}} = \lambda^{-t}dx$$

$$\partial_x f dx + \partial_y f dy + \partial_t f dt + f\log(\lambda)dt = \lambda^{-t}dx$$

$$\Rightarrow \partial_x f = \lambda^{-t}, \partial_y f = 0, \text{ and } \partial_t f + f\log(\lambda) = 0.$$

The general solution of these partial differential equations turns out to be $f(x,y,t) = (x+c)\lambda^{-t}$ for some constant $c$. However, no such function $f$ is continuous on $\mathbb{T}^3_A$ for any constant $c$. Hence $[\nu_{\mathcal{F}}] \in H^1_\kappa(M)$ is a nontrivial Morse-Novikov cohomology class.

If $\nu$ is not a transversal volume form of the involutive distribution $\ker(\nu)$, it may or may not be true that $d\nu = -\omega \wedge \nu$ for some 1-form $\omega$. Here is an example.

**Example 4.39.** Consider the following differential two form in $\mathbb{R}^5$

$$\nu = (1+x_3^2)dx_1 \wedge dx_2 + (1+x_5^2)dx_3 \wedge dx_4.$$

Let $X = X^1\partial_{x_1} + \cdots + X^5\partial_{x_5} \in \Gamma(T\mathbb{R}^5)$ be a vector field. Then

$$X \lrcorner \nu = 0$$

$$\Leftrightarrow X^1(1+x_3^2)dx_2 - X^2(1+x_3^2)dx_1 + X^3(1+x_5^2)dx_4 - X^4(1+x_5^2)dx_3 = 0$$

$$\Leftrightarrow X^1(1+x_3^2) = X^2(1+x_3^2) = X^3(1+x_5^2) = X^4(1+x_3^2) = 0$$

$$\Leftrightarrow X^1 = X^2 = X^3 = X^4 = 0.$$

Therefore $\ker(\nu)$ is the 1-dimensional distribution spanned by the $\partial_{x_5}$. Since a 1-dimensional distribution is integrable, $\ker(\nu)$ is integrable. By computing the differential of $\nu$, we have

$$\begin{aligned} d\nu &= 2x_3 dx_1 \wedge dx_2 \wedge dx_3 + 2x_5 dx_3 \wedge dx_4 \wedge dx_5 \\ &= \frac{2x_3}{1+x_3^2} dx_3 \wedge (1+x_3^2) dx_1 \wedge dx_2 + \frac{2x_5}{1+x_5^2} dx_5 \wedge (1+x_5^2) dx_3 \wedge dx_4 \\ &= \left( \frac{2x_3}{1+x_3^2} dx_3 + \frac{2x_5}{1+x_5^2} dx_5 \right) \wedge \big( (1+x_3^2) dx_1 \wedge dx_2 + (1+x_5^2) dx_3 \wedge dx_4 \big) \\ &\quad - \frac{2x_5(1+x_3^2)}{(1+x_5^2)} dx_1 \wedge dx_2 \wedge dx_5 \\ &= -\omega \wedge \nu + \phi_0, \end{aligned}$$

where $\omega = -\left( \frac{2x_3}{1+x_3^2} dx_3 + \frac{2x_5}{1+x_5^2} dx_5 \right)$, and $\phi_0 = -\frac{2x_5(1+x_3^2)}{(1+x_5^2)} dx_1 \wedge dx_2 \wedge dx_5$. It can be shown that $\phi_0 \neq \tau \wedge \nu$. Assume $\phi_0 = \tau \wedge \nu$ for some 1-form $\tau = \tau_1 dx^1 + \cdots + \tau_5 dx^5$. Then it turns out that $\tau_1 = \cdots = \tau_5 = 0$. Therefore $\ker \nu$ is integrable, but it is not true that $d\nu = -\omega \wedge \nu$ for some 1-form $\omega$.

We would like to investigate the relation between the cosmooth distributions determined by two cohomologous classes $\tau$ and $\eta$. It turns out that they may determine locally different dimensional cosmooth distributions.

**Definition 4.40.** A differential form $\omega$ is called basic relative to a distribution $D$ on a manifold $M$ if it satisfies the equations $X \lrcorner \omega = 0$ and $X \lrcorner d\omega = 0$ for any section $X \in \Gamma(D)$ of the subbundle $D$.

**Remark 4.41.** A differential form $\omega$ is basic relative to a distribution $D$ on a manifold $M$ that locally depends on only the variables that are transverse to the foliation $(M, \mathcal{F})$.

**Example 4.42.** Consider the non-exact closed form $\alpha = dx^2 \wedge dx^3 \wedge dx^4$ on $\mathbb{R}^4/\mathbb{Z}^4$. Then $\ker(\alpha)$ defines a foliation of $\mathbb{R}^4/\mathbb{Z}^4$ given by the integral submanifolds of the vector field

$X = \frac{\partial}{\partial x^1}$. If $\tau = f(x^2)dx^3 \wedge dx^4$ then $d\tau = \frac{\partial f}{\partial x^2}dx^2 \wedge dx^3 \wedge dx^4$ and for any $X \in \ker \alpha$, we have $X \lrcorner \tau = 0$ and $X \lrcorner d\tau = 0$, therefore $\tau$ is a basic differential form relative to the $\ker(\alpha)$.

Suppose $d\alpha = 0$. The basic class of $\alpha$ is the set of all $\alpha^{'} = \alpha + d\tau$ such that $\tau$ is basic relative to the foliation determined by $\ker \alpha$. Then $X \lrcorner \alpha^{'} = X \lrcorner \alpha + X \lrcorner d\tau = 0$ for any $X \in \ker(\alpha)$. Therefore $\ker(\alpha) \subset \ker(\alpha^{'})$. The reverse inclusion may not be true as we will see in the following example. So the foliation determined by the kernel of cohomologous Morse-Novikov classes may not be the same.

**Example 4.43.** For $\alpha = dx^2 \wedge dx^3 \wedge dx^4$ and $\tau = f(x^2)dx^3 \wedge dx^4$ on $\mathbb{R}^4/\mathbb{Z}^4$, and if $X = X^1\frac{\partial}{\partial x^1} + X^2\frac{\partial}{\partial x^2} + X^3\frac{\partial}{\partial x^3} + X^4\frac{\partial}{\partial x^4}$ is an arbitrary vector field on $\mathbb{R}^4/\mathbb{Z}^4$, we have

$$\begin{aligned} X \lrcorner \alpha^{'} &= X \lrcorner \alpha + X \lrcorner d\tau \\ &= X^2(1 + \frac{\partial f}{\partial x^2})dx^3 \wedge dx^4 - X^3(1 + \frac{\partial f}{\partial x^2})dx^2 \wedge dx^4 + X^4(1 + \frac{\partial f}{\partial x^2})dx^2 \wedge dx^3 \\ &= (1 + \frac{\partial f}{\partial x^2})(X^2 dx^3 \wedge dx^4 - X^3 dx^2 \wedge dx^4 + X^4 dx^2 \wedge dx^3). \end{aligned}$$

If we choose $f\,(X^2)$ so that $\frac{\partial f}{\partial x^2} = -1$ at some point $x_0$ of $\mathbb{R}^4/\mathbb{Z}^4$, then $\ker(\alpha + d\tau) = \mathbb{R}^4$ at $x_0$, and $\ker(\alpha + d\tau) = \mathrm{span}(\partial/\partial X^1)$ everywhere else.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be given foliations. A map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is foliated if $f$ maps the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$. In foliation charts $(x, y)$ of $(M, \mathcal{F})$, the $x$-parameter submanifolds are the foliation coordinates, and the $y$-coordinates are the local coordinates of the local quotient of the neighborhood by the leaf plaques. Similarly, $(u, v)$ be a foliation chart of $\left(M^{'}, \mathcal{F}^{'}\right)$. Then a foliated map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is of the form $(u,v) = f(x,y) = (f_1(x,y), f_2(y))$. Let $\mathcal{TF}$ and $\mathcal{TF}'$ be the tangent bundles of the foliations. A foliated map satisfies $f_*(\mathcal{TF}) \subset (\mathcal{TF}')$. Let $\Omega(M,\mathcal{F})$ denote the span of basic forms on $M$ with respect to the foliation.

**Lemma 4.44.** *[17] If $f : (M,\mathcal{F}) \to (M',\mathcal{F}')$ is a foliated map, then $f^*(\Omega(M',\mathcal{F}')) \subset \Omega(M,\mathcal{F})$.*

**Definition 4.45.** Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be two foliated manifolds. The foliated maps $\phi : M \to M'$ and $\psi : M \to M'$ are **foliated homotopic** if there exists a map $H : [0,1] \times M \to M'$ such that for each $t \in [0,1]$, $H(t,.)$ is foliated and $H(0,x) = \phi(x)$ and $H(1,x) = \psi(x)$.

**Definition 4.46.** A foliated map $f : (M,\mathcal{F}) \to (M',\mathcal{F}')$ is a **foliated homotopy equivalence** if there exists a foliated map $g : (M',\mathcal{F}') \to (M,\mathcal{F})$ such that $g \circ f$ is foliated homotopic to $Id_M$ and $f \circ g$ is foliated homotopic to $Id_{M'}$.

**Proposition 4.47.** *Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliations of the same codimension. In local coordinates as above, $\mathcal{F}'$ and $\mathcal{F}$ are the kernels of transverse volume forms $\nu' = g(u,v)dv^1 \wedge \cdots \wedge dv^q$ and $\nu = h(x,y)dy^1 \wedge \cdots \wedge dy^q$ respectively with $g$ and $h$ never zero. Let $f : (M,\mathcal{F}) \to (M',\mathcal{F}')$ be a foliated map of the form $(u,v) = f(x,y) = (f_1(x,y), f_2(y))$ with $\det\left(\frac{\partial f_{2i}}{\partial y_j}\right) \neq 0$. Then $f$ pulls back the volume form $\nu'$ to a volume form $f^*\nu'$ and $\ker(f^*\nu') = \ker(\nu)$.*

*Proof.* We have $\nu' = g(u,v)dv^1 \wedge \cdots \wedge dv^q$. The foliated map is of the form $(u,v) = f(x,y) = (f_1(x,y), f_2(y))$. Then

$$\begin{aligned} f^*\nu' &= (g \circ f)(x,y) df_{21} \wedge \cdots \wedge df_{2q} \\ &= g(f_1(x,y), f_2(y)) \det\left(\frac{\partial f_{2i}}{\partial y_j}\right) dy_1 \wedge \cdots \wedge dy_q. \end{aligned}$$

Therefore $f^*\nu'$ is a volume form of degree $q$ on $M$ if $\det\left(\frac{\partial f_{2i}}{\partial y_j}\right) \neq 0$.

If $X \in \mathcal{TF}$, then $X \in \ker(\nu)$. Then $X \lrcorner f^*\nu' = f_*(X) \lrcorner \nu'$ Since $f_*(X) \in \mathcal{TF}'$, $f_*(X) \lrcorner \nu' = 0$. Hence $\ker(\nu) \subset \ker(f^*\nu')$. Next, suppose $X \in \mathcal{TF}$ and $X \in \ker(f^*\nu')$. Then

$$0 = X \lrcorner \left( g(f_1(x,y), f_2(y)) \det\left(\frac{\partial f_{2i}}{\partial y_j}\right) dy_1 \wedge \cdots \wedge dy_q \right)$$
$$= \left( \frac{g(f_1(x,y), f_2(y)) \det\left(\frac{\partial f_{2i}}{\partial y_j}\right)}{h(x,y)} \right) X \lrcorner \left(h(x,y) dy_1 \wedge \cdots \wedge dy_q\right).$$

Hence $\ker(f^*\nu') \subset \ker(\nu)$. □

**Proposition 4.48.** *Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliations of the same codimension. Let $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a foliated map of the form $(u,v) = f(x,y) = (f_1(x,y), f_2(y))$ with $\det\left(\frac{\partial f_{2i}}{\partial y_j}\right) \neq 0$ in each chart. If $[\nu']$ is a nontrivial Morse-Novikov cohomology class on $M'$, then so is $[f^*\nu']$ on $M$.*

*Proof.* In local coordinates, $\mathcal{F}'$ and $\mathcal{F}$ are the kernels of the transverse volume forms $\nu' = g(u,v)dv^1 \wedge \cdots \wedge dv^q$ and $\nu = h(x,y)dy^1 \wedge \cdots \wedge dy^q$, respectively, with $g$ and $h$ never zero. Since $\mathcal{F} = \ker \nu$, then there exists an 1-form $\kappa$ such that $(d + \kappa)\nu = 0$. where $\kappa$ is the mean curvature form of the transverse distribution. If $\kappa$ is closed, then

$[\nu]$ is a Morse-Novikov cohomology class. Since $f$ is a foliated map, from the previous proposition, we have $f^*\nu' = F\nu$, where

$$F(x,y) = \frac{g(f_1(x,y), f_2(y)) \det\left(\frac{\partial f_{2i}}{\partial y_j}\right)}{h(x,y)} \text{ locally,}$$

We have

$$\begin{aligned}
d(f^*\nu') &= d(F\nu) \\
\Rightarrow d(F\nu) &= d(F)\nu + F d\nu \\
\Rightarrow d(F\nu) &= d(F)\nu + F(-\kappa \wedge \nu) \\
\Rightarrow d(F\nu) &= d(\log(F))F\nu + F(-\kappa \wedge \nu) \\
\Rightarrow d(F\nu) - d(\log(F))F\nu + F(-\kappa \wedge \nu) &= 0 \\
\Rightarrow (d + (\kappa - d(\log(F))))F\nu &= 0 \\
\Rightarrow (d + (\kappa - d(\log(F))))f^*\nu' &= 0 \\
\Rightarrow (d + \kappa')f^*\nu' &= 0,
\end{aligned}$$

where $\kappa' - \kappa = -d\log(F)$. If $\kappa$ is closed, so is $\kappa'$. Homologous closed 1-forms induce isomorphic Morse-Novikov groups, and the isomorphism is multiplication by a function. Therefore $[f^*\nu'] = [F\nu]$ is nontrivial. $\square$

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ABSTRACT

LEAFWISE MORSE-NOVIKOV COHOMOLOGICAL INVARIANTS OF FOLIATIONS

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The idea of Lichnerowicz or Morse-Novikov cohomology groups of a manifold has been utilized by many researchers to study important properties and invariants of a manifold. Morse-Novikov cohomology is defined using the differential $d_\omega = d + \omega\wedge$, where $\omega$ is a closed 1-form. We study Morse-Novikov cohomology in the context of singular distributions given by the kernel of differential forms, and foliations of manifold. The kernel of a $d_\omega$-closed form is involutive and hence gives a foliation of a manifold. A transversely oriented foliation of a Riemannian manifold uniquely determines leafwise Morse-Novikov cohomology groups, which are independent of the choice of metric in the sense that different metrics correspond to isomorphic groups. The relevant 1-form $\omega$, which is always leafwise closed, can be chosen to be the mean curvature 1-form of the transverse distribution of the foliation. In the case of Riemannian foliations, we prove that the reduced leafwise Morse-Novikov cohomology groups satisfy the Hodge theorem and Poincaré duality. We also show that for general singular foliations, the isomorphism classes of the induced leafwise Morse-Novikov cohomology groups are foliated homotopy invariants.