MULTIPLICITY STRUCTURES ON CONICS

by

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Submitted to the Graduate Faculty of the
College of Science and Engineering
Texas Christian University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy in Mathematics

August 2019

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ACKNOWLEDGMENTS

First, I want to thank my advisor Prof. Scott Nollet for accepting me as his PhD student, for teaching me commutative algebra and algebraic geometry, for introducing me to the beautiful subject of curves in $\mathbb{P}^3$, for giving me research problems and ideas, for reading my dissertation numerous times to make it perfect and above all for his incredible patience, encouragement and support all through my graduate student life.

I want to thank all the members of my PhD committee for reading my dissertation carefully and giving me their valuable suggestions to make it more readable. I want to thank all my professors at Texas Christian University who taught me at various levels and gave me a solid foundation in mathematics. I want to thank Prof. Ken Richardson for his warmth, generosity and his numerous hours of help to prepare me for the qualifying exams. I want to thank all my fellow graduate students, especially Luis Aguirre whose work in linkage theory inspired me to extend my project to some extent.

I want to thank Md. Shariful Islam for being my mentor, friend and above all for helping me to reach here at TCU. I want to thank Lubna Pinky for always being there for me and supporting me in some difficult times of my life. Finally, I want to thank my family for always believing in me and having high hopes for me.

Contents

1 Introduction

Algebraic geometry is the study of geometric objects that arise from solutions to polynomial equations over a field, such as curves, surfaces and their higher dimensional analogues. One of the major themes in algebraic geometry is to classify these objects based on certain invariants. In particular, the classification of algebraic space curves based on their degrees and genera has been one of the most fruitful branches of research at least from the later half of the nineteenth century [15–17]. As a subject algebraic geometry went through many revolutions and each generation had its own language and perspective. Until the first half of the twentieth century the notion of curves was restricted to what we now call varieties of dimension 1. Even if we restrict ourselves to this definition, geometric objects that are not varieties arise naturally as limits of families of curves and also via liaison theory [28]. Not knowing what to do with such objects, algebraic geometers tended to avoid such structures. In a desire to include these natural objects in the main frame of study and to unify all earlier developments of algebraic geometry, Alexander Grothendieck came up with his notion of schemes around 1957 [10, VIII]. In this dissertation we use the language of scheme theory to deal with the classfication of multiplicity structures on nonsingular connected curves in $\mathbb{P}^3$.

1.1 Multiplicity structures on curves in $\mathbb{P}^3$

Let $\mathbb{P}^3 = \operatorname{Proj} S$, where $S = k[x, y, z, w]$ and k is an algebraically closed field. If $X \subset \mathbb{P}^3$ is a closed subscheme we denote its ideal sheaf and total ideal by $\mathcal{I}_X$ and I_X respectively. The homogeneous coordinate ring of X is the quotient ring S/I_X of S and is denoted by S_X. We use the abbreviations CM for Cohen-Macaulay and l.c.i. for locally complete intersection. A curve in $\mathbb{P}^3$ is a closed subscheme of dimension 1. In this dissertation we will mainly focus on locally CM curves, i.e., curves having no embedded or isolated points, but which may have multiplicities along their supports. Let $Y \subset \mathbb{P}^3$ be a nonsingular connected curve. A multiplicity structure on Y is a closed subscheme $Z \subset \mathbb{P}^3$ such that $\operatorname{Supp} Z = \operatorname{Supp} Y$. The multiplicity of Z is defined to be the ratio $\deg Z / \deg Y$ and is denoted by $\operatorname{mult}(Z)$. In Proposition 4.2.5 we will prove that if Y is nonsingular connected and Z is CM, then $\operatorname{mult}(Z)$ is a positive integer.

Example 1.1.1. Let $Y \subset \mathbb{P}^3$ be the line with $I_Y = (z, w)$. Let $Z_t \subset \mathbb{P}^3$ be the closed subschemes with $I_{Z_t} = (x^2 - tyz, w)$, where $t \in k$. Then Z_t is a nonsingular conic whenever $t \neq 0$. We have $I_{Z_0} = (x^2, w)$ and $S_{Z_0} \cong k[x, y, z]/(x^2)$. Notice Z_0 is not a variety, since x is a nilpotent element in S_{Z_0}. On the other hand, Z_0 is supported on Y. Moreover, every point of Y comes twice in Z_0. We call Z_0 a double structure on Y. In fact, Z_0 is the simplest kind of multiplicity structures in $\mathbb{P}^3$. Finally since $Z_t \to Z_0$ as $t \to 0$, Z_0 arises as a limit of a family of nonsingular connected curves in $\mathbb{P}^3$.

Multiplicity structures also arise naturally in the study of liaison theory [28, 33]. For example, every smooth connected rational quintic curve in $\mathbb{P}^3$, not lying in a quadric surface, is linked by a complete intersection of two cubic surfaces to a quadruple line [31, Proposition 3.2]. Therefore even if one is primarily interested in the nicest kind of curves, i.e., nonsingular connected curves, multiplicity structures can be crucial and unavoidable. A natural question that arises and leads to a wide open territory of research is as follows.

Problem 1. Let $Y \subset \mathbb{P}^3$ be a nonsingular connected curve.

(a) Classify all CM multiplicity structures on Y.

(b) Find the minimal free resolutions of the total ideals of such structures.

(c) Find the Rao modules of such structures.

(d) Describe the nature of general surfaces containing such a structure.

(e) Describe the irreducible families of such structures.

1.2 Previous work on Problem 1

The systematic study of multiplicity structures on nonsingular connected curves in $\mathbb{P}^3$ began with the pioneering work of Ferrand [12]. In this paper he showed that given a l.c.i. curve $Y \subset \mathbb{P}^3$, there exists a bijection between the set of CM double structures on Y and the set of surjections $\nu_Y \twoheadrightarrow \mathcal{L}$, where ν_Y is the conormal bundle of Y and $\mathcal{L} \in \operatorname{Pic} Y$. Then Bănică and Forster [3] generalized his method to study higher multiplicity structures. More precisely, they showed a way to construct quasi-primitive multiplicity structures (CM curves with generic embedding dimension 2) by introducing the notion of Cohen-Macaulay filtration and an invariant of such extension, called its type.

The total ideals of double lines in $\mathbb{P}^3$ were known to folklore [14, 25]. But their classification has been done by Nollet [29, Proposition 1.4]. The classifications of triple and quadruple lines in $\mathbb{P}^3$ have been done by Nollet [29], and by Nollet and Schlesinger [32] respectively. Manolache [24] and Bănică and Manolache [4] studied double conics in connection with the moduli space of stable rank two vector bundles on $\mathbb{P}^3_{\mathbb{C}}$ with Chern classes $c_1 = -1, c_2 = 2$ and $c_1 = -1, c_2 = 4$ respectively. Finally, Vatne [34] studied CM double structures on twisted cubics in $\mathbb{P}^3$ and gave examples of such curves for all possible arithmetic genus, assuming that char $k \neq 2$.

1.3 Dissertation summary

In this dissertation, we deal with Problem 1 for conics in $\mathbb{P}^3$. More precisely, we give a complete solution to parts (a)-(d) of Problem 1 for double conics and a partial solution to parts (a)-(b) of Problem 1 for triple conics, which are complicated due to new behaviors.

In Chapter 2, we state and prove some results about modules over noetherian rings and finitely supported coherent sheaves in $\mathbb{P}^3$.

In Chapter 3, we carefully prove some well-known results about curves in $\mathbb{P}^3$. In particular, we give three equivalent definitions of CM curves and prove their equivalence. We also prove some nice properties of such curves.

In Chapter 4, we extend the theory of Bănică and Forster [3] from complex analytic three manifolds to $\mathbb{P}^3_k$ over an arbitrary but algebraically closed field k. In particular, we give rigorous proofs of their main statements and theorems. Let Z be a multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. According to Bănică and Forster, $\mathrm{mult}(Z)$ is the rank of $r_*\mathcal{O}_Z$, where $r : Z \to Y$ is a holomorphic retraction and $\mathcal{O}_Z$ is the structure sheaf of Z. But such a holomorphic retraction does not exist on schemes due to the coarse nature of Zariski topology. So we define $\text{mult}(Z)$ as the ratio $\deg Z/\deg Y$ and show in Proposition 4.2.5 that these two definitions yield the same number. We define quasi-primitive extensions on nonsingular connected curves in $\mathbb{P}^3$ and show that each such extension has an invariant, called its type. At the end of this chapter we describe the singularities and class groups of general surfaces containing a quasi-primitive multiplicity structure, following the works of Brevik and Nollet [6].

In Chapter 5, we prove the following classification theorem for CM double conics in $\mathbb{P}^3$.

Theorem 5.2.1. *Let $C \subset \mathbb{P}^3$ be a conic and let $\ell \geq -4$ be an integer such that $\ell \neq -3$. Then each surjection $\psi : \mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{O}_C[\ell]$ defines a CM double conic Z on C with Hilbert polynomial $P_Z(n) = 4n + \ell + 2$ by $\mathcal{I}_Z = \operatorname{Ker} \psi \circ \pi$, where $\pi : \mathcal{I}_C \twoheadrightarrow \mathcal{I}_C/\mathcal{I}_C^2$ is the canonical surjection. Conversely, every CM double conic on C arises from this construction.*

We describe the invariants of double conics, namely their total ideals, Rao modules and minimal free resolutions of their total ideals. We give criteria for two double conics of the same support to be linked by a complete intersection. In particular, we give a criterion for double conics to be self-linked, which extends a well-known theorem of Migliore [27] on self-linkage of double lines to double conics. We also give the criterion for a double conic to be contained in a nonsingular surface. In particular, we show that a double conic of arithmetic genus $-1 - \ell \leq -5$ is contained in a nonsingular surface if and only if ℓ is even. At the end of this chapter we show that a Zariski general surface containing a double conic is normal and the number of its singular points is determined by its degree and the arithmetic genus of the double conic contained in it.

In Chapter 6, we prove the following classification theorem for CM triple conics in $\mathbb{P}^3$.

Theorem 6.1.5. *Let Z be a CM double conic on C of type ℓ, where $\ell \geq -4$ is an integer such that $\ell \neq -3$. Let $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \to \mathcal{O}_C[2\ell + c]$ be a surjection, where $c \geq 0$ is an integer. Then ψ defines a CM triple conic W on C with Hilbert polynomial $P_W(n) = 6n + 3\ell + c + 3$ by $\mathcal{I}_W = \mathrm{Ker}\, \psi \circ \pi$, where $\pi : \mathcal{I}_Z \twoheadrightarrow \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z$ is the canonical surjection. Conversely, every CM triple conic W on C arises from this construction.*

In particular, we determine the range of ℓ and c for which there exists a quasi-primitive triple conic of type (ℓ, c). We give explicit maps which yield the thick triple conics, i.e., triple conics having embedding dimension 3 at each point. For the rest of Chapter 6 we computed total ideals of quasi-primitive triple conics. Let W be a quasi-primitive triple conic on C that arises from a surjection $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell + c]$. Then W is of type (ℓ, c) and has Z as its 2^{nd} CM filtrant. If ℓ is even and $c \geq 3$ then we show that I_W has a nice description. In fact, I_W/I_CI_Z is cyclic in this situation. The classification gets complicated when $\ell \geq 0$ is even and $0 \leq c \leq 2$. For example, if $(\ell, c) = (2a, 0)$, where $a \geq 0$, then I_W can have $7, 8$ or 9 generators. This shows that the cohomology groups jump in the Hilbert scheme of triple conics in $\mathbb{P}^3$, which is not known for any other family of multiplicity structures classified so far. We give criteria to determine I_W for this range of (ℓ, c). The classification of triple conics of type (ℓ, c) is still open whenever ℓ is odd. The main obstacle here is the non-splitting nature of the S_C-module $I_Z \otimes S_C$, which we wish to resolve in our future work.

2 Background

In this chapter we state and prove some well-known results about modules over noetherian rings and finitely supported coherent sheaves in $\mathbb{P}^3$ that we are going to use throughout this exposition.

2.1 Algebraic results

All rings here are assumed commutative with identity. Based on the themes, we split this section into three subsections.

2.1.A. Associated primes

In this subsection we state two important results about associated primes. We prove Lemma 2.1.6 which will be used in Proposition 3.3.6.

Definition 2.1.1. Let A be a noetherian ring and $\mathfrak{a}$ be an ideal in A. Let $\mathfrak{a} = \cap_{i=1}^{n} q_i$ be a primary decomposition of $\mathfrak{a}$ and let $p_i = \sqrt{q_i}$. Then the set $\{p_i\}_{i=1}^{n}$ is independent of the particular decomposition of $\mathfrak{a}$ by [2, Theorem 4.5]. The prime ideals p_i are called the associated primes of $\mathfrak{a}$. The minimal elements of the set $\{p_i\}_{i=1}^{n}$ are called the minimal primes of $\mathfrak{a}$ and the others are called the embedded primes of $\mathfrak{a}$.

Definition 2.1.2. Let A be a ring and M be an A-module. A prime ideal p of A is called an associated prime of M if p is the annihilator of some element $m \in M$. The set of associated primes of M is denoted by $\mathrm{Ass}(M)$. The minimal elements of $\mathrm{Ass}(M)$ are called the isolated associated primes of M and the others are called the embedded associated primes of M.

Remark 2.1.3. Notice the associated primes of $\mathfrak{a}$ as an ideal in A are not the same as the associated primes of $\mathfrak{a}$ as an A-module, rather they are the same as the associated primes of the A-module $A/\mathfrak{a}$.

Proposition 2.1.4. Let A be a noetherian ring and M be a nonzero A-module.

(a) Every maximal element of the family of ideals $F = \{\text{ann}(m) | 0 \neq m \in M\}$ is an associated prime of M, and in particular $\text{Ass}(M) \neq \varnothing$.

(b) The set of zerodivisors for M is the union of all the associated primes of M.

Proof. [26, Theorem 6.1]. □

Proposition 2.1.5. Let $\mathfrak{a}$ is a decomposable homogeneous ideal in a graded ring A. Then the associated primes of $\mathfrak{a}$ are homogeneous.

Proof. [35, Corollary, p. 154] or [5, IV, § 3, Proposition 1] □

Lemma 2.1.6. Let A be a noetherian ring having no embedded associated primes. If I is an ideal in A such that $I_p = 0$ for all minimal primes p of A, then $I = 0$.

Proof. Suppose on the contrary that $I \neq 0$. Then there exists a nonzero element $x \in I$. Let $p_1, \cdots, p_n$ be the minimal primes of A. Then $I_{p_i} = 0$ for $1 \leq i \leq n$. Hence there exist $s_i \in A \setminus p_i$ such that $s_i x = 0$. Let $J = (s_1, \cdots, s_n)$. Then $J \not\subseteq p_i$ for all i, and hence $J \not\subseteq \cup_{i=1}^n p_i$ by the prime avoidance lemma [2, Proposition 1.11 (i)]. Let $s \in J \setminus \cup_{i=1}^n p_i$. Then there exist $a_i \in A$ such that $s = \sum_{i=1}^n a_i s_i$. Notice $sx = \sum_{i=1}^n a_i s_i x = 0$, i.e., $s \in \text{ann}(x)$. Since A is noetherian, by Proposition 2.1.4 (a), there exists an associated prime p of A such that $\text{ann}(x) \subset p$. Notice $p \not\subseteq \cup_{i=1}^n p_i$, since $s \in p \setminus \cup_{i=1}^n p_i$. Hence again by the prime avoidance lemma [2, Proposition 1.11 (i)], $p \not\subseteq p_i$ for all i. Therefore p is an embedded associated prime of A, which is a contradiction. Thus $I = 0$. □

2.1.B. Regular sequence, depth and deviation

The goal of this subsection is to define Cohen-Macaulay and complete intersection rings.

Definition 2.1.7. Let M be an A-module. An element $a \in A$ is said to be M-regular if $ax = 0$ for some $x \in M \Rightarrow x = 0$. In other words, a is M-regular if it is a nonzerodivisor on M. A sequence $\vec{a} = a_1, \cdots, a_n$ of elements in A is an M-regular sequence (or simply an M-sequence) if the following two conditions are satisfied:

1. a_i is $M/(a_1, \cdots, a_{i-1})M$-regular for $1 \le i \le n$ and
2. $M/\vec{a}M \neq 0$, where $\vec{a}M = \sum_{i=1}^{n} a_i M$.

Lemma 2.1.8. Let A be a ring and let $f, g \in A$. Then $\{f, g\}$ is a regular sequence in $A \Leftrightarrow \{f, g + \gamma f\}$ is a regular sequence in A for all $\gamma \in A$.

Proof. Let $\{f, g\}$ be a regular sequence in A. Then f is regular in A. Notice $g + \gamma f \neq 0$ in $A/(f)$, for otherwise $g = 0$ in $A/(f)$ which contradicts the regularity of g in $A/(f)$. Suppose $u(g + \gamma f) = 0$ in $A/(f)$ for some $u \in A$. Then $ug = 0$ in $A/(f)$ and hence $u \in (f)$, since g is regular in $A/(f)$. Therefore $\{f, g + \gamma f\}$ is a regular sequence in A for all $\gamma \in A$. Conversely, if $\{f, g + \gamma f\}$ is a regular sequence in A for all $\gamma \in A$, then taking $\gamma = 0$ we see that $\{f, g\}$ is a regular sequence in A. □

The following lemma will be heavily used in Chapters 5 and 6.

Lemma 2.1.9. Let A be a quotient of a graded polynomial ring such that A is an integral domain with $\dim A = 2$. Let $f, g \in A$ be nonconstant homogeneous polynomials. Then the following statements are equivalent.

(a) $\{f, g\}$ is a regular sequence in A.

(b) $Z(f) \cap Z(g) = \varnothing$.

(c) $A/(f, g)$ has finite length.

Proof. (a)$\Rightarrow$(b): Let $\{f, g\}$ be a regular sequence in A. Then $\operatorname{ht}(f, g) \geq \operatorname{depth}(f, g) = 2$ by [7, Proposition 1.2.14]. Let p be a homogeneous prime ideal in A containing (f, g). Then $\operatorname{ht} p \geq \operatorname{ht}(f, g) \geq 2$. Since $\dim A = 2$, p is the irrelevant maximal ideal in A. Hence $Z(f) \cap Z(g) = \varnothing$.

(b)$\Rightarrow$(c): Let $Z(f) \cap Z(g) = \varnothing$. Let $\mathfrak{a} = (f, g)$ and let $A = S/I$ for some graded polynomial ring S with irrelevant maximal ideal $\mathfrak{m}$. Notice $Z(\mathfrak{a}) = \varnothing$ and hence $\operatorname{Spec} A/\mathfrak{a} = \{\mathfrak{m}\}$. Thus $\dim A/\mathfrak{a} = 0$. Hence $A/\mathfrak{a}$ is Artinian by [2, Theorem 8.5]. Therefore $A/\mathfrak{a}$ has finite length by [11, Theorem 2.13].

(c)$\Rightarrow$(a): Let $A/(f, g)$ have finite length. Suppose $\{f, g\}$ is not a regular sequence in A. Then $\operatorname{depth} A/(f, g) \geq 1$. Let u be a regular element of $A/(f, g)$. Then u is not nilpotent and hence $\cdots \subset (u^n) \subset \cdots \subset (u) \subset A/(f, g)$ is an infinite chain of submodules of $A/(f, g)$. Therefore $A/(f, g)$ has infinite length, which is a contradiction. □

Let A and M be as in Definition (2.1.7) and let I be an ideal in A such that $M/IM \neq 0$. An M-sequence in I is maximal if it cannot be extended to another M-sequence by adding more elements from I. Notice if A is noetherian then a maximal M-sequence in I must have finite length. Moreover, if M is finitely generated over A then every maximal M-sequence in I has the same length by [26, Theorem 16.7].

Definition 2.1.10. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and let $M \neq 0$ be a finitely generated A-module. The *depth* of M is the length of any maximal M-sequence in $\mathfrak{m}$ and is denoted by $\operatorname{depth} M$.

Definition 2.1.11. Let A be a noetherian ring and M be a finitely generated A-module. Then M is called a Cohen-Macaulay (CM henceforth) module if $M \neq 0$ and $\operatorname{depth} M_p = \dim M_p$ for all $p \in \operatorname{Spec} A$, or if $M = 0$.

Definition 2.1.12. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring with $\operatorname{embdim} A = n$. Let $\vec{x} = x_1, \cdots, x_n$ be a minimal basis of $\mathfrak{m}$ and let $E_\bullet = K_{\vec{x}}$ be the Koszul complex corresponding to the regular sequence $\vec{x}$. Then $E_\bullet$ is uniquely determined by A up to isomorphism. Let $H_p(E_\bullet)$ denote the p^{th} homology group of $E_\bullet$. Then $H_p(E_\bullet)$ is a k-vector space, since $\mathfrak{m}H_p(E_\bullet) = 0$ by [26, Theorem 16.4]. The p^{th} deviation of A is defined to be the dimension of the k-vector space $H_p(E_\bullet)$ and is denoted by $\epsilon_p(A)$.

Remark 2.1.13. Notice if A is regular then $\vec{x}$ is an A-sequence and hence $H_p(E_\bullet) = 0$, i.e., $\epsilon_p(A) = 0$ for all $p > 0$ by [26, Theorem 16.5 (i)]. Conversely, if $\epsilon_1(A) = 0$ then $\vec{x}$ is an A-sequence and hence A is regular by [26, Theorem 16.5 (ii)]. Hence A is a regular local ring if and only if $\epsilon_1(A) = 0$. Thus $\epsilon_1(A)$ measures how far A deviates from regularity.

Definition 2.1.14. Let A be a noetherian local ring. Then A is called a complete intersection ring if $\epsilon_1(A) = \operatorname{embdim} A - \dim A$.

Corollary 2.1.15. Every regular local ring is a complete intersection ring.

Proof. Let A be a regular local ring. Then $\operatorname{embdim} A = \dim A$, i.e., $\operatorname{embdim} A - \dim A = 0$. On the other hand, $\epsilon_1(A) = 0$ by Remark 2.1.13. Therefore $\epsilon_1(A) = \operatorname{embdim} A - \dim A$ and hence A is a complete intersection ring. □

Theorem 2.1.16. Let A be a noetherian local ring. If R is a regular local ring such that $A \cong R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ in R, then the minimum number of generators of $\mathfrak{a}$ is $\dim R - \operatorname{embdim} A + \epsilon_1(A)$.

Proof. [26, Theorem 21.1]. □

Corollary 2.1.17. Let A be a noetherian local ring. If A is a complete intersection and if R is a regular local ring such that $A \cong R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ in R, then the minimal number of generators of $\mathfrak{a}$ is $\dim R - \dim A$.

Proof. Since A is a complete intersection, $-\operatorname{embdim} A + \epsilon_1(A) = -\dim A$. Therefore by Theorem 2.1.16, the minimal number of generators of $\mathfrak{a}$ is $\dim R - \dim A$. □

2.1.C. Free resolutions

In this subsection we state two important theorems regarding free resolutions of modules over noetherian rings.

Definition 2.1.18. Let A be a noetherian ring and let $\varphi : F \to G$ be a homomorphism of free A-modules such that $\operatorname{rank} F = n$ and $\operatorname{rank} G = m$. Then φ is given by an $m \times n$ matrix U with respect to the bases of F and G. Let $I_t(U)$ denote the ideal generated by

$t \times t$ minors of U. We define $I_t(\varphi)$ as follows:

$$I_t(\varphi) = \begin{cases} A, & \text{if } t \leq 0, \\ I_t(U), & \text{if } 1 \leq t \leq \min\{m, n\} \\ 0, & \text{if } t > \min\{m, n\}. \end{cases}$$

Then $\operatorname{rank} \varphi = \max \{r | I_r(\varphi) \neq 0\}$. We denote $I_{\operatorname{rank} \varphi}(\varphi)$ by $I(\varphi)$.

Remark 2.1.19. In 1936, Fitting [13] proved that the ideals $I_t(U)$ in Definition 2.1.18 depend only on the module $\operatorname{Coker} \varphi$ and hence are independent of the choice of bases of F and G. These ideals are now called the Fitting ideals of φ or the Fitting invariants of $\operatorname{Coker} \varphi$. See [11, Corollary-Definition 20.4] for a modern proof of this fact.

Theorem 2.1.20 (Buchsbaum-Eisenbud)**.** Let A be a noetherian ring and let

$$0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \tag{1}$$

be a complex of free A-modules. Then (1) is exact if and only if

(a) $\operatorname{rank} F_k = \operatorname{rank} \varphi_k + \operatorname{rank} \varphi_{k+1}$ and

(b) $\operatorname{depth} I(\varphi_k) \geq k$

for $1 \leq k \leq n$.

Proof. [8], [11, Theorem 20.9] or [7, Theorem 1.4.13]. □

Remark 2.1.21. Here we use the convention that the unit ideal has infinite depth. Hence if $I(\varphi_k) = A$ then condition (b) in Theorem 2.1.20 is automatically satisfied.

Theorem 2.1.22 (Hilbert-Burch)**.** Let A be a noetherian ring and let

$$0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} A \to A/I \to 0 \tag{2}$$

be a complex of A-modules.

(a) If (2) is exact, F_2 is free and $F_1 \cong A^{n+1}$ with $n \geq 1$, then $F_2 \cong A^n$ and there exists a nonzerodivisor $u \in A$ such that $I = uI_n(\varphi_2)$. In fact, the i^{th} entry of the matrix for φ_1 is $(-1)^i u$ times the minor obtained from φ_2 by leaving out the i^{th} row. Moreover, $\operatorname{depth} I_n(\varphi_2) = 2$.

(b) Conversely, given any $(n+1) \times n$ matrix φ_2 such that $\operatorname{depth} I_n(\varphi_2) \geq 2$ and a nonzerodivisor u, the map φ_1 obtained as in part (a) makes (2) into a free resolution of A/I, with $I = I_n(\varphi_2)$.

Proof. In 1890 Hilbert [19] proved this theorem for graded ideals of codimension 2 in a polynomial ring. Then in 1968 Burch [9] proved the general case. For a modern proof see [11, Theorem 20.15] or [7, Theorem 1.4.17]. □

2.2 Finitely supported coherent sheaves in $\mathbb{P}^3$

Let k be an algebraically closed field, S be the graded polynomial ring $k[x, y, z, w]$ and $\mathfrak{m}$ be the irrelevant maximal ideal (x, y, z, w) in S. In this section we prove that the Hilbert polynomial of a finitely supported coherent sheaf in $\mathbb{P}^3$ is constant, where $\mathbb{P}^3 = \operatorname{Proj} S$. We also prove that every graded S-module of finite length sheafifies to 0.

Lemma 2.2.1. If $\mathcal{F}$ is a finitely supported coherent sheaf in $\mathbb{P}^3$, then $\mathcal{F} \cong \mathcal{F}(n), \forall n \in \mathbb{Z}$.

Proof. If $\operatorname{Supp}\mathcal{F} = \varnothing$ then $\mathcal{F}$ is the zero sheaf and hence $\mathcal{F} \cong \mathcal{F}(n)$ for all $n \in \mathbb{Z}$. So without loss of generality, we may assume that $\operatorname{Supp}\mathcal{F} \neq \varnothing$. Let $M = H^0_*\mathcal{F}$. Then M is a finitely generated graded S-module with $\operatorname{Supp} M \neq \varnothing$. Notice if $\mathfrak{m}$ is a minimal prime of M, then $\operatorname{Supp} M = \{\mathfrak{m}\}$ by [26, Theorem 6.5] and hence $\operatorname{Supp}\mathcal{F} = \varnothing$. So we may assume that $\mathfrak{m}$ is not a minimal prime of M.

Let $\operatorname{Supp}\mathcal{F} = \{P_1, \cdots, P_r\}$ and let $P \in \mathbb{P}^3$ be a closed point such that $P \notin \operatorname{Supp}\mathcal{F}$. Let π be the projection $\pi : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$. Let $Q \in \mathbb{P}^2 \setminus \cup_{i=1}^r \pi(P_i)$. Notice, $\pi^{-1}(Q)$ is a line in $\mathbb{P}^3$. Let τ be the projection $\tau : \mathbb{P}^2 \setminus \{Q\} \to \mathbb{P}^1$. Define $\phi = \tau \circ \pi$. Then ϕ is a projection from the line $\pi^{-1}(Q)$ to $\mathbb{P}^1$. Let $p_i = \phi(P_i)$. Choose $p \in \mathbb{P}^1 \setminus \{p_1, \cdots, p_r\}$. Let $H = \phi^{-1}(p)$. Then $H \subset \mathbb{P}^3$ is a plane such that $P_i \notin H$ for all i. Let $I_H = (h)$ for some $h \in \mathfrak{m}$. Notice h is not contained in any associated prime of M, since $P_i \notin H$ for all i. Hence by Proposition 2.1.4 (b), h is a nonzerodivisor for M. So we have the exact sequence

$$0 \to M(-1) \xrightarrow{\cdot h} M \to M/hM \to 0. \tag{3}$$

Sheafifying (3) we get the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot h} \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G} = \widetilde{M/hM}$. Since M is finitely generated, we have $\operatorname{Supp} M = V(\operatorname{ann} M)$ and $\operatorname{Supp} M/hM = V(\operatorname{ann} M/hM)$. Also $V(\operatorname{ann} M/hM) = V((h) + \operatorname{ann} M)$ by [2, Exercise 3.19 (vii)], hence $\operatorname{Supp} M/hM = V((h) + \operatorname{ann} M)$. Since $\mathcal{F}$ is finitely supported and

$\mathfrak{m}$ is not a minimal prime of M, we have $\dim \operatorname{Supp} M = \dim V(\operatorname{ann} M) = 1$ in $\operatorname{Spec} S$. Since h is not contained in any associated prime of M, $h \notin \operatorname{ann} M$. Hence $\operatorname{ht} \operatorname{ann} M < \operatorname{ht}((h) + \operatorname{ann} M)$. Thus $\dim V((h) + \operatorname{ann} M) \leq \dim V(\operatorname{ann} M) - 1 = 1 - 1 = 0$, i.e., $\dim V((h) + \operatorname{ann} M) = 0$ in $\operatorname{Spec} S$. Therefore $\operatorname{Supp} M/hM$ is either $\varnothing$ or $\{\mathfrak{m}\}$. Hence $\operatorname{Supp} \mathcal{G} = \varnothing$, i.e., $\mathcal{G}$ is the zero sheaf. Therefore $\mathcal{F}(-1) \cong \mathcal{F}$. Thus $\mathcal{F} \cong \mathcal{F}(1)$ and hence $\mathcal{F} \cong \mathcal{F}(n)$ for all $n \in \mathbb{Z}$. □

Corollary 2.2.2. Let $\mathcal{F}$ be a finitely supported coherent sheaf in $\mathbb{P}^3$. Then the Hilbert polynomial of $\mathcal{F}$ is constant.

Proof. Let $P(z) \in \mathbb{Q}[z]$ be the Hilbert polynomial of $\mathcal{F}$. Then $\chi\mathcal{F}(n) = P(n)$ for all $n \in \mathbb{Z}$. Since $\mathcal{F}$ is finitely supported, $\chi\mathcal{F}(n) = \chi\mathcal{F}(0)$ for all $n \in \mathbb{Z}$ by Lemma 2.2.1. Thus $P(n) = P(0)$ for all $n \in \mathbb{Z}$. Hence $P(n)$ and therefore $P(z)$ is constant. □

Lemma 2.2.3. Let E be a simple graded S-module of length 1. Then $E \cong (S/\mathfrak{m})(n)$ for some $n \in \mathbb{Z}$.

Proof. Since E is a simple module of length 1, it is generated by a single nonzero element, say $e \in N$. Let $\phi : S \to E$ be the map given by $1 \mapsto e$. Then ϕ is surjective. Notice $\operatorname{Ker} \phi = \operatorname{ann}(e)$. Let $P = \operatorname{ann}(e)$. We have the chain $0 \subset S/\mathfrak{m} \subset S/P$ of submodules of S/P. Since S/P has length 1, we have $S/\mathfrak{m} = S/P$, i.e., $P = \mathfrak{m}$. Therefore $E \cong (S/\mathfrak{m})(n)$ for some $n \in \mathbb{Z}$. □

Lemma 2.2.4. Let M be a graded S-module. Then M has finite length $\Leftrightarrow \widetilde{M} = 0$.

Proof. Let M have finite length and let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a composition series of M, where n is the length of M. Then each M_i/M_{i-1} is a simple graded S-module of length 1. Therefore by Lemma 2.2.3, $M_i/M_{i-1} \cong (S/\mathfrak{m})(n_i) = k(n_i)$ for some $n_i \in \mathbb{Z}$. By [18, II, Proposition 8.13], we have the exact sequence

$$0 \to \Omega_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(-1)^4 \to \mathcal{O}_{\mathbb{P}^3} \to 0, \tag{4}$$

where $\Omega_{\mathbb{P}^3}$ is the sheaf of differentials of $\mathbb{P}^3$. Taking the long exact cohomology sequence in (4) we get the exact sequence

$$0 \to H^0\Omega_{\mathbb{P}^3} \to S(-1)^4 \xrightarrow{\phi} S. \tag{5}$$

Let $\{e_i\}_{i=1}^4$ be a basis of $S(-1)^4$. Then ϕ is given by $e_1 \mapsto x, e_2 \mapsto y, e_3 \mapsto z$ and $e_4 \mapsto w$. Thus $\operatorname{Coker}\phi = S/\mathfrak{m} = k$ and hence we get the exact sequence

$$0 \to H^0\Omega_{\mathbb{P}^3} \to S(-1)^4 \xrightarrow{\phi} S \to k \to 0 \tag{6}$$

from (5). Sheafifying (6) we get the exact sequence (4). Therefore $\widetilde{k} = 0$. Since $\widetilde{k}$ is a finitely supported coherent sheaf on $\mathbb{P}^3$, $\widetilde{k} \cong \widetilde{k}(n)$ for all $n \in \mathbb{Z}$ by Lemma 2.2.1. Hence

$$\widetilde{M_i/M_{i-1}} \cong \widetilde{k(n_i)} \cong \widetilde{k}(n_i) = 0$$

for all $1 \leq i \leq n$. We have the short exact sequece of S-modules

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0. \tag{7}$$

Sheafifying (7) we get the short exact sequence of sheaves

$$0 \to \widetilde{M_{i-1}} \to \widetilde{M_i} \to \widetilde{M_i/M_{i-1}} \to 0.$$

Since $\widetilde{M_i/M_{i-1}} = 0$, we have $\widetilde{M_i} \cong \widetilde{M_{i-1}}$ for all i. Therefore $\widetilde{M} = \widetilde{M_n} \cong \cdots \cong \widetilde{M_0} = 0$. Conversely, let $\widetilde{M} = 0$. If $M = 0$ then it has length 0. So let's suppose $M \neq 0$. Since $\widetilde{M} = 0$, we have $\operatorname{Supp} M = \{\mathfrak{m}\}$ and hence $\operatorname{Ass}(M) = \{\mathfrak{m}\}$. Thus $\mathfrak{m} = \operatorname{ann} M$. Since $S/\operatorname{ann} M = S/\mathfrak{m} = k$ is an Artinian ring, M has finite length by [11, Corollary 2.17]. $\square$

3 Curves in $\mathbb{P}^3$

Let $\mathbb{P}^3 = \operatorname{Proj} S$, where $S = k[x, y, z, w]$ and k is an algebraically closed field. If $X \subseteq \mathbb{P}^3$ is a closed subscheme we denote its ideal sheaf and total ideal by $\mathcal{I}_X$ and I_X respectively. The homogeneous coordinate ring of X is defined to be the quotient ring S/I_X of S and is denoted by S_X. A curve in $\mathbb{P}^3$ is a closed subscheme of dimension 1. In this chapter we carefully prove some well-known results about curves in $\mathbb{P}^3$.

3.1 Preliminaries

Let $X \subset \mathbb{P}^3$ be a curve. Then X is a complete intersection if the total ideal I_X of X is generated by 2 elements. We say that X is a locally complete intersection if the ideal sheaf $\mathcal{I}_X$ of X is generated by 2 elements at every point.

Proposition 3.1.1. Let $X \subset \mathbb{P}^3$ be a complete intersection curve with $I_X = (F, G)$. Then

$$0 \to S(-d-e) \xrightarrow{\begin{pmatrix} G \\ -F \end{pmatrix}} S(-d) \oplus S(-e) \xrightarrow{\begin{pmatrix} F & G \end{pmatrix}} I_X \to 0 \tag{8}$$

is a minimal S-resolution of I_X, where $d = \deg F$ and $e = \deg G$.

Proof. Let $\varphi = \begin{pmatrix} G \\ -F \end{pmatrix}$. Then $\operatorname{rank} \varphi = 1$ and hence $I(\varphi) = (F, G)$. Since $\{F, G\}$ is a regular sequence in S, $\operatorname{depth} I(\varphi) = 2$. Therefore by the Hilbert-Burch theorem 2.1.22, (8) is an S-resolution of I_X. Since none of the entries of φ and (F, G) is a unit, (8) is a minimal S-resolution of I_X. $\square$

Lemma 3.1.2. Let X and X' be curves in $\mathbb{P}^3$ with the same Hilbert polynomial. If $X' \subseteq X$ then $X' = X$.

Proof. Since $X' \subseteq X$, we have $I_X \subseteq I_{X'}$ and hence $\mathcal{I}_X \subseteq \mathcal{I}_{X'}$. Therefore we have the canonical surjection $\xi : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{X'}$. Set $\mathcal{F} := \operatorname{Ker} \xi$. Then we have the exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0. \tag{9}$$

Twisting by n and taking the Euler characteristics of the sheaves in (9) we see that $\chi\mathcal{F}(n) = \chi\mathcal{O}_X(n) - \chi\mathcal{O}_{X'}(n)$. Since X and X' have the same Hilbert polynomial, we have $\chi\mathcal{O}_X(n) = \chi\mathcal{O}_{X'}(n)$ for all $n \in \mathbb{Z}$. Therefore $\chi\mathcal{F}(n) = 0$ for all $n \in \mathbb{Z}$. Hence $\mathcal{F} = 0$, i.e., $\mathcal{O}_X \cong \mathcal{O}_{X'}$ and therefore $X = X'$. □

Proposition 3.1.3. Let $X \subset \mathbb{P}^3$ be a curve. If $H \subset \mathbb{P}^3$ is a plane that does not contain any component of X then $\deg X = l(X \cap H)$, where $l(X \cap H)$ denotes the length of the scheme $X \cap H$.

Proof. Let $I_X = \cap_{i=1}^r q_i$ be a primary decomposition of I_X and let $p_i = \sqrt{q_i}$ be the associated primes of I_X. By Proposition 2.1.5, each p_i is a homogeneous ideal in S. Let $H \subset \mathbb{P}^3$ be a plane not containing any component of X. Let $I_H = (h)$, where h is some linear homogeneous polynomial in S. Then $h \notin p_i, \forall i$. Hence by Proposition 2.1.4 (ii), h is not a zerodivisor in S_X. Therefore $I_X = [I_X :_S h]$. So we have

$$\frac{I_X}{I_X \cap (h)} = \frac{I_X}{h \cdot [I_X :_S h]} = \frac{I_X}{h \cdot I_X}.$$

Also by [2, Proposition 2.1 (ii)] we have

$$\frac{I_X + (h)}{(h)} \cong \frac{I_X}{I_X \cap (h)}.$$

Therefore we have the isomorphism

$$\frac{I_X + (h)}{(h)} \cong \frac{I_X}{h \cdot I_X}$$

and hence the exact sequence

$$0 \to I_X(-1) \xrightarrow{\cdot h} I_X \to \frac{I_X + (h)}{(h)} \to 0. \tag{10}$$

Sheafifying (10) we get the exact sequence

$$0 \to \mathcal{I}_X(-1) \xrightarrow{\cdot h} \mathcal{I}_X \to \mathcal{I}_{(X \cap H)|H} \to 0. \tag{11}$$

Twisting by n and taking the Euler characteristics of the sheaves in (11) we get

$$\chi \mathcal{I}_{(X \cap H)|H}(n) = \chi \mathcal{I}_X(n) - \chi \mathcal{I}_X(n-1).$$

Also from the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

we have $\chi\mathcal{I}_X(n) = \chi\mathcal{O}_{\mathbb{P}^3}(n) - \chi\mathcal{O}_X(n)$. Hence

$$\chi\mathcal{I}_{(X\cap H)|H}(n) = \chi\mathcal{O}_{\mathbb{P}^3}(n) - \chi\mathcal{O}_{\mathbb{P}^3}(n-1) - [\chi\mathcal{O}_X(n) - \chi\mathcal{O}_X(n-1)].$$

Since X is a curve, $\chi\mathcal{O}_X(n) = (\deg X)n + 1 - p_a(X)$, hence $\chi\mathcal{O}_X(n) - \chi\mathcal{O}_X(n-1) = \deg X$. Therefore

$$\chi\mathcal{I}_{(X\cap H)|H}(n) = \chi\mathcal{O}_{\mathbb{P}^3}(n) - \chi\mathcal{O}_{\mathbb{P}^3}(n-1) - \deg X. \tag{12}$$

We also have the exact sequence

$$0 \to (h) \to I_X + (h) \to \frac{I_X + (h)}{(h)} \to 0. \tag{13}$$

Sheafifying (13) we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{I}_{X\cap H} \to \mathcal{I}_{(X\cap H)|H} \to 0. \tag{14}$$

Twisting by n and taking the Euler characteristics of the sheaves in (14) we get

$$\chi\mathcal{I}_{X\cap H}(n) = \chi\mathcal{I}_{(X\cap H)|H}(n) + \chi\mathcal{O}_{\mathbb{P}^3}(n-1). \tag{15}$$

Combining (12) and (15) we get

$$\chi\mathcal{O}_{\mathbb{P}^3}(n) - \chi\mathcal{I}_{X\cap H}(n) = \deg X.$$

Finally, from the exact sequence

$$0 \to \mathcal{I}_{X\cap H} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{X\cap H} \to 0$$

we see that $\chi\mathcal{O}_{X\cap H}(n) = \chi\mathcal{O}_{\mathbb{P}^3}(n) - \chi\mathcal{I}_{X\cap H}(n)$. Therefore $\chi\mathcal{O}_{X\cap H}(n) = \deg X$ and hence is independent of n. Since $X \cap H$ is a zero dimensional scheme, $\chi\mathcal{O}_{X\cap H}$ is the length of $X \cap H$. Therefore $\deg X = l(X \cap H)$. □

3.2 Vector bundles on curves in $\mathbb{P}^3$

In this section we prove two lemmas regarding the Hilbert polynomials of vector bundles on curves in $\mathbb{P}^3$.

Lemma 3.2.1. Let $\mathcal{L}$ be a line bundle on a reduced curve $Y \subset \mathbb{P}^3$. Then there exists a constant $c \in \mathbb{Z}$ such that

$$\chi\mathcal{L}(n) = n \deg Y + c, \ \forall n \in \mathbb{Z}.$$

Proof. Let Y_i be the irreducible components of Y, where $1 \leq i \leq r$. Since Y is reduced, Y_i is integral. Therefore $\operatorname{Sing} Y_i$ is a proper closed subset of Y_i by [18, I, Corollary 8.16]. Choose $P_i \in Y_i$ such that $P_i \notin Y_j$ and $P_i \notin \operatorname{Sing} Y_i$. This is possible since both $\operatorname{Sing} Y_i$ and $\cup_{j\neq i}\{Y_i \cap Y_j\}$ are finite sets of points. Let m be a positive integer such that $\mathcal{L}(m)$ is generated by global sections. Then there exist global sections $s_i \in H^0(Y, \mathcal{L}(m))$ such that s_i generates the stalk of $\mathcal{L}(m)$ at P_i. Hence $s_i \otimes k(P_i) \neq 0$ in $\mathcal{L}(m) \otimes k(P_i) \cong k(P_i) \cong k$, since $\mathcal{L}(m) \otimes k(P_i)$ is a one dimensional vector space. Multiplying by suitable scalars we may assume that $s_i \otimes k(P_i) = 1$ for all i. Set $a_{i,j} := s_i \otimes k(P_j)$. Notice $a_{i,i} = 1$. Let $\tau : k^r \to k^r$ be the map given by the matrix $M = (a_{i,j})_{i,j=1}^r$. Let $s = \sum_{i=1}^r b_i s_i$, where $b_i \in k$. Then s is a global section of $\mathcal{L}(m)$. Let L_i be the linear forms $a_{1,i}x_1 + \cdots a_{r,i}x_r$ for $1 \leq i \leq r$. Notice $L_i \neq 0$, since $a_{i,i} = 1$. Hence each $Z(L_i)$ is a hyperplane in $\mathbb{A}_k^r$. Notice $s \otimes k(P_i) = 0 \Leftrightarrow (b_1, \cdots, b_r) \in Z(L_i)$. Since k is algebraically closed and since each $Z(L_i) \subset \mathbb{A}_k^r$ is a hyperplane, $\cup_{i=1}^r Z(L_i) \subsetneq \mathbb{A}_k^r$. Let $(b_1, \cdots, b_r) \in \mathbb{A}_k^r \setminus \cup_{i=1}^r Z(L_i)$. Then $s \otimes k(P_i) \neq 0$, i.e., $s \otimes k(P_i)$ is a unit for all i. Define the map $\phi : \mathcal{O}_Y \xrightarrow{\cdot s} \mathcal{L}(m)$. Let $\mathcal{K} = \operatorname{Ker} \phi$ and $\mathcal{C} = \operatorname{Coker} \phi$. So we have the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y \xrightarrow{\cdot s} \mathcal{L}(m) \to \mathcal{C} \to 0. \tag{16}$$

Since $s \otimes k(P_i)$ is a unit for all i, ϕ_{P_i} is an isomorphism for all i. Hence $\mathcal{K}_{P_i} = \mathcal{C}_{Pi} = 0$, i.e., $\mathcal{K}$ and $\mathcal{C}$ are not supported on $\{P_i\}_{i=1}^r$. Thus $\operatorname{Supp} \mathcal{K}$ and $\operatorname{Supp} \mathcal{C}$ are proper closed subsets of Y. Therefore $\mathcal{K}$ and $\mathcal{C}$ are finitely supported on Y. Hence by Corollary 2.2.2, there exist constants $c_1, c_2 \in k$ such that $\chi\mathcal{K}(l) = c_1$ and $\chi\mathcal{C}(l) = c_2$ for all $l \in \mathbb{Z}$. Twisting by $n - m$ and taking the Euler characteristics of the sheaves in (16) we get

$$\chi\mathcal{L}(n) = \chi\mathcal{O}_Y(n-m) + \chi\mathcal{C}(n-m) - \chi\mathcal{K}(n-m) = (\deg Y)(n-m) + 1 - p_a(Y) + c_2 - c_1.$$

Set $c := -m \deg Y + 1 - p_a(Y) + c_2 - c_1$. Then $\chi\mathcal{L}(n) = n \deg Y + c, \forall n \in \mathbb{Z}$. □

Lemma 3.2.2. Let $\mathcal{L}$ be a vector bundle on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Then there exists a constant $c \in \mathbb{Z}$ such that

$$\chi\mathcal{L}(n) = n(\operatorname{rank}\mathcal{L})\deg Y + c,\ \forall n \in \mathbb{Z}.$$

Proof. Let η be the generic point of Y and let $m \in \mathbb{N}$ be such that $\mathcal{L}(m)$ is generated by global sections. Then at the stalk at η we have the isomorphism

$$\mathcal{O}^r_{Y,\eta} \xrightarrow{\sim} \mathcal{L}_\eta(m),$$

where $r = \operatorname{rank}\mathcal{L}$. Hence there exist global sections $\{s_i\}_{i=1}^r$ of $\mathcal{L}(m)$ such that $\{s_{i,\eta}\}_{i=1}^r$ generate $\mathcal{L}_\eta(m)$. Therefore $\{s_i\}_{i=1}^r$ defines a map $\phi : \mathcal{O}^r_Y \to \mathcal{L}(m)$. Let $\mathcal{K} = \operatorname{Ker}\phi$ and $\mathcal{C} = \operatorname{Coker}\phi$. Then we have the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}^r_Y \xrightarrow{\phi} \mathcal{L}(m) \to \mathcal{C} \to 0. \tag{17}$$

Notice ϕ_η is an isomorphism, hence $\mathcal{K}$ and $\mathcal{C}$ are not supported at the generic point of Y. Therefore $\mathcal{K}$ and $\mathcal{C}$ are finitely supported on Y. Hence $\chi\mathcal{K}(n)$ and $\chi\mathcal{C}(n)$ are constants by Corollary 2.2.2. Twisting by $n-m$ and taking the Euler characteristics of the sheaves in (17) we get

$$\chi\mathcal{L}(n) = r\chi\mathcal{O}_Y(n-m) + \chi\mathcal{C}(n) - \chi\mathcal{K}(n).$$

Since $\chi\mathcal{O}_Y(n-m) = (n-m)\deg Y + 1 - p_a(Y)$, we have

$$\chi\mathcal{L}(n) = n(\operatorname{rank}\mathcal{L})\deg Y + c,$$

where $c = \chi\mathcal{C}(n) - \chi\mathcal{K}(n) - r(m\deg Y - 1 + p_a(Y)) \in \mathbb{Z}$ is a constant. □

3.3 Cohen-Macaulay curves

In this section we give three equivalent definitions of Cohen-Macaulay curves and prove their equivalence. We also prove some nice properties of such curves. In particular, we show that every extension by locally free sheaves of a Cohen-Macaulay curve, having an integral support, is also a Cohen-Macaulay curve of the same support.

Definition 3.3.1. Let $X \subset \mathbb{P}^3$ be a curve. The graded S-module $H^1_*\mathcal{I}_X$ is called the *Rao module* of X and is denoted by M_X.

Proposition 3.3.2. Let $X \subset \mathbb{P}^3$ be a curve. The following conditions are equivalent.

(a) X has pure dimension 1, i.e., X has no embedded or isolated points.

(b) $\mathcal{O}_{X,P}$ is CM of dimension 1 for all closed points $P \in X$.

(c) M_X has finite length.

Proof. (a)⇒(b): Suppose X has pure dimension 1. Let $P \in X$ be a closed point. Then $\dim\mathcal{O}_{X,P} = 1$, and hence $\operatorname{depth}\mathcal{O}_{X,P} \leq 1$. If $\operatorname{depth}\mathcal{O}_{X,P} = 0$ then $O_{X,P}$ has no regular element. Hence every nonzero element in $\mathfrak{m}_{X,P}$ is a zerodivisor. Let $p_1, \cdots, p_n$ be the associated primes of $\mathcal{O}_{X,P}$. Let $x \in \mathfrak{m}_{X,P}$. Then there exists an element $u \in \mathfrak{m}_{X,P}$ such that $u \neq 0$ but $xu = 0$, i.e., $x \in \operatorname{ann}(u)$. Since $\mathcal{O}_{X,P}$ is noetherian, by Proposition 2.1.4 we have $\operatorname{ann}(u) \subseteq p_i$ for some i. Thus $x \in p_i$ and hence $\mathfrak{m}_{X,P} \subseteq \cup_{i=1}^{n} p_i$. Hence by the prime avoidance lemma [2, Proposition 1.11 (i)] we have $\mathfrak{m}_{X,P} \subseteq p_i$, i.e., $\mathfrak{m}_{X,P} = p_i$ for some i. Thus $m_{X,P}$ is an associated prime of $\mathcal{O}_{X,P}$. Since $\dim \mathcal{O}_{X,P} = 1$, $\mathfrak{m}_{X,P}$ is not a minimal prime. Therefore $\mathfrak{m}_{X,P}$ is an embedded associated prime of $\mathcal{O}_{X,P}$, i.e., P is an embedded point of X, which is a contradiction. Therefore $\operatorname{depth} \mathcal{O}_{X,P} = 1$ and hence $\mathcal{O}_{X,P}$ is CM of dim 1 for all closed points $P \in X$.

(b)⇒(a): Conversely, let $\mathcal{O}_{X,P}$ be CM of dimension 1 for all closed points $P \in X$. Then P is not an isolated point of X, for otherwise $\dim \mathcal{O}_{X,P} = 0$. Suppose P is an embedded point of $\mathcal{O}_{X,P}$. Then there exist $u \in \mathcal{O}_{X,P}$ such that $u \neq 0$ and $\operatorname{ann}(u) = \mathfrak{m}_{X,P}$. But then every element of $\mathfrak{m}_{X,P}$ is a zerodivisor, i.e., $\mathcal{O}_{X,P}$ has no regular element. Therefore $\operatorname{depth} \mathcal{O}_{X,P} = 0$, which contradicts the fact that $\mathcal{O}_{X,P}$ is CM of dimension 1. Thus P is not an embedded point and hence X has no embedded or isolated points, i.e., X has pure dimension 1.

(a)⇒(c): Suppose X has pure dimension 1. Since $\mathcal{I}_X$ is a coherent sheaf on X, by Serre's theorem [18, III, Theorem 5.2 (b)] we have $H^1\mathcal{I}_X(n) = 0$ for $n \gg 0$. So it remains to show that $H^1\mathcal{I}_X(n) = 0$ for $n \ll 0$. Notice $\operatorname{proj\,dim} S_X \leq 4$ by the Hilbert Syzygy Theorem [11, Corollary 19.7]. Hence $\operatorname{proj\,dim} I_X \leq 3$. Let

$$0 \to L_3 \to L_2 \to L_1 \to L_0 \xrightarrow{\tau} I_X \to 0 \tag{18}$$

be a minimal free resolution of I_X and let $E = \operatorname{Ker} \tau$. From (18) we get the exact sequence

$$0 \to E \to L_0 \to I_X \to 0. \tag{19}$$

Sheafifying (19) we get the exact sequence

$$0 \to \mathcal{E} \to \mathcal{L}_0 \to \mathcal{I}_X \to 0, \tag{20}$$

where $\mathcal{E} = \widetilde{E}$ and $\mathcal{L}_0 = \widetilde{L_0}$. Localizing (20) at a closed point $P \in X$ we get the exact sequence

$$0 \to \mathcal{E}_P \to \mathcal{L}_{0,P} \to \mathcal{I}_{X,P} \to 0. \tag{21}$$

By the Auslander-Buchsbaum Theorem [26, Theorem 19.1] we have

$$\operatorname{proj\,dim} \mathcal{O}_{X,P} + \operatorname{depth} \mathcal{O}_{X,P} = \operatorname{depth} \mathcal{O}_{\mathbb{P}^3,P}. \tag{22}$$

Notice both $\mathcal{O}_{\mathbb{P}^3,P}$ and $\mathcal{O}_{X,P}$ are CM rings. Hence $\operatorname{depth} \mathcal{O}_{\mathbb{P}^3,P} = \dim \mathcal{O}_{\mathbb{P}^3,P} = 3$ and $\operatorname{depth} \mathcal{O}_{X,P} = \dim \mathcal{O}_{X,P} = 1$. Therefore $\operatorname{proj\,dim} \mathcal{O}_{X,P} = 2$. We have the exact sequence

$$0 \to \mathcal{I}_{X,P} \to \mathcal{O}_{\mathbb{P}^3,P} \to \mathcal{O}_{X,P} \to 0. \tag{23}$$

Let N be an $\mathcal{O}_{\mathbb{P}^3,P}$-module. Applying $\operatorname{Hom}(-, N)$ to (23) we get the long exact sequence

$$\cdots \to \operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3,P}, N) \to \operatorname{Ext}^2(\mathcal{I}_{X,P}, N) \to \operatorname{Ext}^3(\mathcal{O}_{X,P}, N) \to \cdots . \tag{24}$$

Since $\operatorname{proj\,dim} \mathcal{O}_{X,P} = 2$, $\operatorname{Ext}^3(\mathcal{O}_{X,P}, N) = 0$ by [26, § 19, Lemma 2]. On the other hand,

$\mathcal{O}_{\mathbb{P}^3,P}$ is free and hence projective. Therefore $\operatorname{Ext}^2(\mathcal{O}_{\mathbb{P}^3,P}, N) = 0$ by [20, Proposition 7.2]. Thus we get $\operatorname{Ext}^2(\mathcal{I}_{X,P}, N) = 0$ and hence $\operatorname{proj\,dim} \mathcal{I}_{X,P} \leq 1$ by [26, § 19, Lemma 2]. Similarly, applying $\operatorname{Hom}(-, N)$ to (21) and using the fact that $\operatorname{proj\,dim} \mathcal{I}_{X,P} \leq 1$, we have $\operatorname{proj\,dim} \mathcal{E}_P = 0$, i.e., $\mathcal{E}_P$ is a free $\mathcal{O}_{\mathbb{P}^3,P}$-module. Hence $\mathcal{E}$ is locally free by [18, II, Exercise 5.7 (b)]. Therefore by the Serre Duality Theorem [18, III, Theorem 7.6 (b)(ii)], $H^2(\mathbb{P}^3, \mathcal{E}(n)) = 0$ for $n \ll 0$. Twisting by n and taking the long exact cohomology sequence of (20), we get $H^1(\mathbb{P}^3, \mathcal{I}_X(n)) \cong H^2(\mathbb{P}^3, \mathcal{E}(n))$ for all n. Hence $H^1(\mathbb{P}^3, \mathcal{I}_X(n)) = 0$ for $n \ll 0$. Therefore M_X has finite length.

(c)⇒(a): Suppose M_X has finite length. Let Y be the largest subcurve of X having pure dimension 1, i.e., Y is obtained by removing all the embedded and isolated points of X. We will show that $Y = X$, i.e., X has pure dimension 1. Suppose on the contrary that $Y \subsetneq X$. Let $\mathcal{I}_{Y|X}$ be the ideal sheaf of Y in X. Then we have the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_Y \to \mathcal{I}_{Y|X} \to 0. \tag{25}$$

Twisting by $-n$ and taking the long exact cohomology sequence in (25) we get the exact sequence

$$H^0\mathcal{I}_Y(-n) \to H^0\mathcal{I}_{Y|X}(-n) \to H^1\mathcal{I}_X(-n) \to H^1\mathcal{I}_Y(-n) \to 0. \tag{26}$$

Since Y has pure dimension 1, M_Y has finite length and hence $H^1\mathcal{I}_Y(-n) = 0$ for $n \gg 0$. On the other hand, $H^0_*\mathcal{I}_Y$ is the total ideal I_Y of Y. Hence $H^0\mathcal{I}_Y(-n) = 0$ for $n > 0$. Therefore from (26) we see that $H^0\mathcal{I}_{Y|X}(-n) \cong H^1\mathcal{I}_X(-n)$ for $n \gg 0$. Since $\mathcal{I}_{Y|X}$ is supported on a finite set, $\mathcal{I}_{Y|X}(-n) \cong \mathcal{I}_{Y|X}$ by Lemma 2.2.1. Therefore $H^0\mathcal{I}_{Y|X}(-n) \neq 0$

for $n \gg 0$. Hence $H^1\mathcal{I}_X(-n) \neq 0$ for $n \gg 0$, which contradicts the fact that M_X has finite length. Therefore $Y = X$ and hence X has pure dimension 1. □

Definition 3.3.3. A curve in $\mathbb{P}^3$ is called *Cohen-Macaulay* (CM henceforth) if it satisfies any one of the three equivalent conditions in Proposition 3.3.2.

Example 3.3.4. Let $m \geq n$ be integers and let $W \subset \mathbb{P}^3$ be the closed subscheme given by the total ideal $I_W = (x, y, z)^m \cap (x, y^n)$. Notice (x, y, z) is an embedded associated prime of I_W. Hence $(x, y, z)^m$ defines an embedded point on W at $(0, 0, 0, 1)$ of multiplicity m. Hence W is not a CM curve by Proposition 3.3.2. Throwing away these embedded points we get a CM curve $Z \subset \mathbb{P}^3$ with total ideal $I_Z = (x, y^n)$. Notice, Z is the largest CM curve contained in W. This is an example of a Cohen-Macaulay filtration which we will see in Section 4.2.

Lemma 3.3.5. Let $Y \subset \mathbb{P}^3$ be an integral curve, $\mathcal{F}$ be a locally free sheaf on Y and Z be a CM curve such that $\operatorname{Supp} Z = Y$. If $W \subset \mathbb{P}^3$ is a closed subscheme such that the sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{F} \to 0 \tag{27}$$

is exact, then W is a CM curve with $\operatorname{Supp} W = Y$.

Proof. From (27) we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_W & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_W & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0. \\ & & \downarrow & & & & & & \\ & & \mathcal{F} & & & & & & \end{array} \tag{28}$$

Applying the snake lemma to (28) we get the exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_W \to \mathcal{O}_Z \to 0.$$

Let $P \in \mathbb{P}^3$ be a closed point. Then at the stalk at P we get the exact sequence

$$0 \to \mathcal{F}_P \to \mathcal{O}_{W,P} \to \mathcal{O}_{Z,P} \to 0. \tag{29}$$

Notice $\operatorname{Supp} \mathcal{F} = Y$, since $\mathcal{F}$ is locally free on Y. Therefore $\mathcal{O}_{W,P} \neq 0 \Leftrightarrow \mathcal{O}_{Z,P} \neq 0$, i.e., $\operatorname{Supp} W = \operatorname{Supp} Z = Y$.

Now let P be a closed point of Y. Since Y is integral, it is CM. Hence $\operatorname{depth} \mathcal{O}_{Y,P} = 1$. Let $z \in \mathcal{O}_{Y,P}$ be a regular element. Since Z is supported on Y, $\mathcal{I}_{Z,P} \subset \mathcal{I}_{Y,P}$ and hence there exists a surjection $\mathcal{O}_{Z,P} \twoheadrightarrow \mathcal{O}_{Y,P}$. Let $u \in \mathcal{O}_{Z,P}$ be such that $u \mapsto z$ under this surjection. Notice if $u = 0$ then $z = 0$, which contradicts the regularity of z in $\mathcal{O}_{Y,P}$. Hence $u \neq 0$. Now if u is a zerodivisor in $\mathcal{O}_{Z,P}$ then there exists a nonzero element $a \in \mathcal{O}_{Z,P}$ such that $au = 0$ in $\mathcal{O}_{Z,P}$. Since Y is integral, $\mathcal{I}_{Y,P}$ is a prime ideal in $\mathcal{O}_{\mathbb{P}^3,P}$. Hence $\mathcal{I}_{Z,P}$ is $\mathcal{I}_{Y,P}$-primary, since $\sqrt{\mathcal{I}_{Z,P}} = \mathcal{I}_{Y,P}$. Since $au \in \mathcal{I}_{Z,P}$ but $a \notin \mathcal{I}_{Z,P}$, we therefore have $u^n \in \mathcal{I}_{Z,P}$ for some $n \in \mathbb{N}$. Thus u is nilpotent in $\mathcal{O}_{Z,P}$ and hence z is nilpotent in $\mathcal{O}_{Y,P}$, which contradicts the regularity of z in $\mathcal{O}_{Y,P}$. Therefore u is regular in $\mathcal{O}_{Z,P}$. From (29) we have the surjection $\mathcal{O}_{W,P} \twoheadrightarrow \mathcal{O}_{Z,P}$. Let $v \in \mathcal{O}_{W,P}$ be such that $v \mapsto u$ under this surjection. Notice if $v = 0$ then $u = 0$, which contradicts the regularity of u in $\mathcal{O}_{Z,P}$. Therefore $v \neq 0$. From (29) we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_P & \longrightarrow & \mathcal{O}_{W,P} & \longrightarrow & \mathcal{O}_{Z,P} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_P} & & \downarrow{\scriptstyle \cdot v} & & \downarrow{\scriptstyle \cdot u} & & \\ 0 & \longrightarrow & \mathcal{F}_P & \longrightarrow & \mathcal{O}_{W,P} & \longrightarrow & \mathcal{O}_{Z,P} & \longrightarrow & 0, \end{array} \tag{30}$$

where ϕ_P is the restriction of the map $\cdot v$ on $\mathcal{F}_P$. Notice $\operatorname{Ker}(\cdot u) = 0$, since u is regular in $\mathcal{O}_{Z,P}$. Also $\cdot v$ is not the zero map, since $v \neq 0$. Since $\mathcal{F}$ is locally free on Y, there exists an integer $r \in \mathbb{N}$ such that $\mathcal{F}_P \cong \mathcal{O}_{Y,P}^r$. Then ϕ_P is given by the $r \times r$ diagonal matrix

$$\begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & z & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z \end{pmatrix}.$$

Hence $\det \phi_P = z^r$. Since z is regular in $\mathcal{O}_{Y,P}$, $\det \phi_P = z^r \neq 0$ and hence $\operatorname{Ker} \phi_P = 0$. Applying the snake lemma to (30) we therefore have $\operatorname{Ker}(\cdot v) = 0$. Thus if $vw = 0$ for some $w \in \mathcal{O}_{W,P}$ then $w = 0$. Therefore v is regular in $\mathcal{O}_{W,P}$ and $\operatorname{depth} \mathcal{O}_{W,P} \geq 1$. Since $\operatorname{depth} \mathcal{O}_{W,P} \leq \dim \mathcal{O}_{W,P} = 1$, we have $\operatorname{depth} \mathcal{O}_{W,P} = 1$. Thus $\mathcal{O}_{W,P}$ is CM of dimension 1. Since $P \in Y$ was arbitrary, W is a CM curve by Proposition 3.3.2. Thus W is a CM curve with $\operatorname{Supp} W = Y$. □

Proposition 3.3.6. Let $X \subset \mathbb{P}^3$ be a CM curve. If $\mathcal{I}$ is an ideal sheaf in $\mathcal{O}_X$ that is not supported at any generic point of X, then $\mathcal{I} = 0$.

Proof. Let Y be the closed subscheme of X defined by the ideal sheaf $\mathcal{I}$. Then we have the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Let $P \in X$ be a closed point. At the stalk at P we get the exact sequence

$$0 \to \mathcal{I}_P \to \mathcal{O}_{X,P} \to \mathcal{O}_{Y,P} \to 0.$$

Since X is CM, $\mathcal{O}_{X,P}$ is a CM ring of dimension 1. Set $I := \mathcal{I}_P$ and $A = \mathcal{O}_{X,P}$. Since $\mathcal{I}$ is not supported at any generic point of X, $I_q = 0$ for all minimal primes q of A. Since A is CM, it has no embedded associated prime. Hence by Lemma 2.1.6, $I = 0$, i.e., $\mathcal{I}_P = 0$ for all closed points $P \in X$. Therefore $\mathcal{I} = 0$. □

Corollary 3.3.7. Let $X \subset \mathbb{P}^3$ be an irreducible CM curve. If U is an open set such that $X \cap U$ is nonempty, then $X = \overline{X \cap U}$.

Proof. Let $Y = \overline{X \cap U}$. Then Y is a closed subscheme of X. Notice Y is dense in X, since $X \cap U \neq \varnothing$. Hence $X \setminus Y$ consists of finitely many points, since X is irreducible. Let $\mathcal{I}$ be the ideal sheaf of Y in X. Then $\operatorname{Supp}(\mathcal{I}) \subseteq X \setminus Y$, i.e., $\mathcal{I}$ is not supported at the generic point of X. Hence $\mathcal{I} = 0$ by Proposition 3.3.6. Therefore $X = Y = \overline{X \cap U}$. □

Corollary 3.3.8. Let $X, X' \subset \mathbb{P}^3$ be irreducible CM curves. Then $X = X'$ if and only if $X \cap U = X' \cap U$ for some open set U such that $X \cap U$ and $X' \cap U$ are nonempty.

Proof. If $X = X'$ then of course $X \cap U = X' \cap U$ for all open sets U. Conversely, let U be an open set such that $X \cap U$ and $X' \cap U$ are nonempty with $X \cap U = X' \cap U$. Then by Corollary 3.3.7 $X = \overline{X \cap U} = \overline{X' \cap U} = X'$. □

4 Multiplicity structures on curves in $\mathbb{P}^3$

In this chapter we describe general behaviors of multiplicity structures on nonsingular connected curves in $\mathbb{P}^3$ having generic embedding dimension 2. In Sections 4.1 − 4.3 we extend the theory of Bănică and Forster [3] from complex analytic three manifolds to $\mathbb{P}^3_k$ over an arbitrary but algebraically closed field k. In particular, we give rigorous proofs of their claims and statements. In Section 4.4 we give an independent proof of Ferrand's construction of doubling a locally complete intersection curve in the context of nonsingular connected curves in $\mathbb{P}^3$. We also prove that every Cohen-Macaulay double structure on nonsingular connected curves in $\mathbb{P}^3$ arises from this construction. In Section 4.5 we describe the singularities and class groups of general surfaces containing multiplicity structures, following the works of Brevik and Nollet [6].

4.1 Primitive extensions

In this section we describe multiplicity structures on nonsingular connected curves in $\mathbb{P}^3$ having embedding dimension at most 2 at every point.

Definition 4.1.1. Let $Y \subset \mathbb{P}^3$ be a nonsingular connected curve. A *multiplicity structure* on Y, or an *extension* of Y, is a CM curve $Z \subset \mathbb{P}^3$ such that $\operatorname{Supp} Z = \operatorname{Supp} Y$. The multiplicity of Z is defined by $\deg Z / \deg Y$ and is denoted by $\operatorname{mult}(Z)$. We say Z is a multiplicity m-structure on Y or an m-extension of Y if $\operatorname{mult}(Z) = m$. If Z is a 1-extension of Y, i.e., if $Y = Z$, then we say Z is a trivial extension of Y.

Example 4.1.2. Let $Y \subset \mathbb{P}^3$ be the line with $I_Y = (x, y)$ and $Z \subset \mathbb{P}^3$ be the closed subscheme with $I_Z = (x, y^n)$ for some $n \in \mathbb{N}$. Then Z is a multiplicity n-structure on Y.

Lemma 4.1.3. Let $Y \subset \mathbb{P}^3$ be a nonsingular curve and let $P \in Y$ be a closed point. Then there exists a regular system of parameters $\{x, y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $\mathcal{I}_{Y,P} = (x, y)$. Moreover, if there exists a regular element $x' \in \mathcal{O}_{\mathbb{P}^3,P}$ such that $x' \in \mathcal{I}_{Y,P}$ but $x' \notin \mathfrak{m}^2_{\mathbb{P}^3,P}$, then there exists a regular system of parameters $\{x', y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ with $\mathcal{I}_{Y,P} = (x', y)$.

Proof. We have the exact sequence

$$0 \to \mathcal{I}_{Y,P} \to \mathcal{O}_{\mathbb{P}^3,P} \to \mathcal{O}_{Y,P} \to 0. \tag{31}$$

Since Y is nonsingular, $\mathcal{O}_{Y,P}$ is a regular local ring and hence a complete intersection ring by Corollary 2.1.15. Therefore by Corollary 2.1.17, the minimal number of generators of $\mathcal{I}_{Y,P}$ is $\dim \mathcal{O}_{\mathbb{P}^3,P} - \dim \mathcal{O}_{Y,P} = 3 - 1 = 2$. Let $k(P) = O_{\mathbb{P}^3,P}/\mathfrak{m}_{\mathbb{P}^3,P}$ be the residue field at P. Then $\dim \mathcal{I}_{Y,P} \otimes k(P) = 2$ by Nakayama's lemma [2, Proposition 2.8]. From (31) we get the exact sequence

$$0 \to \mathcal{I}_{Y,P} \to \mathfrak{m}_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P} \to 0. \tag{32}$$

Tensoring (32) with $k(P)$ we get the exact sequence

$$\mathcal{I}_{Y,P} \otimes k(P) \to \mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P} \to 0. \tag{33}$$

Let $K_P = \mathrm{Ker}(\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P})$. Since (33) is exact in the middle, K_P is equal to the image of $\mathcal{I}_{Y,P} \otimes k(P)$. Hence there exists a surjection $\mu_P : \mathcal{I}_{Y,P} \otimes k(P) \twoheadrightarrow K_P$. Since Y is nonsingular, $\dim \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P} = 1$ and hence $\dim K_P = 2$. Thus μ_P is a surjection of two-dimensional vector spaces, hence is an isomorphism and therefore $\mathcal{I}_{Y,P} \otimes k(P) \cong K_P$. Hence (33) is left exact. Combining (32) and (33) we get the commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathfrak{m}^2_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}^2_{Y,P} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{Y,P} & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{Y,P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{Y,P} \otimes k(P) & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P} & \longrightarrow & 0. \end{array} \tag{34}$$

Let $\{x, y, z\}$ be a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$. Then $\{\bar{x}, \bar{y}, \bar{z}\}$ is a basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, where $\bar{x}, \bar{y}, \bar{z}$ are the images of x, y, z in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$ respectively. By a change of basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, if necessary, we may assume that $\bar{z}$ is a basis of $\mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P}$. Then z generates $\mathfrak{m}_{Y,P}$ by Nakayama's lemma [2, Proposition 2.8]. Let ϕ_P denote the map $\mathfrak{m}_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}$ in (32). Then $\phi_P(x) = az, \phi_P(y) = bz$ and $\phi_P(z) = z$ for some $a, b \in \mathcal{O}_{\mathbb{P}^3,P}$. Let $x' = x - az, y' = y - bz$. Let $\bar{x}', \bar{y}'$ be the images of x', y' in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Then

$$A \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z} \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice A is invertible and hence $\{\bar{x}', \bar{y}', \bar{z}\}$ is a basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, i.e., $\{x', y', z\}$ is a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$. Making this change of basis we therefore have $\phi_P(x') = \phi_P(y') = 0$. Thus $(x', y') \subset \mathcal{I}_{Y,P}$ and hence $\bar{x}', \bar{y}' \in \mathcal{I}_{Y,P} \otimes k(P)$. Since $\bar{x}'$ and $\bar{y}'$ are linearly independent and since $\dim \mathcal{I}_{Y,P} \otimes k(P) = 2$, $\{\bar{x}', \bar{y}'\}$ is a basis of $\mathcal{I}_{Y,P} \otimes k(P)$. Therefore $\{x', y'\}$ is a minimal basis of $\mathcal{I}_{Y,P}$ by Nakayama's lemma [2, Proposition 2.8].

Hence $\mathcal{I}_{Y,P} = (x', y')$. Denoting x' by x and y' by y we get $\mathcal{I}_{Y,P} = (x, y)$.

Now suppose x' is a regular element in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $x' \in \mathcal{I}_{Y,P}$ but $x' \notin \mathfrak{m}^2_{\mathbb{P}^3,P}$. Let $\bar{x}'$ be the image of x' in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Then $\bar{x}' \neq 0$. Hence $\bar{x}'$ is a basis element of $\mathrm{Ker}(\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P})$ in (34). Let $y', z \in \mathfrak{m}_{\mathbb{P}^3,P}$ be such that $\{\bar{x}', \bar{y}'\}$ is a basis of $\mathrm{Ker}(\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P})$ and $\bar{z}$ is a basis of $\mathfrak{m}_{Y,P}/\mathfrak{m}^2_{Y,P}$, where $\bar{y}'$ and $\bar{z}$ are the images of y' and z in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Then $\{\bar{x}', \bar{y}', \bar{z}\}$ is a basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$ and hence $\{x', y', z\}$ is a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$. Let ϕ_P denote the map $\mathfrak{m}_{\mathbb{P}^3,P} \to \mathfrak{m}_{Y,P}$ in (32). Then $\phi_P(x') = 0, \phi_P(y') = cz$ and $\phi_P(z) = z$ for some $c \in \mathcal{O}_{\mathbb{P}^3,P}$. Let $y = y' - cz$. Then by the same argument in the previous paragraph, $\{x', y, z\}$ is a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$ with $\mathcal{I}_{Y,P} = (x', y)$. □

Corollary 4.1.4. Let $Y \subset \mathbb{P}^3$ be a nonsingular curve and let $P \in Y$ be a closed point. If $\mathcal{I}_{Y,P} = (\widetilde{x}, \widetilde{y})$ for some $\widetilde{x}, \widetilde{y} \in \mathcal{O}_{\mathbb{P}^3,P}$ then there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that $\mathcal{I}_{Y|U} = (x, y)$ and $x_P = \widetilde{x}, y_P = \widetilde{y}$.

Proof. Let $V = \mathrm{Spec}\, A$ be an open affine neighborhood of P and let p be the prime ideal in A corresponding to P. Let $\mathfrak{m}_p$ denote the maximal ideal in A_p. Then $\widetilde{x}, \widetilde{y} \in \mathfrak{m}_p$. So there exist $x, y \in A$ and $a, b \in A \setminus p$ such that $\widetilde{x} = x/a$ and $\widetilde{y} = y/b$. Let $\psi : A^2_{ab} \to \mathcal{I}_{Y|V_{ab}}$ be the map given by the matrix $\begin{pmatrix} x & y \end{pmatrix}$, where $V_{ab} = \mathrm{Spec}\, A_{ab}$. Let $C = \mathrm{Coker}\, \psi$. Then $C_p = 0$, since ψ_p is a surjection. Let $\{c_1, \cdots, c_r\}$ be a generating set for C. Then there exist $s_1, \cdots, s_r \in A \setminus p$ such that $s_i c_i = 0$ for $1 \leq i \leq r$. Hence $\psi_s : A^2_{abs} \to \mathcal{I}_{Y|V_{abs}}$ is a surjection, where $s = \Pi_{i=1}^r s_i$ and $V_{abs} = \mathrm{Spec}\, A_{abs}$. Let $U = V_{abs}$. Then $\mathcal{I}_{Y|U} = (x, y)$. Moreover, $x_P = x_p = \widetilde{x}$ and $y_P = y_p = \widetilde{y}$. □

Definition 4.1.5. Let Z be a multiplicity structure on a nonsingular connected curve Y. Then Z is a *primitive extension* of Y if $\operatorname{embdim}_P Z \leq 2$ for all closed points $P \in Y$.

Lemma 4.1.6. Let $Z \subset \mathbb{P}^3$ be a curve and let $P \in Z$ be a closed point. Then $\operatorname{embdim}_P(Z) \leq 2$ if and only if there exists a regular system of parameters $\{x, y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $x \in \mathcal{I}_{Z,P}$. Moreover, in that case there exists an open affine neighborhood U of P such that $x \in \mathcal{O}_U$ and the ideal (x) defines a nonsingular surface $F \subset U$ with $\mathcal{I}_F = (x) \subset \mathcal{I}_{Z|U}$.

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathfrak{m}^2_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}^2_{Z,P} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z,P} & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{Z,P} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_P & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{Z,P}/\mathfrak{m}^2_{Z,P} & \longrightarrow & 0,
\end{array} \tag{35}$$

where $K_P = \operatorname{Ker}(\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \to \mathfrak{m}_{Z,P}/\mathfrak{m}^2_{Z,P})$. Let $\operatorname{embdim}_P(Z) \leq 2$. If $\operatorname{embdim}_P Z = 1$ then $\mathcal{O}_{Z,P}$ is a regular local ring of dimension 1. Hence by Lemma 34, there exists a regular system of paramenters $\{x, y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $\mathcal{I}_{Z,P} = (x, y)$ and hence $x \in \mathcal{I}_{Z,P}$. Now suppose $\operatorname{embdim}_P Z = 2$. Let $\{x, y, z\}$ be a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$. Then $\{\bar{x}, \bar{y}, \bar{z}\}$ is a basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, where $\bar{x}, \bar{y}, \bar{z}$ are the images of x, y, z in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$ respectively. By a change of basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, if necessary, we may assume that $\{\bar{y}, \bar{z}\}$ is a basis of $\mathfrak{m}_{Z,P}/\mathfrak{m}^2_{Z,P}$. Then y and z generate $\mathfrak{m}_{Z,P}$ by Nakayama's lemma [2, Proposition 2.8]. Let ϕ_P denote the map $\mathfrak{m}_{\mathbb{P}^3,P} \to \mathfrak{m}_{Z,P}$ in (35). Then there exist $a, b \in \mathcal{O}_{\mathbb{P}^3,P}$ such that $\phi_P(x) = ay + bz, \phi_P(y) = y$ and $\phi_P(z) = z$. Let $x' = x - ay - bz$.

Let $\bar{x}'$ be the image of x' in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Then

$$A\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}\bar{x}'\\ \bar{y}'\\ \bar{z}\end{pmatrix}, \text{ where } A = \begin{pmatrix}1 & -a & -b\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Notice A is invertible and hence $\bar{x}', \bar{y}, \bar{z}$ is a basis of $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$, i.e., $\{x', y, z\}$ is a regular system of parameters of $\mathcal{O}_{\mathbb{P}^3,P}$. Making this change of basis we see that $\phi_P(x') = 0$. Hence $x' \in \mathcal{I}_{Z,P}$. Denoting x' by x we get $x \in \mathcal{I}_{Z,P}$.

Conversely, let $\{x, y, z\}$ be a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $x \in \mathcal{I}_{Z,P}$. Let $\bar{x}$ be the image of x in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Notice $\bar{x} \notin \mathfrak{m}^2_{\mathbb{P}^3,P}$ and hence $\bar{x}$ is a nonzero element of K_P. Therefore $\dim K_P \geq 1$ and hence $\operatorname{embdim}_P(Z) = \dim \mathfrak{m}_{Z,P}/\mathfrak{m}^2_{Z,P} \leq 2$.

Let $V = \operatorname{Spec} A$ be an open affine neighborhood of P. Let p be the prime ideal in A corresponding to the point P. Let $\mathfrak{m}_p$ be the maximal ideal in the local ring A_p. Let $\operatorname{embdim}_P(Z) \leq 2$. Then $x \in \mathcal{I}_{Z,P}$ and hence there exist $a \in \mathcal{I}_Z(V)$ and $\xi \in A \setminus p$ such that $x = a/\xi$. Let $E \subset V_\xi$ be the surface defined by the ideal (x). Then $\mathcal{I}_E = (x)$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{E,P} & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{E,P} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{E,P}/\mathfrak{m}^2_{E,P} & \longrightarrow & 0. \end{array} \tag{36}$$

Let $\bar{x}$ be the image of x in $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P}$. Since $\mathcal{I}_{E,P} = (x)$, $\bar{x} \mapsto 0 \in \mathfrak{m}_{E,P}/\mathfrak{m}^2_{E,P}$. Therefore $\dim \mathfrak{m}_{E,P}/\mathfrak{m}^2_{E,P} = 2$ and hence E is nonsingular at P. By [18, II, Corollary 8.16], there exists an open dense set $V' \subset V_\xi$ such that $E \cap V'$ is nonsingular. Let $U = V'$ and $F = E \cap V'$. Then $F \subset U$ is a nonsingular surface with $\mathcal{I}_F = (x) \subset \mathcal{I}_{Z|U}$. $\square$

The following lemma has been taken from a series of lectures given by Scott Nollet at our algebraic geometry seminar in TCU.

Lemma 4.1.7. Let $Y \subset Z \subset F$ be such that F is a nonsingular affine surface, Y is a nonsingular connected curve and Z is a multiplicity n-structure on Y. Then $I_Z = I_Y^d$ for some $d \in \mathbb{N}$, where I_Y and I_Z denote the ideals of Y and Z in F respectively.

Proof. Let $F = \operatorname{Spec} A$. Notice Y is a c.i., since it is nonsingular. Hence I_Y is generated by $\operatorname{codim}(Y, F) = 1$ element. Let $I_Y = (f)$ for some $f \in A$. Let $I_Z = (a_1, \cdots, a_s)$, where $a_i \in A$. Since $I_Z \subset I_Y$, $f \mid a_i$ for all i. Let d be the largest integer such that $f^d \mid a_i$ for all i. Then $f^{d+1} \nmid a_i$ for some i. Without loss of generality we may assume that $f^{d+1} \nmid a_1$. Let $a_i = f^d b_i$. Then $I_Z = (f^d)\mathfrak{b}$, where $\mathfrak{b} = (b_1, \cdots, b_s)$. Notice $f \nmid b_1$. We will show that $\mathfrak{b} = A$. Suppose on the contrary that $\mathfrak{b} \neq A$. Then $1 \notin \mathfrak{b}$ and hence $f^d \notin I_Z$. Let $I_Z = \cap_{j=1}^n q_j$ be a primary decomposition of I_Z. Since Z is CM, I_Z has no embedded associated prime. Therefore $\sqrt{q_j} = (f)$ for all j, since $\operatorname{Supp} Z = Y$ and Y is connected. Since $f^d b_1 \in I_Z$ but $f^d \notin I_Z$, there exists j_0 such that $f^d b_1 \in q_{j_0}$ but $f^d \notin q_{j_0}$. Since q_{j_0} is primary we therefore have $b_1^m \in q_{j_0}$ for some $m \in \mathbb{N}$. Hence $b_1 \in (f)$, since $\sqrt{q_{j_0}} = (f)$. But then $f \mid b_1$, which is a contradiction. Therefore $\mathfrak{b} = A$ and hence $I_Z = (f^d) = I_Y^d$. $\square$

Proposition 4.1.8. Let Z be a multiplicity n-structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Then Z is a primitive extension of Y if and only if for each closed point $P \in Y$ there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that the ideal (x) defines a nonsingular surface $F \subset U$ with $\mathcal{I}_F = (x), \mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z|U} = (x, y^n)$.

Proof. Let Z be a primitive extension of Y and let $P \in Y$ be a closed point. Hence $\operatorname{embdim}_P(Z) \leq 2$. By Lemma 4.1.6, there exists a regular system of parameters $\{x, y', z'\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $x \in \mathcal{I}_{Z,P}$. Moreover, there exists an open affine neighborhood U_1 of P such that $x \in \mathcal{O}_{U_1}$ and the ideal (x) defines a nonsingular surface $F' \subset U_1$ with $\mathcal{I}_{F'} = (x) \subset \mathcal{I}_{Z|U_1}$. We have $x \in \mathcal{I}_{Y,P}$, since $\mathcal{I}_{Z,P} \subset \mathcal{I}_{Y,P}$. Notice $x \notin \mathfrak{m}^2_{\mathbb{P}^3,P}$, since $\{x, y', z'\}$ is a regular system of parameters in $\mathcal{O}_{\mathbb{P}^3,P}$. Therefore by Lemma 4.1.3, there exists a regular system of parameters $\{x, y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $\mathcal{I}_{Y,P} = (x, y)$. Hence by Proposition 4.1.4, there exists an open affine neighborhood U_2 of P such that $x, y \in \mathcal{O}_{U_2}$ and $\mathcal{I}_{Y|U_2} = (x, y)$. Let $U = U_1 \cap U_2$ and $F = F' \cap U$. Then $\mathcal{I}_{Y|U} = (x, y)$ and $F \subset U$ is a nonsingular affine surface with $\mathcal{I}_F = (x) \subset \mathcal{I}_{Z|U}$. It remains to show that $\mathcal{I}_{Z|U} = (x, y^n)$. Let $\mathcal{I}_{Y|F}$ and $\mathcal{I}_{Z|F}$ denote the ideal sheaves of Y and Z in F respectively. Then $\mathcal{I}_{Y|F} = (y)$ and hence $\mathcal{I}_{Z|F} = (y^d)$ for some $d \in \mathbb{N}$ by Lemma 4.1.7. Therefore $\mathcal{I}_{Z|U} = (x, y^d)$. Since U is a nonempty open set, $Y \cap U$ is dense in Y. Hence $Y \setminus (Y \cap U)$ has finitely many points. Therefore a general plane will miss every point of $Y \setminus (Y \cap U)$. Let $H \subset \mathbb{P}^3$ be a plane such that $Y \cap H = \{Q_i\}_{i=1}^r \subset U$ and H intersects Y transversely at each Q_i. Let $Q \in \{Q_i\}_{i=1}^r$. Then $\mathcal{I}_{H,Q} = (h)$ for some $h \in \mathcal{O}_U$. Notice $\{x, y, h\}$ is a regular sequence in $\mathcal{O}_{\mathbb{P}^3,Q}$ since H intersects Y transversely at Q. We have $\mathcal{I}_{Y \cap H,Q} = (x, y, h)$ and $\mathcal{I}_{Z \cap H,Q} = (x, y^d, h)$. Therefore $\mathcal{I}_{Y \cap H,Q}$ has the filtration

$$\mathcal{I}_{Z \cap H,Q} = (x, y^d, h) \subset (x, y^{d-1}, h) \subset \cdots \subset (x, y, h) = \mathcal{I}_{Y \cap H,Q}$$

and hence we have the exact sequences

$$0 \to \frac{(x, y^m, h)}{(x, y^{m+1}, h)} \to \frac{\mathcal{O}_{\mathbb{P}^3,Q}}{(x, y^{m+1}, h)} \to \frac{\mathcal{O}_{\mathbb{P}^3,Q}}{(x, y^m, h)} \to 0, \tag{37}$$

where $1 \leq m \leq d-1$. Notice $\mathfrak{m}_Q$ annihilates $(x, y^m, h)/(x, y^{m+1}, h)$, where $\mathfrak{m}_Q = (x, y, h)$ is the maximal ideal in $\mathcal{O}_{\mathbb{P}^3,Q}$. Hence $(x, y^m, h)/(x, y^{m+1}, h)$ is a $k(Q)$-vector space, where $k(Q) = \mathcal{O}_{\mathbb{P}^3,Q}/\mathfrak{m}_Q$ is the residue field at Q. Also notice $(x, y^m, h)/(x, y^{m+1}, h)$ is generated by a single element, i.e., by the image of y^m in $\mathcal{O}_{\mathbb{P}^3,Q}/(x, y^{m+1}, h)$. Therefore $\dim(x, y^m, h)/(x, y^{m+1}, h) = 1$ for all m. Since $\mathcal{O}_{\mathbb{P}^3,Q}/(x, y, h) = k(Q)$, we have $\dim \mathcal{O}_{\mathbb{P}^3,Q}/(x, y, h) = 1$ and hence $\dim \mathcal{O}_{\mathbb{P}^3,Q}/(x, y^d, h) = d$ by induction on m. Therefore

$$l(Z \cap H) = \sum_{i=1}^{r} \dim \frac{\mathcal{O}_{\mathbb{P}^3,Q_i}}{(x, y^d, h)} = \sum_{i=1}^{r} d = d \sum_{i=1}^{r} 1 = d \sum_{i=1}^{r} \dim \frac{\mathcal{O}_{\mathbb{P}^3,Q_i}}{(x, y, h)} = d \cdot l(Y \cap H),$$

where $l(Z \cap H)$ and $l(Y \cap H)$ denote the lengths of $Z \cap H$ and $Y \cap H$ respectively. Therefore $\deg Z = d \cdot \deg Y$ by Proposition 3.1.3. Since $\mathrm{mult}(Z) = n$, we have $d = n$. Thus $\mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z|U} = (x, y^n)$.

Conversely, let for each closed point $P \in Y$ there exist an open affine neighborhood U of P and $x, y \in O_U$ such that the ideal (x) defines a nonsingular surface $F \subset U$ with $\mathcal{I}_F = (x), \mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z|U} = (x, y^n)$. Then $x \in \mathcal{I}_{Z,P}$ and hence $\mathrm{embdim}_P(Z) \leq 2$ by Lemma 4.1.6. Therefore Z is a primitive extension of Y. □

Remark 4.1.9. From Proposition 4.1.8 we see that every primitive extension of a nonsingular connected curve in $\mathbb{P}^3$ is a locally complete intersection.

Corollary 4.1.10. Let Z be a primitive n-extension of a nonsingular connected curve $Y \subset \mathbb{P}^3$ and let $Z_j \subseteq Z$ be a multiplicity j-structure on Y. Then for each closed point $P \in Y$ there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that $\mathcal{I}_{Z_j} = (x, y^j)$. Moreover, Z_j is the unique multiplicity j-structure on Y contained in Z.

Proof. Let $P \in Y$ be a closed point. Since Z is a primitive n-extension of Y by Proposition 4.1.8, there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that $\mathcal{I}_F = (x)$, $\mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z|U} = (x, y^n)$. Since $Y \subseteq Z_j \subseteq Z$, we have $(x, y^n) \subseteq \mathcal{I}_{Z_j|U} \subseteq (x, y)$ and hence $x \in \mathcal{I}_{Z_j|U}$. Therefore Z_j is a primitive j-extension of Y by Lemma 4.1.6 and hence $\mathcal{I}_{Z_j|U} = (x, y^j)$ by Lemma 4.1.8. Now if Z'_j is another multiplicity j-structure of Y contained in Z then by the same analysis we get $\mathcal{I}_{Z'_j|U} = (x, y^j) = \mathcal{I}_{Z_j|U}$, i.e., $Z'_j \cap U = Z_j \cap U$. Therefore $Z'_j = Z_j$ by Corollary 3.3.8. Hence Z_j is the unique multiplicity j-structure of Y contained in Z. □

Remark 4.1.11. From Corollary 4.1.10 we see that if Z is a primitive n-extension of a nonsingular connected curve $Y \subset \mathbb{P}^3$, then Z has a unique filtration by the primitive j-extensions $Z_j \subset Z$ of Y.

4.2 Cohen-Macaulay filtrations

Although primitive extensions are the nicest extensions, most multiplicity structures are not primitive. To deal with general kinds of extensions Bănică and Forster introduced the notion of Cohen-Macaulay filtration, which we describe next.

Let Z be a CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Let $Y^{(j)}$ be the j^{th}-infinitesimal neighborhood of Y, where $\mathcal{I}_{Y^{(j)}} = \mathcal{I}_Y^j$. Then $\mathcal{I}_{Z \cap Y^{(j)}} = \mathcal{I}_Z + \mathcal{I}_Y^j$

is the ideal sheaf of the intersection $Z \cap Y^{(j)}$. Let $\cap_{i=1}^{n} Q_i$ be a primary decomposition of $\mathcal{I}_{Z\cap Y^{(j)}}$. Let $P_i = \sqrt{Q_i}$ be the associated primes of $\mathcal{I}_{Z\cap Y^{(j)}}$. Notice, $Z \cap Y^{(j)}$ has an embedded point if and only if P_i is an embedded prime for some i. Throwing away all the embedded primary components of $\mathcal{I}_{Z\cap Y^{(j)}}$ we obtain a unique ideal $\mathcal{I}_j$, by [2, Corollary 4.11]. Let Z_j be the subscheme defined by the ideal sheaf $\mathcal{I}_j$. Then Z_j has no embedded or isolated point and hence is CM by Proposition 3.3.2. By construction, Z_j is the largest CM curve contained in $Z\cap Y^{(j)}$ and hence is uniquely determined by the j^{th}-infinitesimal neighborhood $Y^{(j)}$ of Y. Now if $Z_j \subsetneq Z$ for all $j \in \mathbb{N}$ then $\deg Z > \deg Z_j \geq j \deg Y \geq j$ for all $j \in \mathbb{N}$, i.e., $\deg Z = \infty$, which is impossible. Hence there exists a positive integer n such that $Z_j = Z$ for all $j \geq n$. Thus we get a flitration of Z by CM curves as follows:

$$Y = Z_1 \subset \cdots \subset Z_n = Z. \tag{38}$$

Definition 4.2.1. We call (38) the Cohen-Macaulay (CM henceforth) filtration of Z.

Notation 4.2.2. Let Γ_j denote the set of embedded points in $Z\cap Y^{(j)}$. Then $\dim \Gamma_j = 0$ and $Z_j = Z\cap Y^{(j)}$ on $Y \setminus \Gamma_j$. In other words, $\mathcal{I}_{Z_j} = \mathcal{I}_Z + \mathcal{I}_Y^j$ on $Y \setminus \Gamma_j$. Set $\Gamma := \cup_{j=1}^{n-1} \Gamma_j$.

Example 4.2.3. Let Y be the line with total ideal $I_Y = (x, y)$ and let Z be the multiplicity n-structure on Y with total ideal $I_Z = (x, y^n)$. If Z_j is the j^{th} CM filtrant of Z, then $I_{Z_j} = (x, y^j)$.

Example 4.2.4. Let $Y \subset \mathbb{P}^3$ be the line given by $I_Y = (z, w)$. Let $Z \subset \mathbb{P}^3$ be the curve given by $I_Z = (z^2, yz - w^2)$. Notice $\mathcal{I}_Z \subset \mathcal{I}_Y$, hence $\sqrt{\mathcal{I}_Z} \subset \sqrt{\mathcal{I}_Y} = \mathcal{I}_Y$. On the other hand, $z \in \sqrt{\mathcal{I}_Z}$ since $z^2 \in \mathcal{I}_Z$. Therefore $w \in \sqrt{\mathcal{I}_Z}$, hence $\mathcal{I}_Y \subset \sqrt{\mathcal{I}_Z}$, i.e., $\sqrt{\mathcal{I}_Z} = \mathcal{I}_Y$.

Notice Z is CM, since it is a complete intersection. Thus Z is a CM multiplicity 4-structure on Y.

We have $\mathcal{I}_Z + \mathcal{I}_Y^3 = (z^2, yz - w^2, w^3) \subseteq \mathfrak{a} \cap \mathfrak{b}$, where $\mathfrak{a} = (y, z^2, w)$, $\mathfrak{b} = (yz - w^2, zw, z^2)$ are primary ideals. Let Z_3 be the subscheme defined by the total ideal $I_{Z_3} = \mathfrak{b}$. Then I_{Z_3} has the S-resolution

$$0 \to S(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ w & z \\ -y & -w \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} yz - w^2 & zw & z^2 \end{pmatrix}} I_{Z_3} \to 0. \tag{39}$$

From (39) we see that Z_3 is ACM and hence CM. Sheafifying (39) and augmenting by $\mathcal{O}_{\mathbb{P}^3}$, we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{Z_3} \to 0. \tag{40}$$

Twisting by 1 and taking the Euler characteristics of the sheaves in (40) we get $\chi\mathcal{O}_{Z_3}(1) = \chi\mathcal{O}_{\mathbb{P}^3}(1) - 3\chi\mathcal{O}_{\mathbb{P}^3}(-1) + 2\chi\mathcal{O}_{\mathbb{P}^3}(-2)$. Similarly, $\chi\mathcal{O}_{Z_3} = \chi\mathcal{O}_{\mathbb{P}^3} - 3\chi\mathcal{O}_{\mathbb{P}^3}(-2) + 2\chi\mathcal{O}_{\mathbb{P}^3}(-3)$. Notice $n < 0 \Rightarrow h^0\mathcal{O}_{\mathbb{P}^3}(n) = 0$ and $n \geq -3 \Rightarrow h^3\mathcal{O}_{\mathbb{P}^3}(n) = 0$. Hence $\chi\mathcal{O}_{Z_3}(1) = h^0\mathcal{O}_{\mathbb{P}^3}(1) = 4$ and $\chi\mathcal{O}_{Z_3} = h^0\mathcal{O}_{\mathbb{P}^3} = 1$. Therefore $\deg Z_3 = \chi\mathcal{O}_{Z_3}(1) - \chi\mathcal{O}_{Z_3} = 3$. Hence Z_3 is a CM triple structure on Y contained in Z. Since CM filtration is unique, Z_3 is the 3^{rd} CM filtrant of Z. Notice $\mathfrak{a}$ yields an embedded point at the origin since $\sqrt{\mathfrak{a}} = (y, z, w)$. Similarly, $\mathcal{I}_Z + \mathcal{I}_Y^2 = (yz, z^2, zw, w^2) \subseteq \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p} = (y, z, w)^2$, $\mathfrak{q} = (z, w^2)$ are primary ideals. As above, $\mathfrak{p}$ yields an embedded point at the origin since $\sqrt{\mathfrak{p}} = (y, z, w)$. Let Z_2

be the curve defined by the total ideal $I_{Z_2} = \mathfrak{q}$. Then Z_2 is a complete intersection and hence CM. Also Z_2 is supported on Y with $\deg Z_2 = 2$. Therefore Z_2 is the 2^{nd} CM filtrant of Z and hence $Y = Z_1 \subset Z_2 \subset Z_3 \subset Z_4 = Z$ is the CM filtration of Z.

Proposition 4.2.5. Let Z be a CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$ with the CM filtration $Y = Z_1 \subset \cdots \subset Z_n = Z$. Set $\mathcal{I}_j := \mathcal{I}_{Z_j}$ for $1 \leq j \leq n$ and $\mathcal{L}_j := \mathcal{I}_j/\mathcal{I}_{j+1}$ for $1 \leq j \leq n-1$. Then $\mathcal{L}_j$ is a quotient sheaf on $Z_{j+1}, \forall j$.

(a) $\mathcal{I}_i\mathcal{I}_j \subseteq \mathcal{I}_{i+j}$ and each $\mathcal{L}_j$ can be considered as an $\mathcal{O}_{Z_i}$-module. In particular, for $i = 1$ we have $\mathcal{I}_Y\mathcal{I}_j \subseteq \mathcal{I}_{j+1}$ and each $\mathcal{L}_j$ can be considered as an $\mathcal{O}_Y$-module.

(b) Each $\mathcal{L}_j$ is torsion free as an $\mathcal{O}_Y$-module, hence a vector bundle on Y.

(c) The sequence

$$0 \to \mathcal{L}_j \to \mathcal{O}_{Z_{j+1}} \to \mathcal{O}_{Z_j} \to 0$$

is exact for all j.

(d) The multiplicity of Z is given by

$$\operatorname{mult}(Z) = 1 + \sum_{j=1}^{n-1} \operatorname{rank} \mathcal{L}_j.$$

(e) There exist natural maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$ for all $1 \leq i, j \leq n-1$.

Proof. (a) Set $\Gamma_{ij} := \Gamma_i \cup \Gamma_j$ and $\mathcal{K}_{ij} := (\mathcal{I}_i\mathcal{I}_j + \mathcal{I}_{i+j})/\mathcal{I}_{i+j}$. Apart from Γ_{ij} we have

$$\mathcal{I}_i\mathcal{I}_j = (\mathcal{I}_Z + \mathcal{I}_Y^i)(\mathcal{I}_Z + \mathcal{I}_Y^j) \subseteq \mathcal{I}_Z + \mathcal{I}_Y^{i+j}.$$

So the statement holds in $Y \setminus \Gamma_{ij}$ and $\operatorname{Supp} \mathcal{K}_{ij} \subseteq \Gamma_{ij}$. Thus $\mathcal{K}_{ij}$ is an ideal sheaf in $\mathcal{O}_{Z_{i+j}}$, which is not supported at the generic points of Z_{j+1}. Since Z_{i+j} is a CM curve, we have $\mathcal{K}_{ij} = 0$ by Proposition 3.3.6. Therefore $\mathcal{I}_i \mathcal{I}_j \subseteq \mathcal{I}_{i+j}$. Since $\mathcal{I}_{i+j} \subseteq \mathcal{I}_{j+1}$, each $\mathcal{L}_j$ is annihilated by $\mathcal{I}_i$ and hence can be considered as an $\mathcal{O}_{Z_i}$-module.

(b) Let $\mathcal{F}_j$ be the torsion subsheaf of $\mathcal{L}_j$ on Y. I.e., $\mathcal{F}_j$ is the sheaf associated to the presheaf

$$U \mapsto \operatorname{Tor} \mathcal{L}_j(U),$$

where U is an open subset of Y and $\operatorname{Tor} \mathcal{L}_j(U)$ is the torsion submodule of $\mathcal{L}_j(U)$. Then $\operatorname{Supp} \mathcal{F}_j$ is a closed subset of Y. Let η be the generic point of Y. Then $\mathcal{L}_{j,\eta}$ is a finitely generated module over $\mathcal{O}_{Y,\eta}$. Since Y is integral, $\mathcal{O}_{Y,\eta}$ is a field. Hence $\mathcal{L}_{j,\eta}$ is a finite dimensional vector space over the field $\mathcal{O}_{Y,\eta}$. Thus $\mathcal{F}_{j,\eta} = (\operatorname{Tor} \mathcal{L}_j)_\eta = \operatorname{Tor} \mathcal{L}_{j,\eta} = 0$. Hence $\mathcal{F}_j$ is supported on a proper closed subset of Y. Hence $\mathcal{F}_j$ is an ideal sheaf in $\mathcal{O}_{Z_{j+1}}$, which is not supported at the generic points of Z_{j+1}. Since Z_{j+1} is CM, we have $\mathcal{F}_j = 0$ by Proposition 3.3.6. Therefore each $\mathcal{L}_j$ is torsion free on Y.

Let $P \in Y$ be a closed point. Then $\mathcal{O}_{Y,P}$ is a DVR and hence a PID, since Y is nonsingular. Hence $\mathcal{L}_{j,P}$ is a finitely generated module over a PID. Now every finitely generated module over a PID is a direct sum of its torsion submodule and a free submodule [21, Theorem 3.10]. Therefore $\mathcal{L}_{j,P} = \mathcal{G}_{j,P} \oplus \mathcal{F}_{j,P}$, where $\mathcal{G}_{j,P}$ is a free $\mathcal{O}_{Y,P}$-module and $\mathcal{F}_{j,P}$ is the stalk at P of the torsion subsheaf $\mathcal{F}_j$ of $\mathcal{L}_j$. But $\mathcal{F}_j = 0$ from the previous paragraph and hence $\mathcal{L}_{j,P}$ is a free $\mathcal{O}_{Y,P}$-module. Therefore each $\mathcal{L}_j$ is locally free on Y and hence a vector bundle on Y, since there exists a one-to-one correspondence between locally free sheaves and vector bundles on a scheme [18, II, Exercise 5.18].

(c) We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{j+1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{Z_{j+1}} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\| & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_j & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{Z_j} & \longrightarrow & 0. \\ & & \big\downarrow & & & & & & \\ & & \mathcal{L}_j & & & & & & \end{array} \tag{41}$$

Applying the snake lemma to (41) we get the exact sequence

$$0 \to \mathcal{L}_j \to \mathcal{O}_{Z_{j+1}} \to \mathcal{O}_{Z_j} \to 0. \tag{42}$$

(d) Twisting by n and taking the Euler characteristics of the sheaves in (42) we get

$$\chi\mathcal{O}_{Z_{j+1}}(n) = \chi\mathcal{O}_{Z_j}(n) + \chi\mathcal{L}_j(n),$$

and hence

$$\chi\mathcal{O}_Z(n) = \chi\mathcal{O}_Y(n) + \sum_{j=1}^{n-1} \chi\mathcal{L}_j(n). \tag{43}$$

By Lemma 3.2.2, we have

$$\chi\mathcal{L}_j(n) = n(\operatorname{rank}\mathcal{L}_j)\deg Y + c_j, \tag{44}$$

where $c_j \in k$ is some constant. Combining (43) and (44) we get

$$\chi\mathcal{O}_Z(n) = \chi\mathcal{O}_Y(n) + n(\sum_{j=1}^{n-1} \operatorname{rank}\mathcal{L}_j)\deg Y + \sum_{j=1}^{n-1} c_j.$$

Therefore

$$n \deg Z + 1 - p_a(Z) = n(1 + \sum_{j=1}^{n-1} \operatorname{rank} \mathcal{L}_j) \deg Y + 1 - p_a(Y) + \sum_{j=1}^{n-1} c_j. \tag{45}$$

Equating the coefficients of n in (45) we see that

$$\deg Z = (1 + \sum_{j=1}^{n-1} \operatorname{rank} \mathcal{L}_j) \deg Y,$$

and hence $\operatorname{mult}(Z) = 1 + \sum_{j=1}^{n-1} \operatorname{rank} \mathcal{L}_j$.

(e) Let U be an open affine subset of Y. Set $I_i := \mathcal{I}_i(U)$. Then $\mathcal{L}_i(U) = I_i/I_{i+1}$. We define the maps $\phi_{i,j} : \mathcal{L}_i(U) \times \mathcal{L}_j(U) \to I_{i+j}/I_{i+j+1}$ by

$$(a + I_{i+1}, b + I_{j+1}) \mapsto ab + I_{i+j+1},$$

where $a \in I_i, b \in I_j$. The map is well defined. To show this, suppose $(a+I_{i+1}, b+I_{j+1}) = (a' + I_{i+1}, b' + I_{j+1})$. Then $a - a' \in I_{i+1}, b - b' \in I_{j+1}$. Now $a(b-b') \in I_i I_{j+1} \subset I_{i+j+1}$ and $(a-a)b' \in I_{i+1} I_j \subset I_{i+j+1}$ by part (a). Therefore $ab - a'b' = a(b-b') + (a-a')b' \in I_{i+j+1}$ and the map is well-defined. Again by part (a), we have the inclusion maps $I_i I_j \subset I_{i+j}$ and hence the inclusion maps $\tau_{i,j} : I_i I_j / I_{i+j+1} \hookrightarrow I_{i+j}/I_{i+j+1}$. Let $\psi_{i,j} = \tau_{i,j} \circ \phi_{i,j}$. Then $\psi_{i,j} : \mathcal{L}_i(U) \times \mathcal{L}_j(U) \to \mathcal{L}_{i+j}(U)$ are bilinear maps and hence factor through the tensor products $\mathcal{L}_i(U) \otimes \mathcal{L}_j(U)$. Therefore we get the maps $\mathcal{L}_i(U) \otimes \mathcal{L}_j(U) \to \mathcal{L}_{i+j}(U)$ given by $(a + I_{i+1}) \otimes (b + I_{j+1}) \mapsto ab + I_{i+j+1}$, where $a \in I_i$ and $b \in I_j$. Gluing these maps we get the maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$. □

Remark 4.2.6. From Proposition 4.2.5 (d), we see that if Z is a CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$, then $\operatorname{mult}(Z)$ is a positive integer.

Notation 4.2.7. Let Y, Z and $\mathcal{L}_j$ be as in Proposition 4.2.5. Set $\mathcal{L} := \mathcal{L}_1$ and $\mathcal{L}^j := \mathcal{L}^{\otimes j}$, where $\mathcal{L}^{\otimes j}$ denotes the j^{th} tensor power of $\mathcal{L}$ as an $\mathcal{O}_Y$-module.

Corollary 4.2.8. Let Z be a primitive extension of a nonsingular connected curve Y. Let $\mathcal{L}_j$ be the vector bundles on Y as in Proposition 4.2.5.

(a) Each $\mathcal{L}_j$ is a line bundle on Y.

(b) Let $\mathcal{L}^j$ be as in (4.2.7) and let Z_2 be the 2^{nd} CM filtrant of Z. Then

$$\mathcal{L}^j \cong \mathcal{L}_j \cong \mathcal{I}_Y^j / \mathcal{I}_{Z_2} \mathcal{I}_Y^{j-1}.$$

Proof. (a) Let $P \in Y$ be a closed point and let Z_j be the j^{th} CM filtrant of Z. By Corollary 4.1.10, there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that $\mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z_j|U} = (x, y^j)$. Hence $\mathcal{L}_j(U)$ is generated by a single element, namely $\bar{y}^j$, where $\bar{y}^j$ is the image of y^j in $\mathcal{O}_{Z_{j+1}}$. Therefore $\mathcal{L}_j$ is a line bundle on Y.

(b) Let $\psi_j(U) : \mathcal{L}^j(U) \to \mathcal{L}_j(U)$ be the map given by $\bar{y}^{\otimes j} \mapsto \bar{y}^j$. Notice $\psi_j(U)$ is surjective. Gluing these maps we get a map of line bundles $\psi_j : \mathcal{L}^j \to \mathcal{L}_j$. At the stalk at P we get the map $\psi_{j,P} : \mathcal{L}^j_P \to \mathcal{L}_{j,P}$. Notice $\psi_{j,P}$ is surjective, since U is a neighborhood of P and $\psi_j(U)$ is surjective. Since $\mathcal{L}_{j,P}$ and $\mathcal{L}_P$ are line bundles on Y, $\mathcal{L}^j_P \cong \mathcal{O}_{Y,P}$ and $\mathcal{L}_{j,P} \cong \mathcal{O}_{Y,P}$. Hence $\psi_{j,P}$ takes the form $\mathcal{O}_{Y,P} \xrightarrow{\cdot b} \mathcal{O}_{Y,P}$ for some $b \in \mathcal{O}_{Y,P}$. Since $\psi_{j,P}$ is surjective, b is a unit in $\mathcal{O}_{Y,P}$. Therefore $\psi_{j,P}$ is an isomorphism. Since $P \in Y$ is arbitrary, ψ_j is an isomorphism and therefore $\mathcal{L}^j \cong \mathcal{L}_j$.

Let $\mathcal{E}_j = \mathcal{I}_Y^j/\mathcal{I}_{Z_2}\mathcal{I}_Y^{j-1}$. Let P and U be as above. Then $\mathcal{I}_{Y|U}^j$ is generated by $x^{j-l}y^l$ and y^j, where $0 \leq l \leq j-1$. Notice $x^{j-l}y^l \in \mathcal{I}_{Z_2|U}\mathcal{I}_{Y|U}^{j-1}$ but $y^j \notin \mathcal{I}_{Z_2|U}\mathcal{I}_{Y|U}^{j-1}$. Therefore $\mathcal{E}_j(U)$ is generated by the class of y^j. Let $\phi_j(U) : \mathcal{L}_j(U) \to \mathcal{E}_j(U)$ be the map given by $\bar{y}^j \mapsto y^j + \mathcal{I}_{Z_2|U}\mathcal{I}_{Y|U}^{j-1}$. Notice $\phi_j(U)$ is surjective. Glueing these maps we get a map $\phi_j : \mathcal{L}_j \to \mathcal{E}_j$. At the stalk at P we get the map $\phi_{j,P} : \mathcal{L}_{j,P} \to \mathcal{E}_{j,P}$. Notice $\phi_{j,P}$ is surjective, since U is a neighborhood of P and $\phi_j(U)$ is surjective. Since $\mathcal{L}_{j,P}$ is a line bundle on Y, $\phi_{j,P}$ takes the form $\mathcal{O}_{Y,P} \xrightarrow{\cdot c} \mathcal{E}_{j,P}$ for some $c \in \mathcal{O}_{Y,P}$. Notice $c \neq 0$, since $\phi_{j,P}$ is surjective. Therefore c is not a zerodivisor, since $\mathcal{O}_{Y,P}$ is an integral domain. Thus $\phi_{j,P}$ is injective and hence an isomorphism for all closed points $P \in Y$. Therefore ϕ_j is an isomorphism and $\mathcal{L}_j \cong \mathcal{E}_j$. Thus $\mathcal{L}^j \cong \mathcal{L}_j \cong \mathcal{E}_j$. □

4.3 Quasi-primitive and thick extensions

Let Y, Z and $\mathcal{L}_j$ be as in Proposition 4.2.5. Let $\mathcal{L}^j$ be as in (4.2.7). Then $\mathcal{L} = \mathcal{I}_Y/\mathcal{I}_{Z_2}$. Since $\mathcal{I}_Y^2 \subseteq \mathcal{I}_{Z_2}$, we always have the surjection $\nu_Y \twoheadrightarrow \mathcal{L}$, where $\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2$ is the conormal bundle of Y. Thus $\operatorname{rank}\mathcal{L} \leq \operatorname{rank}\nu_Y = 2$. If $\operatorname{rank}\mathcal{L} = 0$ then $\mathcal{L} = 0$, hence $\mathcal{I}_Y = \mathcal{I}_Z$, i.e., $Y = Z$. Therefore for nontrivial extensions we must have $1 \leq \operatorname{rank}\mathcal{L} \leq 2$. Notice if $\operatorname{rank}\mathcal{L} = 2$ then $\nu_Y \cong \mathcal{L}$, i.e., $\mathcal{I}_Y^2 = \mathcal{I}_{Z_2}$ and hence $Y^{(2)} \subset Z$.

Definition 4.3.1. Let Z be a CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Let $\mathcal{L}$ be the vector bundle on Y as above. Then Z is a *quasi-primitive extension* of Y if $\operatorname{rank}\mathcal{L} = 1$. On the other hand, if $\operatorname{rank}\mathcal{L} = 2$, i.e., if $Y^{(2)} \subset Z$ then Z is a *thick extension* of Y.

Corollary 4.3.2. Let Z be a CM double structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Then $\mathcal{I}_Y/\mathcal{I}_Z$ is a line bundle on Y and Z is a primitive extension of Y.

Proof. Let $\mathcal{L} = \mathcal{I}_Y/\mathcal{I}_Z$. Notice $Y \subset Z$ is the CM filtration of Z. Hence by Proposition 4.2.5, $\mathcal{L}$ is a vector bundle on Y with $\operatorname{rank} \mathcal{L} = \operatorname{mult}(Z) - 1 = 2 - 1 = 1$, i.e., $\mathcal{L}$ is a line bundle on Y. By Proposition 4.2.5 (c), we get the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0.$$

Let $P \in Y$ be a closed point. Then at the stalk at P we have the exact sequence

$$0 \to \mathcal{L}_P \to \mathcal{O}_{Z,P} \to \mathcal{O}_{Y,P} \to 0$$

and hence the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_P\mathcal{L}_P & \longrightarrow & \mathfrak{m}_{Z,P} & \longrightarrow & \mathfrak{m}_{Y,P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathfrak{m}_P\mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P & \longrightarrow & \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2 & \longrightarrow & \mathfrak{m}_{Y,P}/\mathfrak{m}_{Y,P}^2 & \longrightarrow & 0. \end{array}$$

Now $\dim \mathfrak{m}_{Y,P}/\mathfrak{m}_{Y,P}^2 = 1$, since Y is nonsingular. Also $\dim \mathfrak{m}_P\mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P = 1$, since $\mathcal{L}$ is a line bundle on Y. Therefore $\dim \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2 \leq 2$ and hence Z is primitive. $\square$

In the following proposition we give a criterion for a multiplicity structure to be a quasi-primitive extension based on its generic embedding dimension.

Proposition 4.3.3. Let Z be a nontrivial CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Let Γ be as in (4.2.2). Then Z is a quasi-primitive extension of Y if and only if $Z \setminus \Gamma$ is a primitive extension of $Y \setminus \Gamma$, i.e., $\operatorname{embdim}_P Z = 2$ for all closed points $P \in Y \setminus \Gamma$, i.e., Z has generic embedding dimension 2.

Proof. Let Z be a nontrivial quasi-primitive extension of Y. Let Z_2 be the 2^{nd} CM filtrant of Z with the ideal sheaf $\mathcal{I}_{Z_2}$. Then $\mathcal{I}_Y^2 \subsetneq \mathcal{I}_{Z_2}$, since $\operatorname{rank} \mathcal{L} = 1$. Therefore Z_2 is a CM double structure on Y. Since Y is nonsingular, Z_2 is a primitive extension of Y by Corollary 4.3.2. Therefore by Proposition 4.1.8, given a closed point $P \in Y$ there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$ such that the ideal (x) defines a nonsingular surface $F \subset U$ with $\mathcal{I}_F = (x), \mathcal{I}_{Y|U} = (x, y), \mathcal{I}_{Z_2|U} = (x, y^2)$, where $x, y \in \mathcal{O}_U$. Now if $P \notin \Gamma$ then $\mathcal{I}_{Z_2|U} = \mathcal{I}_{Z|U} + \mathcal{I}_{Y|U}^2$. Since $x \in \mathcal{I}_{Z_2|U}$ but $x \notin \mathcal{I}_{Y|U}^2$, we must have $x \in \mathcal{I}_{Z|U}$, i.e., $\mathcal{I}_F \subset \mathcal{I}_{Z|U}$. Therefore we have the surjection $\mathcal{O}_F \twoheadrightarrow \mathcal{O}_{Z|U}$ and hence the surjections $\mathcal{O}_{F,P} \twoheadrightarrow \mathcal{O}_{Z,P}$, $\mathfrak{m}_{F,P} \twoheadrightarrow \mathfrak{m}_{Z,P}$ and finally

$$\mathfrak{m}_{F,P}/\mathfrak{m}_{F,P}^2 \twoheadrightarrow \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2,$$

where $\mathfrak{m}_{F,P}$ and $\mathfrak{m}_{Z,P}$ are the maximal ideals in $\mathcal{O}_{F,P}$ and $\mathcal{O}_{Z,P}$ respectively. Notice $\dim \mathfrak{m}_{F,P}/\mathfrak{m}_{F,P}^2 = 2$, since F is nonsingular at P. Thus $\dim \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2 \leq 2$. Notice $\dim \mathfrak{m}_{Z_2,P}/\mathfrak{m}_{Z_2,P}^2 = 2$, since Z_2 is a double structure on Y. Therefore $\dim \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2 \geq 2$ and hence $\dim \mathfrak{m}_{Z,P}/\mathfrak{m}_{Z,P}^2 = \operatorname{embdim}_P Z = 2$ for all $P \in Y \setminus \Gamma$, i.e., $Z \setminus \Gamma$ is a primitive extension of $Y \setminus \Gamma$.

Conversely, let $Z \setminus \Gamma$ be a primitive extension of $Y \setminus \Gamma$. Let $P \in Y \setminus \Gamma$ be a closed point. Then by Proposition 4.1.8, there exist an open affine neighborhood U of P and $x, y \in \mathcal{O}_U$

such that $\mathcal{I}_{Y|U} = (x, y)$ and $\mathcal{I}_{Z|U} = (x, y^n)$, where $n = \text{mult}(Z)$. Let Z_2 be the 2^{nd} CM filtrant of Z. Then $\mathcal{I}_{Z_2|U} = (x, y^2)$ by Corollary 4.1.10. Therefore $\operatorname{rank} \mathcal{L}|_U = 1$, i.e., $\operatorname{rank} \mathcal{L} = 1$, hence Z is quasi-primitive. □

Corollary 4.3.4. Let Z be a CM multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$. Then Z is a thick extension of $Y \Leftrightarrow \text{embdim}_P Z = 3$ for all closed points $P \in Y$.

Proof. Z is a thick extension of Y if and only if Z is not a quasi-primitive extension of Y if and only if $\text{embdim}_P Z > 2$, i.e., $\text{embdim}_P Z = 3$ for all closed points $P \in Y$. □

Proposition 4.3.5. Let Z be a quasi-primitive extension of a nonsingular connected curve $Y \subset \mathbb{P}^3$. Let $\mathcal{L}_j$ be the vector bundles on Y as in Proposition 4.2.5. Set $\mathcal{L} := \mathcal{L}_1$ and $\mathcal{L}^j := \mathcal{L}^{\otimes j}$, where $\mathcal{L}^{\otimes j}$ denotes the j^{th} tensor power of $\mathcal{L}$ as an $\mathcal{O}_Y$-module.

(a) The maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$ defined in Proposition 4.2.5 (e) are surjective on $Y \setminus \Gamma$.

(b) Each $\mathcal{L}_j$ is a line bundle on Y, and hence each map $\mathcal{L}^j \to \mathcal{L}_j$ is injective.

(c) There exist effective Cartier divisors D_j on Y such that $\mathcal{L}_j = \mathcal{L}^j(D_j)$ with $D_1 = 0$ and $D_i + D_j \leq D_{i+j}$.

Proof. (a) By Proposition 4.3.3, $Z \setminus \Gamma$ is primitive extension of $Y \setminus \Gamma$. Hence $\mathcal{L}^j \cong \mathcal{L}_j$ on $Y \setminus \Gamma$ by Corollary 4.2.8. Therefore $\mathcal{L}_i \otimes \mathcal{L}_j \cong \mathcal{L}^i \otimes \mathcal{L}^j = \mathcal{L}_{i+j} \cong \mathcal{L}_{i+j}$ on $Y \setminus \Gamma$ and hence the maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$ are surjective on $Y \setminus \Gamma$.

(b) $\mathcal{L}$ is a line bundle on Y by definition of quasi-primitive extension. Hence each $\mathcal{L}^j$ is a line bundle on Y. The maps $\mathcal{L}^j \to \mathcal{L}_j$ are generically surjective by part (a). Therefore each $\mathcal{L}_j$ is a line bundle on Y. Let $P \in Y$ be a closed point. Then at the stalk at P the map $\mathcal{L}^j \to \mathcal{L}_j$ takes the form $\mathcal{O}_{Y,P} \xrightarrow{\cdot b} \mathcal{O}_{Y,P}$, where b is some nonzero element of $\mathcal{O}_{Y,P}$.

Notice $\mathrm{Ker}(\cdot b) = 0$, since $b \neq 0$ and $\mathcal{O}_{Y,P}$ is a regular local ring, hence an integral domain. Thus the map $\mathcal{L}^j_P \to \mathcal{L}_{j,P}$ is injective for all $P \in Y$. Hence the map $\mathcal{L}^j \to \mathcal{L}_j$ is injective.

(c) Let $\mathcal{F}_j$ be the cokernel of the map $\mathcal{L}^j \to \mathcal{L}_j$. Then we have the exact sequence

$$0 \to \mathcal{L}^j \to \mathcal{L}_j \to \mathcal{F}_j \to 0. \tag{46}$$

Tensoring (46) with $\mathcal{L}_j^{-1}$ we get the exact sequence

$$0 \to \mathcal{L}^j \otimes \mathcal{L}_j^{-1} \to \mathcal{O}_Y \to \mathcal{F}_j \otimes \mathcal{L}_j^{-1} \to 0. \tag{47}$$

The sequence (47) is exact on the left since Y is nonsingular and $\mathcal{L}_j^{-1}$ is a line bundle on Y. Notice $\mathcal{L}^j \otimes \mathcal{L}_j^{-1}$ is an ideal sheaf in $\mathcal{O}_Y$. Let D_j be the subscheme of Y defined by the ideal sheaf $\mathcal{L}^j \otimes \mathcal{L}_j^{-1}$. Notice $\mathrm{Supp}\, D_j = \mathrm{Supp}\, \mathcal{F}_j \otimes \mathcal{L}_j^{-1}$ and hence $D_j \subset \Gamma$, i.e., D_j is supported on a finite subset of Y. Since Y is nonsingular, $\mathcal{O}_{Y,P}$ is a regular local ring for all closed point $P \in Y$. Therefore $O_{Y,P}$ is a DVR and hence a PID. Hence every closed subscheme of Y is locally principal, i.e., an effective Cartier divisor. Therefore D_j is an effective Cariter divisor on Y for all j. Since $\mathcal{I}_{D_j} = \mathcal{O}_Y(-D_j)$, we have $\mathcal{L}^j \otimes \mathcal{L}_j^{-1} \cong \mathcal{O}_Y(-D_j)$ and hence $\mathcal{L}^j \otimes \mathcal{L}_j^{-1}(D_j) \cong \mathcal{O}_Y$, i.e., $\mathcal{L}_j \cong \mathcal{L}^j(D_j)$. Notice $D_1 = 0$, since $\mathcal{L}_1 = \mathcal{L} = \mathcal{L}^1$.

The maps $\mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_{i+j}$ are surjective on $Y \setminus \Gamma$ by part (a). By the same token, these maps are injective and the cokernels have finite support which yield effective Cartier divisors E_{ij} on Y. Hence $\mathcal{L}_{i+j} \cong \mathcal{L}_i \otimes \mathcal{L}_j(E_{ij})$, and therefore by the paragraph above we have $\mathcal{L}^{i+j}(D_{i+j}) \cong \mathcal{L}^{i+j}(D_i + D_j + E_{ij})$. Thus $D_{i+j} = D_i + D_j + E_{ij}$ and hence $D_i + D_j \leq D_{i+j}$, since $E_{ij} \geq 0$. □

Remark 4.3.6. For $i \leq j$ we have $D_i \leq D_j$, since $D_i = D_i + (j-i)D_1 \leq D_j$ and $D_1 = 0$.

Definition 4.3.7. Let $Y \subset Z \subset \mathbb{P}^3$ be a quasi-primitive extension and let $\mathcal{L}_j$ be the line bundles on Y, where $j = 1, \cdots, n-1$. By Proposition 4.3.5 there exist effective Cartier divisors D_j on Y such that $\mathcal{L}_j \cong \mathcal{L}^j(D_j)$, where $\mathcal{L} = \mathcal{L}_1$ and $D_1 = 0$. Set $d_i := \deg D_i$. We call $(\mathcal{L}, d_2, d_3, \cdots, d_{n-1})$ the *type* of the extension.

Example 4.3.8. Let $Y \subset \mathbb{P}^3$ be the line with total ideal $I_Y = (x, y)$. Let Z and W be curves in $\mathbb{P}^3$ with total ideals $I_Z = (x, y^2)$ and $I_W = (x^2, xy, y^3, y^2z - w^2x)$. Then W is a quasi-primitive triple structure on Y of type $(\mathcal{O}_Y(-1), 2)$, having Z as the 2$^{\text{nd}}$ CM filtrant. See [29, Proposition 2.1] or [30, Example 2.17] for details.

4.4 Construction of Cohen-Macaulay double structures

In this section we describe the construction of CM double structures on nonsingular connected curves in $\mathbb{P}^3$.

Theorem 4.4.1 (Ferrand)**.** Let $Y \subset \mathbb{P}^3$ be a l.c.i. curve and $\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2$ be its conormal bundle. Let $\mathcal{L}$ be a line bundle on Y and $\beta : \nu_Y \twoheadrightarrow \mathcal{L}$ be a surjection. Then $\operatorname{Ker}\beta = \mathcal{I}_Z/\mathcal{I}_Y^2$ for a CM double structure Z on Y. Moreover, if Z is given by some other line bundle $\mathcal{L}'$ on Y and some surjection $\beta' : \nu_Y \twoheadrightarrow \mathcal{L}'$, then there exists an isomorphism $\phi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\beta' = \phi \circ \beta$.

Proof. [12, Proposition 2]. □

Remark 4.4.2. The converse of Proposition 4.4.1 is false in general. For example, let $I_Y = (x^6, y^6)$ and $I_Z = (x^8, y^9)$. Then Y is a complete intersection and Z is a double structure on Y. But Z doesn't arise from Ferrand's construction, since $\mathcal{I}_Y^2 \not\subset \mathcal{I}_Z$. This example was given by Manaresi [23]. In that paper she proved that if Y is a l.c.i. codimension 2 analytic subspace of a complex manifold with $\operatorname{embdim} Y \leq \dim Y + 1$, then every l.c.i. double structure on Y arises by Ferrand's construction. Bănică and Forster [3, § 1] stated without proof that every CM double structure on a nonsingular connected curve in complex three manifold can be obtained by this construction.

Next we give an independent proof of Theorem 4.4.1 for nonsingular connected curves in $\mathbb{P}^3$. We also prove that its converse holds in this situation.

Theorem 4.4.3. Let $Y \subset \mathbb{P}^3$ be a nonsingular connected curve and let $\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2$ be its conormal bundle. Then the set of CM double structures on Y are in one-to-one correspondence with the set of pairs $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a line bundle on Y and $\beta : \nu_Y \twoheadrightarrow \mathcal{L}$ is a surjection, modulo the equivalence relation: $(\mathcal{L}, \beta) \sim (\mathcal{L}', \beta')$ if there exists an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ such that $\beta' = \phi \circ \beta$.

Proof. Let Z be a CM double structure on Y. Set $\mathcal{L} := \mathcal{I}_Y/\mathcal{I}_Z$. Then $\mathcal{L}$ is a line bundle on Y by Corollary 4.3.2. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_Y^2 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_Y^2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Y^2 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \nu_Y & \longrightarrow & 0. \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{L} & & & &
\end{array} \tag{48}$$

Applying the snake lemma to (48) we get the exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Y^2 \to \nu_Y \to \mathcal{L} \to 0. \tag{49}$$

Let β be the surjection in (49). Then $\operatorname{Ker}\beta = \mathcal{I}_Z/\mathcal{I}_Y^2$ and hence the pair $(\mathcal{L}, \beta)$ defines the CM double structure Z on Y. If Z is given by some other pair $(\mathcal{L}', \beta')$, where $\mathcal{L}'$ is a line bundle on Y and $\beta' : \nu_Y \twoheadrightarrow \mathcal{L}'$ is a surjection, then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_Y^2 & \longrightarrow & \nu_Y & \xrightarrow{\beta} & \mathcal{L} & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_Y^2 & \longrightarrow & \nu_Y & \xrightarrow{\beta'} & \mathcal{L}' & \longrightarrow & 0. \end{array} \tag{50}$$

Applying the snake lemma to (50) we see that $\mathcal{L} \cong \mathcal{L}'$. Hence there exists an isomorphism $\phi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\beta' = \phi \circ \beta$, i.e., $(\mathcal{L}, \beta) \sim (\mathcal{L}', \beta')$.

Conversely, let $\mathcal{L}$ be a line bundle on Y and $\beta : \nu_Y \twoheadrightarrow \mathcal{L}$ be a surjection. Then $\operatorname{Ker}\beta$ has the form $\mathcal{I}/\mathcal{I}_Y^2$, where $\mathcal{I}$ is an ideal sheaf in $\mathcal{O}_{\mathbb{P}^3}$. Let Z be the closed subscheme defined by the ideal sheaf $\mathcal{I}$. Then $\mathcal{I}_Z = \mathcal{I}$ and we have the exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_Y \to \mathcal{L} \to 0. \tag{51}$$

Therefore Z is a CM multiplicity structure on Y by Lemma 3.3.5. From (51) we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0. \\ & & \downarrow & & & & & & \\ & & \mathcal{L} & & & & & & \end{array} \tag{52}$$

Applying the snake lemma to (52) we get the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0. \tag{53}$$

Twisting by n and taking the Euler characteristics of the sheaves in (53) we get

$$\chi\mathcal{O}_Z(n) = \chi\mathcal{O}_Y(n) + \chi\mathcal{L}(n). \tag{54}$$

Now $\chi\mathcal{O}_Z(n) = n\deg Z + 1 - p_a(Z), \chi\mathcal{O}_Y(n) = n\deg Y + 1 - p_a(Y)$ and by Lemma 3.2.1, $\chi\mathcal{L}(n) = n\deg Y + c$, where $c \in k$ is some constant. Hence from (54) we get

$$n\deg Z + 1 - p_a(Z) = 2n\deg Y + c + 1 - p_a(Y). \tag{55}$$

Equating the coefficients of n in (55) we get $\deg Z = 2\deg Y$. Therefore Z is a CM double structure on Y induced by the pair $(\mathcal{L}, \beta)$. Finally, let Z' be a double structure on Y induced by some pair $(\mathcal{L}', \beta') \sim (\mathcal{L}, \beta)$. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_Y^2 & \longrightarrow & \nu_Y & \xrightarrow{\beta} & \mathcal{L} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow \phi & & \\ 0 & \longrightarrow & \mathcal{I}_{Z'}/\mathcal{I}_Y^2 & \longrightarrow & \nu_Y & \xrightarrow{\beta'} & \mathcal{L}' & \longrightarrow & 0, \end{array} \tag{56}$$

where $\phi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ is an isomorphism such that $\beta' = \phi \circ \beta$. Therefore $\operatorname{Ker} \beta = \operatorname{Ker} \beta'$. Thus $\mathcal{I}_Z/\mathcal{I}_Y^2 = \mathcal{I}_{Z'}/\mathcal{I}_Y^2$ and hence $\mathcal{I}_Z = \mathcal{I}_{Z'}$, i.e., $Z = Z'$. □

4.5 Surfaces containing quasi-primitive extensions

In this section we describe the singularities and class groups of general surfaces containing quasi-primitive extensions of nonsingular connected curves in $\mathbb{P}^3$.

Lemma 4.5.1. Let F be a surface containing a nonsingular connected curve $Y \subset \mathbb{P}^3$. Then $\operatorname{Sing} F \supseteq Y$ if and only if $\mathcal{I}_F \subset \mathcal{I}_Y^2$.

Proof. Let $\operatorname{Sing} F \supseteq Y$. Then $\operatorname{embdim}_P(F) = 3$ for all closed points $P \in Y$. Suppose on the contrary that $\mathcal{I}_F \not\subset \mathcal{I}_Y^2$. Let $W \subset \mathbb{P}^3$ be the closed subscheme defined by the ideal sheaf $\mathcal{I}_W = \mathcal{I}_F + \mathcal{I}_Y^2$. Then W is a curve supported on Y. Throwing away the embedded points of W we get a well-defined CM multiplicity structure Z on Y. Notice $Z \subset Y^{(2)}$ and hence $Y \subset Z$ is the CM filtration of Z. Let $\mathcal{L} = \mathcal{I}_Y/\mathcal{I}_Z$. Then $\mathcal{L}$ is a vector bundle on Y by Proposition 4.2.5 (b). We have the surjection $\nu_Y \twoheadrightarrow \mathcal{L}$, where $\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2$ is the conormal bundle of Y. Now if Z is a thick extension then $\nu_Y \cong \mathcal{L}$ and hence $\mathcal{I}_Y^2 = \mathcal{I}_Z$. But then $\mathcal{I}_Y^2 = \mathcal{I}_Z \supset \mathcal{I}_F + \mathcal{I}_Y^2 \supset \mathcal{I}_Y^2$, i.e., $\mathcal{I}_F \subset \mathcal{I}_Y^2$, which is a contradiction. Therefore Z is a quasi-primitive extension. Thus $\operatorname{rank} \mathcal{L} = 1$ and hence $\operatorname{mult}(Z) = 2$ by

Proposition 4.2.5 (d). Therefore Z is a primitive extension of Y by Proposition 4.3.2. Hence $\operatorname{embdim}_P(Z) = 2$ for all closed points $P \in Y$. Let Γ be the set of embedded points thrown away in the process of CM filtration of Z. Let $Q \in Y \setminus \Gamma$ be a closed point. Then $\mathcal{I}_{Z,Q} = \mathcal{I}_{F,Q} + \mathcal{I}^2_{Y,Q}$. Let $\mathfrak{m}_{\mathbb{P}^3,Q}$ be the maximal ideal in $\mathcal{O}_{\mathbb{P}^3,Q}$. Since $\operatorname{embdim}_Q(F) = 3$, we have $\mathcal{I}_{F,Q} \subset \mathfrak{m}^2_{\mathbb{P}^3,Q}$. On the other hand, $\mathcal{I}^2_{Y,Q} \subset \mathfrak{m}^2_{\mathbb{P}^3,Q}$. Hence $\mathcal{I}_{Z,Q} \subset \mathfrak{m}^2_{\mathbb{P}^3,Q}$, i.e., $\operatorname{embdim}_Q(Z) = 3$, which is a contradiction. Therefore $\mathcal{I}_F \subset \mathcal{I}^2_Y$.

Conversely, let $\mathcal{I}_F \subset \mathcal{I}^2_Y$. Let $P \in Y$ be a closed point. Then we have the exact sequence

$$0 \to \mathcal{I}_{F,P} \to \mathfrak{m}_{\mathbb{P}^3,P} \to \mathfrak{m}_{F,P} \to 0$$

and hence the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{F,P} & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P} & \longrightarrow & \mathfrak{m}_{F,P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_P & \longrightarrow & \mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} & \xrightarrow{\phi_P} & \mathfrak{m}_{F,P}/\mathfrak{m}^2_{F,P} & \longrightarrow & 0, \end{array} \tag{57}$$

where $K_P = \operatorname{Ker} \phi_P$. Since $\mathcal{I}_F \subset \mathcal{I}^2_Y$, from (57) we get $\mathcal{I}_{F,P} \subset \mathcal{I}^2_{Y,P} \subset \mathfrak{m}^2_{\mathbb{P}^3,P}$. Therefore $K_P = 0$, i.e., $\mathfrak{m}_{\mathbb{P}^3,P}/\mathfrak{m}^2_{\mathbb{P}^3,P} \cong \mathfrak{m}_{F,P}/\mathfrak{m}^2_{F,P}$, hence $\dim \mathfrak{m}_{F,P}/\mathfrak{m}^2_{F,P} = 3$. Thus F is singular at every closed point $P \in Y$. Hence $\operatorname{Sing} F \supseteq Y$. □

Lemma 4.5.2. Let $Z \subset \mathbb{P}^3$ be a curve such that $\mathcal{I}_Z(d-1)$ is generated by global sections. Let $\delta \subset |H^0\mathcal{O}_{\mathbb{P}^3}(d)|$ be the incomplete linear system corresponding to the vector space $H^0\mathcal{I}_Z(d)$. Then δ separates points and tangent vectors of X, where $X = \mathbb{P}^3 \setminus Z$.

Proof. Let $P \in X$ be a closed point. Then $\mathcal{O}_{Z,P} = 0$. Hence from the exact sequence

$$0 \to \mathcal{I}_{Z,P} \to \mathcal{O}_{\mathbb{P}^3,P} \to \mathcal{O}_{Z,P} \to 0$$

we get $\mathcal{I}_{Z,P} \cong \mathcal{O}_{\mathbb{P}^3,P}$ and therefore $\mathcal{I}_{Z,P}(d-1) \cong \mathcal{O}_{\mathbb{P}^3,P}(d-1)$. Hence for each closed point $P \in X$ there exists an element $s \in H^0\mathcal{I}_Z(d-1)$ such that $s_P \mapsto 1 \in \mathcal{O}_{\mathbb{P}^3,P}(d-1)$, since $\mathcal{I}_Z(d-1)$ is generated by global sections. Let σ be the complete linear system corresponding to the vector space $H^0\mathcal{O}_{\mathbb{P}^3}(1)$. Since $\mathcal{O}_{\mathbb{P}^3}(1)$ is very ample, σ separates points and tangent vectors of $\mathbb{P}^3$. Notice $st \in H^0\mathcal{I}_Z(d)$ for all $s \in H^0\mathcal{I}_Z(d-1)$ and $t \in H^0\mathcal{O}_{\mathbb{P}^3}(1)$, since $\mathcal{I}_Z(d) \cong \mathcal{I}_Z(d-1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)$.

Let $Q \in X$ be a closed point distinct from P. Since σ separates points of $\mathbb{P}^3$, there exists $t \in H^0\mathcal{O}_{\mathbb{P}^3}(1)$ such that $t_P \in \mathfrak{m}_P$ but $t_Q \notin \mathfrak{m}_Q$. Let $s \in H^0\mathcal{I}_Z(d-1)$ such that $s_Q \mapsto 1$, where 1 is the generator of $\mathcal{O}_{\mathbb{P}^3,Q}(d-1)$. Then $st \in H^0\mathcal{I}_Z(d)$ and $(st)_P \in \mathfrak{m}_P$ but $(st)_Q \notin \mathfrak{m}_Q$. Hence δ separates points of X.

Since σ separates tangent vectors of $\mathbb{P}^3$, the set $\{t \in H^0\mathcal{O}_{\mathbb{P}^3}(1) | t_P \in \mathfrak{m}_P\}$ spans the vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$. Let $s \in H^0\mathcal{I}_Z(d-1)$ be such that $s_P \mapsto 1 \in \mathcal{O}_{\mathbb{P}^3,P}(d-1)$. Then $st \in H^0\mathcal{I}_Z(d)$. Moreover, $(st)_P = s_P t_P \in \mathfrak{m}_P \Leftrightarrow t_P \in \mathfrak{m}_P$, since s_P is a unit in $\mathcal{O}_{\mathbb{P}^3,P}$. Also for the same reason the sets $\{t \in H^0\mathcal{O}_{\mathbb{P}^3}(1) | t_P \in \mathfrak{m}_P\}$ and $\{st \in H^0\mathcal{I}_Z(d) | (st)_P \in \mathfrak{m}_P\}$ span the same vector space, i.e., $\mathfrak{m}_P/\mathfrak{m}_P^2$. Therefore δ separates tangent vectors of X. □

Corollary 4.5.3. Let $Z \subset \mathbb{P}^3$ be a curve such that $\mathcal{I}_Z(d-1)$ is generated by global sections. Let $\delta \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be the incomplete linear system corresponding to the vector subspace $V = H^0\mathcal{I}_Z(d)$. If $F \in \delta$ is general, then $\operatorname{Sing} F \subseteq \operatorname{Supp} Z$.

Proof. Notice Z is the base locus of δ since $\mathcal{I}_Z(d)$ is generated by global sections. Let

$X = \mathbb{P}^3 \setminus Z, Y = \mathbb{P}V$ and $\varphi : X \to Y$ be the map corresponding to δ. Let $x \in X$ and $y = \varphi(x) \in Y$. At the level of stalks we have the ring homomorphism $\varphi_y^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Let $\mathfrak{m}_{X,x}$ and $\mathfrak{m}_{Y,y}$ be the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively. Since $\mathcal{I}_Z(d-1)$ is generated by global sections, δ separates points and tangent vectors of X by Lemma 4.5.2. Thus $\varphi_y^{\#}(\mathfrak{m}_{Y,y})$ generates the Zariski cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ of X, and hence generates $\mathfrak{m}_{X,x}$ by Nakayama's lemma [2, Proposition 2.8]. Therefore $\mathfrak{m}_{Y,y} \cdot \mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$. Finally, $k(x)$ is a separable algebraic extension of $k(y)$, since $k(x) \cong k(y) \cong k$. Therefore φ is unramified. Let $F \in \delta$ be general. Then F is nonsingular on X by Bertini theorem [22, Proposition 6.3 (2)], since X is nonsingular and φ is unramified. Therefore $\operatorname{Sing} F \subseteq \operatorname{Supp} Z$, since $X = \mathbb{P}^3 \setminus Z$. □

Proposition 4.5.4. Let Z be a quasi-primitive multiplicity structure on a nonsingular connected curve $Y \subset \mathbb{P}^3$ such that $\mathcal{I}_Z(d-1)$ is generated by global sections. Let F be a Zariski general surface of degree d containing Z. Then

(a) $\operatorname{Sing} F \subsetneq Y$ is finite and F is normal.

(b) If $Z' \subset F$ is a multiplicity structure on Y with $\operatorname{mult} Z' = \operatorname{mult} Z$, then $Z' = Z$.

(c) If $\operatorname{char} k = 0$ and F is very general in the linear system $|\mathcal{I}_Z(d)|$, then $\operatorname{Cl} F$ is freely generated by Y and $\mathcal{O}_F(1)$.

Proof. (a) Let $\delta \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be the linear system corresponding to the vector subspace $V = H^0\mathcal{I}_Z(d)$. Let $F \in \delta$ be general. Since $\mathcal{I}_Z(d-1)$ is generated by global sections, $\operatorname{Sing} F \subseteq \operatorname{Supp} Z = Y$ by Corollary 4.5.3. Hence $\operatorname{Sing} F$ is a closed subscheme of Y. If $\operatorname{Sing} F \neq Y$ then $\operatorname{Sing} F$ is a proper closed subset of Y, i.e., is a finite set of points. Hence

F is regular in codimension 1. Therefore F is normal by [18, II, Proposition 8.23].

Now suppose $\operatorname{Sing} F = Y$. Let $W = H^0\mathcal{I}_Y^2(d) \cap H^0\mathcal{I}_Z(d)$ and let δ' be the linear system corresponding to W. Since Z is a quasi-primitive extension, $Z \setminus \Gamma$ is a primitive extension of $Y \setminus \Gamma$ by Proposition 4.3.3. Let $P \in Y \setminus \Gamma$ be a closed point. Then there exist an open affine neighborhood U of P and a nonsingular surface $E \subset U$ such that $Z \cap U \subset E$ by Proposition 4.1.8. If necessary, we can replace E by EE', where E' is some surface not vanishing along Y, so that $\mathcal{I}_E \subset \mathcal{I}_Z(d)$. Since E is nonsingular along $Y \cap U$, $\mathcal{I}_E \not\subset \mathcal{I}_Y^2(d)$ by Lemma 4.5.1. Thus $E \in \delta \setminus \delta'$, i.e., $\delta' \neq \delta$. Let $F' \in \delta \setminus \delta'$ be Zariski general. Then $\mathcal{I}_{F'} \not\subset \mathcal{I}_Y^2$ and hence $\operatorname{Sing} F' \not\supseteq Y$ by Lemma 4.5.1. Therefore $\operatorname{Sing} F' \subsetneq Y$ by Bertini theorem [22, Theoreme 6.3 (2)]. Replacing F' by F we get a c.i. which is regular in codimension 1. Therefore F is normal by [18, II, Proposition 8.23].

(b) Let $\operatorname{mult}(Z) = n$ and let $Z' \subset F$ be a multiplicity structure on Y with $\operatorname{mult} Z' = n$. Let $P \in Y \setminus (\operatorname{Sing} F \cup \Gamma)$. Then F is nonsingular at P and hence there exists a regular system of parameters $\{x, y, z\}$ in $\mathcal{O}_{\mathbb{P}^3,P}$ such that $\mathcal{I}_{F,P} = (x)$ and $\mathcal{I}_{Y,P} = (x, y)$. Notice, $Z \setminus (\operatorname{Sing} F \cap \Gamma)$ and $Z' \setminus (\operatorname{Sing} F \cap \Gamma)$ are primitive extensions of $Y \setminus (\operatorname{Sing} F \cap \Gamma)$. Therefore by Proposition 4.1.8, there exists an open affine neighborhood U of P such that $\mathcal{I}_{Z|U} = (x, y^n) = \mathcal{I}_{Z'|U}$. Thus $Z \cap U = Z' \cap U$ and hence $Z = Z'$ by Corollary 3.3.8.

(c) [6, Theorem 1.1]. □

5 Double Conics in $\mathbb{P}^3$

Let $\mathbb{P}^3 = \operatorname{Proj} S$, where $S = k[x, y, z, w]$ and k is an algebraically closed field. In this chapter we describe all CM double structures on conics in $\mathbb{P}^3$. In Section 5.1 we show that each conic in $\mathbb{P}^3$ has a canonical form after a change of coordinates. In Section 5.2 we describe the classification of double conics. In Section 5.3 we describe the invariants of double conics, namely their total ideals, Rao modules and minimal free resolutions of their total ideals. In Section 5.4 we give criteria for two double conics of the same support to be linked by complete intersection. In particular, we give a criterion for double conics to be self-linked. Finally in Section 5.5 we describe singular loci and class groups of general surfaces containing double conics.

5.1 Conics in $\mathbb{P}^3$

In this section we show that every conic in $\mathbb{P}^3$ is, after a suitable change of coordinates, a nondegenerate plane section of the quadric cone in $\mathbb{P}^3$.

Definition 5.1.1. A conic in $\mathbb{P}^3$ is a degree 2 integral curve.

Proposition 5.1.2. Every conic in $\mathbb{P}^3$ is planar.

Proof. Let $C \subset \mathbb{P}^3$ be conic and $P \in C$ be a closed point. Let $F \subset \mathbb{P}^3$ be a plane that intersects C transversely at P. Since $\deg C = 2$, F intersects C at exactly one other point, say Q, with multiplicity one by Bézout's theorem [18, I, Theorem 7.7]. Therefore $C \cap F = \{P, Q\}$. Let $R \in C \setminus \{P, Q\}$. Now if $\overline{PQ} = \overline{QR}$ then we must have $R \in C \cap F$, which is a contradiction. Therefore $\overline{PQ} \neq \overline{QR}$. Let H be the plane spanned by $\overline{PQ}$ and

$\overline{QR}$. Now if $C \not\subset H$ then $C \cap H$ must consist of 2 points, counting with multiplicities, by Proposition 3.1.3. But this contradicts the fact that $C \cap H$ contains at least three distinct points, namely P, Q, R. Therefore $C \subset H$, i.e., C is planar. □

Proposition 5.1.3. Let $g \in k[y, z]$ be irreducible of degree 2. Then up to a change of coordinate $g = y^2 - z$ or $yz - 1$.

Proof. Let $g = ay^2 + byz + cz^2 + dy + ez + f$, where $a, b, c, d, e, f \in k$. Denote the homogeneous quadratic part of g by G, i.e., $G = ay^2 + byz + cz^2$. Notice $G \neq 0$, since $\deg g = 2$. First we show that G factors into linear terms. If $a = 0$ then $G = z(by + cz)$ and we are done. Now suppose $a \neq 0$. Then we can write $G = Qz^2$, where $Q = au^2+bu+c$ and $u = y/z$. Since $Q \in k[u]$ and k is algebraically closed, we must have $Q = ll'$, where $l, l' \in k[u]$ are linear polynomials. Therefore $G = (lz)(l'z)$, i.e., G factors into linear forms. Let $G = LL'$, where $L, L' \in k[y, z]$ are linear forms. Then $g = LL' + dy + ez + f$. Suppose L and L' are independent. Then we can make a change of coordinates by mapping $y \mapsto L$ and $z \mapsto L'$. Let's denote L and L' by Y and Z respectively. Then we have $g = YZ + DY + EZ + f = (Y + E)(Z + D) - (DE - f)$ for some $D, E \in k$. Notice $DE - f \neq 0$, since g is irreducible. Taking $Y' = (DE - f)(Y + E)$ and $Z' = Z + D$ we see that g takes the form $Y'Z' - 1$ up to scalar.

Now suppose L and L' are dependent. Let $L = \alpha y + \beta z$ and $L' = \mu L$, where $\alpha, \beta, \mu \in k$ and $\mu \neq 0$. Then $g = \mu L^2 + dy + ez + f$. Taking $Y = \sqrt{\mu}L$ and $Z = -dy - ez - f$ we see that g takes the form $Y^2 - Z$. □

Proposition 5.1.4. Let $C \subset \mathbb{P}^3$ be conic. Then up to a change of coordinate $I_C = (x, q)$, where $q = yz - w^2$.

Proof. By Proposition 5.1.2, $C \subset H$, where H is some plane in $\mathbb{P}^3$. By a change of coordinate we may assume that $I_H = (x)$, i.e., $H \cong \mathbb{P}^2 = \operatorname{Proj} k[y, z, w]$. Let $U \subset \mathbb{P}^2$ be the open affine $\operatorname{Spec} k[y, z]$. Then $C \cap U$ is given by an irreducible polynomial $g \in k[y, z]$ of degree 2. By Proposition 5.1.3, $g = y^2 - z$ or $yz - 1$. Homogenizing g we get $y^2 - zw$ or $yz - w^2$. Interchanging the variables y and w we see that $y^2 - zw$ becomes $-(yz - w^2)$. Therefore up to a change of coordinate we have $I_C = (x, q)$, where $q = yz - w^2$. □

Let $\mathbb{P}^1 = \operatorname{Proj} T$, where $T = k[s, t]$. Let $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the composition of the 2-uple embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ and the inclusion $\mathbb{P}^2 \subset \mathbb{P}^3$ as a plane.

Proposition 5.1.5. The image of the closed immersion i is a conic in $\mathbb{P}^3$. Conversely, every conic in $\mathbb{P}^3$ arises in this way up to automorphisms of $\mathbb{P}^3$.

Proof. Let $\mathbb{P}^2 = \operatorname{Proj} k[y, z, w]$ and let $\rho_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ be the 2-uple embedding given by $(s, t) \mapsto (s^2, t^2, st)$. Let υ be the inclusion of $\mathbb{P}^2$ into $\mathbb{P}^3$ as the plane $\{x = 0\}$. Let $i = \upsilon \circ \rho_2$ and let $\theta : S \to T$ be the map of graded rings corresponding to i, i.e., $\theta : S \to T$ is given by $x \mapsto 0, y \mapsto s^2, z \mapsto t^2$ and $w \mapsto st$. Then $(x, q) \subseteq \operatorname{Ker} \theta$, where $q = yz - w^2$. Notice $S/\operatorname{Ker} \theta \cong k[s^2, st, t^2]$ is an integral domain. Hence $\operatorname{Ker} \theta$ is a prime ideal in S. Since $\dim S = 4$ and $\dim \operatorname{Im} \theta = \dim k[s^2, st, t^2] = 2$, we have $\operatorname{ht} \operatorname{Ker} \theta = 2$. On the other hand (x, q) is a height 2 prime ideal in S. Therefore $\operatorname{Ker} \theta = (x, q)$. By Proposition 5.1.4, $\operatorname{Ker} \theta$ is the total ideal of some conic $C \subset \mathbb{P}^3$. Therefore $\operatorname{Im}(i)$ is a conic in $\mathbb{P}^3$.

Conversely, let $C \subset \mathbb{P}^3$ be a conic. By Proposition 5.1.4, up to an automorphism of $\mathbb{P}^3$ the total ideal C has the form $I_C = (x, q)$, where $q = yz - w^2$. Let $\theta : S \to T$ be the map given by $x \mapsto 0, y \mapsto s^2, z \mapsto t^2$ and $w \mapsto st$. Then $\operatorname{Ker}\theta = I_C$. Let $i : \mathbb{P}^1 \to \mathbb{P}^3$ be the map corresponding to θ. Then i is an embedding of C with the desired property, since i factors through $\mathbb{P}^2$ as $(s,t) \mapsto (s^2, t^2, st) \mapsto (0, s^2, t^2, st)$. □

Corollary 5.1.6. Let $C \subset \mathbb{P}^3$ be a conic with total ideal $I_C = (x, q)$, where $q = yz - w^2$, and let $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be an embedding of C.

(a) $C \cong \mathbb{P}^1$ and hence is nonsingular.

(b) $\operatorname{Pic} C = \langle i_* \mathcal{O}_{\mathbb{P}^1}(1) \rangle \cong \mathbb{Z}$.

(c) $i^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2)$.

(d) $S_C \cong T^e$, where $T^e = k[s^2, st, t^2] \subset T$ is the even subalgebra.

(e) $I_C / I_C^2 \cong S_C(-1) \oplus S_C(-2)$ and $\mathcal{I}_C / \mathcal{I}_C^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2)$.

Proof. Since $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is an embedding of C, we have $C \cong \mathbb{P}^1$. Hence C is nonsingular and $\operatorname{Pic} C \cong \operatorname{Pic} \mathbb{P}^1 = \langle \mathcal{O}_{\mathbb{P}^1}(1) \rangle = \mathbb{Z}$. Thus $\operatorname{Pic} C$ is generated by $i_* \mathcal{O}_{\mathbb{P}^1}(1)$. By Proposition 5.1.5, $i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is given by $(s,t) \mapsto (0, s^2, t^2, st)$. In other words, i is given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ of $\mathbb{P}^1$ and its sections $i^*x = 0, i^*y = s^2, i^*z = t^2$ and $i^*w = st$. Therefore $i^* \mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Let $\theta : S \to T$ be the morphism of rings corresponding to the embedding i of C. Then by Proposition 5.1.5, $\operatorname{Ker}\theta = I_C$ and hence $S_C \cong \operatorname{Im}\theta = T^e$, where T^e consists of the even degree pieces of T, i.e., $T^e = k[s^2, st, t^2]$. Finally, since C is a complete intersection with $I_C = (x, q)$,

$$0 \to S(-3) \xrightarrow{\begin{pmatrix} q \\ -x \end{pmatrix}} S(-1) \oplus S(-2) \xrightarrow{\begin{pmatrix} x & q \end{pmatrix}} I_C \to 0 \tag{58}$$

is a minimal S-resolution of I_C by Proposition 3.1.1. Tensoring (58) with S_C we get

$$I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2). \tag{59}$$

Sheafifying (59) we get $\mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2)$. □

Notation 5.1.7. For the rest of this exposition we fix a conic $C \subset \mathbb{P}^3$ with total ideal $I_C = (x, q)$, where $q = yz - w^2$. We denote the embedding of C by i and the corresponding map of graded rings by θ. Then θ defines an injective map $\bar{\theta} : S_C \to T$. We have $I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$ by Corollary 5.1.6 (e). Therefore $\bar{\theta}$ induces the inclusion $I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2) \hookrightarrow T(-2) \oplus T(-4)$. We denote this inclusion by j.

Notation 5.1.8. Let $\mathcal{L} \in \operatorname{Pic} C$. Then $\mathcal{L} = i_*\mathcal{O}_{\mathbb{P}^1}(\ell)$ for some $\ell \in \mathbb{Z}$, by Corollary 5.1.6 (b). We use the notations $\mathcal{O}_C[\ell]$ and $S_C[\ell]$ to denote $i_*\mathcal{O}_{\mathbb{P}^1}(\ell)$ and $H^0_* i_*\mathcal{O}_{\mathbb{P}^1}(\ell)$ respectively. If ℓ is even, say $\ell = 2a$, then by Corollary 5.1.6 (c), $\mathcal{O}_C[2a] = i_*\mathcal{O}_{\mathbb{P}^1}(2a) \cong i_*i^*\mathcal{O}_{\mathbb{P}^3}(a) = \mathcal{O}_C(a)$. Thus $S_C[2a] = S_C(a)$. If ℓ is odd, then $S_C[\ell] \cong T^o(\ell)$ as graded k-vector spaces, where T^o consists of the odd degree pieces of T.

Definition 5.1.9. Let $m, n, l \in \mathbb{Z}$. We define the sets $\mathcal{A}^l_{m,n}, \mathcal{B}^l_{m,n}, \mathcal{C}^l_{m,n}, \mathcal{D}^l_{m,n}$ as follows:

$$\mathcal{A}^l_{m,n} = \{\psi \in \operatorname{Hom}_{S_C}(S_C[m] \oplus S_C[n], S_C[l]) |\operatorname{Coker}\psi \text{ has finite length}\},$$

$$\mathcal{B}^l_{m,n} = \{\mu \in \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{O}_C[m] \oplus \mathcal{O}_C[n], \mathcal{O}_C[l]) | \mu \text{ is a surjection}\},$$

$$\mathcal{C}^l_{m,n} = \{\varepsilon \in \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(l)) | \varepsilon \text{ is a surjection}\},$$

$$\mathcal{D}^l_{m,n} = \{\tau \in \operatorname{Hom}_T(T(m) \oplus T(n), T(l)) |\operatorname{Coker}\tau \text{ has finite length}\}.$$

Lemma 5.1.10. Let $\mathcal{A}^l_{m,n}, \mathcal{B}^l_{m,n}, \mathcal{C}^l_{m,n}, \mathcal{D}^l_{m,n}$ be the sets defined in (5.1.9). Then $\mathcal{A}^l_{m,n} \cong \mathcal{B}^l_{m,n} \cong \mathcal{C}^l_{m,n} \cong \mathcal{D}^l_{m,n}$ as sets.

Proof. Let $\psi \in \mathcal{A}^l_{m,n}$, $N = \operatorname{Ker}\psi$ and $M = \operatorname{Coker}\psi$. Then we have the exact sequence

$$0 \to N \to S_C[m] \oplus S_C[n] \xrightarrow{\psi} S_C[l] \to M \to 0. \tag{60}$$

Since M has finite length, $\widetilde{M} = 0$ by Proposition 2.2.4. Hence sheafifying (60) we get the short exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_C[m] \oplus \mathcal{O}_C[n] \xrightarrow{\widetilde{\psi}} \mathcal{O}_C[l] \to 0, \tag{61}$$

where $\mathcal{N} = \widetilde{N}$. Therefore if $\psi \in \mathcal{A}^l_{m,n}$ then $\widetilde{\psi} \in \mathcal{B}^l_{m,n}$. Let $\widetilde{\ }: \mathcal{A}^l_{m,n} \to \mathcal{B}^l_{m,n}$ be the map given by $\psi \mapsto \widetilde{\psi}$. Now let $\mu \in \mathcal{B}^l_{m,n}$ and $\mathcal{N} = \operatorname{Ker}\mu$. Then we have the exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_C[m] \oplus \mathcal{O}_C[n] \xrightarrow{\mu} \mathcal{O}_C[l] \to 0. \tag{62}$$

Applying H^0_* to (62) we get the exact sequence

$$0 \to N \to S_C[m] \oplus S_C[n] \xrightarrow{H^0_*\mu} S_C[l] \to H^1_*\mathcal{N}, \tag{63}$$

where $N = H^0_*\mathcal{N}$. Sheafifying (63) we get the exact sequence (62). Therefore $\operatorname{Coker} H^0_*\mu$ must have finite length, since μ is a surjection. Hence if $\mu \in \mathcal{B}^l_{m,n}$ then $H^0_*\mu \in \mathcal{A}^l_{m,n}$. Let $H^0_* : \mathcal{B}^l_{m,n} \to \mathcal{A}^l_{m,n}$ be the map given by $\mu \mapsto H^0_*\mu$. By the functoriality of H^0_* and $\widetilde{\ }$ we have $H^0_* \widetilde{\ } = \mathrm{id}_{\mathcal{A}^l_{m,n}}$ and $\widetilde{H^0_*} = \mathrm{id}_{\mathcal{B}^l_{m,n}}$. Hence $\mathcal{A}^l_{m,n} \cong \mathcal{B}^l_{m,n}$. Similarly, $\mathcal{C}^l_{m,n} \cong \mathcal{D}^l_{m,n}$.

Finally, let $i_* : \mathcal{B}^l_{m,n} \to \mathcal{C}^l_{m,n}$ be given by $\mu \mapsto i_*\mu$ and let $i^* : \mathcal{C}^l_{m,n} \to \mathcal{B}^l_{m,n}$ be given by $\epsilon \mapsto i^*\epsilon$. Since $C \cong \mathbb{P}^1$ we have $i^*i_* = \mathrm{id}_{\mathcal{B}^l_{m,n}}$ and $i_*i^* = \mathrm{id}_{\mathcal{C}^l_{m,n}}$. Hence $\mathcal{B}^l_{m,n} \cong \mathcal{C}^l_{m,n}$. Therefore $\mathcal{A}^l_{m,n} \cong \mathcal{B}^l_{m,n} \cong \mathcal{C}^l_{m,n} \cong \mathcal{D}^l_{m,n}$. □

5.2 Classification of double conics

Let C be the conic as in (5.1.7). According to Theorem 4.4.3, giving a CM double structure on C is equivalent to giving a line bundle $\mathcal{L}$ on C and a surjection $\mathcal{I}_C \twoheadrightarrow \mathcal{L}$. Let Z be a CM double conic on C corresponding to $\mathcal{L}$. We have the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{L} \to 0. \tag{64}$$

Tensoring (64) with $\mathcal{O}_C$ we get the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{I}_C & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \pi & & \| & & \\ & & \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z & \longrightarrow & \mathcal{I}_C/\mathcal{I}_C^2 & \longrightarrow & \mathcal{L} & \longrightarrow & 0, \end{array} \tag{65}$$

where $\pi : \mathcal{I}_C \twoheadrightarrow \mathcal{I}_C/\mathcal{I}_C^2$ is the canonical surjection. Thus every surjection $\mathcal{I}_C \twoheadrightarrow \mathcal{L}$ factors through the conormal bundle $\mathcal{I}_C/\mathcal{I}_C^2$ of C. Therefore every CM double conic Z on C arises from a surjection $\mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{L}$. Since $\mathcal{L}$ is a line bundle on C, $\mathcal{L} = \mathcal{O}_C[\ell]$ for some $\ell \in \mathbb{Z}$ by (5.1.8). We call Z a CM double conic on C of type ℓ.

By Corollary 5.1.6 (e) we have

$$\begin{aligned}
\operatorname{Hom}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C[\ell]) &\cong \operatorname{Hom}(\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-2), \mathcal{O}_C[\ell]) \\
&\cong \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-4), \mathcal{O}_{\mathbb{P}^1}(\ell)) \\
&\cong \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(\ell+2))\oplus\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(\ell+4)) \\
&\cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell+2))\oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell+4)).
\end{aligned}$$

Let $\tau : T(-2)\oplus T(-4) \to T(\ell)$ be a map. Then $\tau = (f, g)$, where f and g are homogeneous polynomials in T with $\deg f = \ell+2$ and $\deg g = \ell+4$. By Lemma 2.2.4, τ sheafifies to a surjection if and only if $\operatorname{Coker}\tau$ has finite length. Also by Lemma 2.1.9, $\operatorname{Coker}\tau$ has finite length if and only if f and g have no common zeros. Therefore defining a surjection $\mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{O}_C[\ell]$ is equivalent to giving a map $\tau = (f, g)$, where f and g are homogeneous polynomials in T with $\deg f = \ell+2$ and $\deg g = \ell+4$, having no common zeros. Notice if $\ell < -4$ then τ is the zero map and hence cannot sheafify to a surjection. Also notice if $\ell = -3$ then $f = 0$ and g is linear. Thus every zeros of g is also a common zero of f and g. Hence $\ell = -3 \Rightarrow \operatorname{Coker}\tau$ has infinite length. Therefore to define a surjection $\mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{O}_C[\ell]$ we must have $\ell \geq -4$ and $\ell \neq -3$.

Theorem 5.2.1. Let $C \subset \mathbb{P}^3$ be a conic and let $\ell \geq -4$ be an integer such that $\ell \neq -3$. Then each surjection $\psi : \mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{O}_C[\ell]$ defines a CM double conic Z on C with Hilbert polynomial $P_Z(n) = 4n+\ell+2$ by $\mathcal{I}_Z = \operatorname{Ker}\psi\circ\pi$, where $\pi : \mathcal{I}_C \twoheadrightarrow \mathcal{I}_C/\mathcal{I}_C^2$ is the canonical surjection. Conversely, every CM double conic on C arises from this construction.

Proof. Let $\varphi : \mathcal{I}_C \twoheadrightarrow \mathcal{O}_C[\ell]$ be the surjection $\varphi = \psi \circ \pi$. Then by Proposition 4.4.1, $\operatorname{Ker}\psi = \mathcal{I}_Z/\mathcal{I}_Y^2$, i.e., $\operatorname{Ker}\varphi = \mathcal{I}_Z$ for some CM double structure Z on C. By Proposition 4.3.2, Z is a primitive extension of C. Thus we have the exact sequence

$$0 \to \mathcal{O}_C[\ell] \to \mathcal{O}_Z \to \mathcal{O}_C \to 0 \tag{66}$$

by Proposition 4.2.8 (4). Twisting by n and taking the Euler characteristics of the sheaves in (66) we get

$$P_Z(n) = \chi\mathcal{O}_Z(n) = \chi\mathcal{O}_C(n) + \chi\mathcal{O}_C[\ell](n) = \chi\mathcal{O}_C(n) + \chi\mathcal{O}_{\mathbb{P}^1}(2n+\ell) = 4n + \ell + 2.$$

Conversely, let Z be a CM double conic on C with Hilbert polynomial $P_Z(n) = 4n + \ell + 2$. Since C is nonsingular, by Theorem 4.4.3 there exists a line bundle $\mathcal{L}$ on C and a surjection $\psi : \mathcal{I}_C/\mathcal{I}_C^2 \twoheadrightarrow \mathcal{L}$ such that $\operatorname{Ker}\psi = \mathcal{I}_Z/\mathcal{I}_Y^2$. Hence $\mathcal{I}_Z = \operatorname{Ker}\psi \circ \pi$. Since $\mathcal{L} \in \operatorname{Pic} C$, there exists $\ell' \in \mathbb{Z}$ such that $\mathcal{L} = \mathcal{O}_C[\ell']$ by 5.1.8. By Proposition 4.3.2, Z is a primitive extension of C. Hence by Proposition 4.2.8 (4) we have the exact sequence

$$0 \to \mathcal{O}_C[\ell'] \to \mathcal{O}_Z \to \mathcal{O}_C \to 0. \tag{67}$$

Twisting by n and taking the Euler characteristics of the sheaves in (67) we see that $P_Z(n) = 4n + \ell' + 2$. Therefore $\ell = \ell'$. □

Remark 5.2.2. If Z is a double conic on C of type ℓ then $p_a(Z) = 1 - P_Z(0) = -1 - \ell$, since $P_Z(n) = 4n + \ell + 2$ by Theorem 5.2.1.

5.3 Invariants of double conics

In this section we compute the total ideals and Rao modules of double conics. We also compute minimal free resolutions of their total ideals.

Let $C \subset \mathbb{P}^3$ be a conic and let $\ell \geq -4$ be an integer such that $\ell \neq -3$. Let $f, g \in T$ be homogeneous polynomials with $\deg f = \ell+2$ and $\deg g = \ell+4$, having no common zeros. Let $\tau : T(-2) \oplus T(-4) \to T(\ell)$ be the map given by $\tau = (f, g)$. Then $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Let $\psi : S_C(-1) \oplus S_C(-2) \to S_C[\ell]$ be the map corresponding to τ as in Lemma 5.1.10. Define $\phi = \psi \circ \pi$, where $\pi : I_C \twoheadrightarrow I_C/I_C^2$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccccc} \operatorname{Ker}\phi & \hookrightarrow & I_C & \xrightarrow{\phi} & S_C[\ell] \\ \downarrow & & \downarrow \pi & & \| \\ \operatorname{Ker}\psi & \hookrightarrow & I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2) & \xrightarrow{\psi} & S_C[\ell] \\ \downarrow & & \downarrow j & & \downarrow \\ \operatorname{Ker}\tau & \hookrightarrow & T(-2) \oplus T(-4) & \xrightarrow{\tau} & T(\ell), \end{array} \tag{68}$$

where j is the inclusion $S_C(-1) \oplus S_C(-2) \hookrightarrow T(-2) \oplus T(-4)$ as in (5.1.7).

Theorem 5.3.1. *In the setting of Diagram (68), ϕ defines a CM double conic Z on C of type ℓ, with $I_Z = \operatorname{Ker}\phi = I_C^2 + (\pi \circ j)^{-1} \operatorname{Ker}\tau$ and $H^1_* \mathcal{I}_Z = \operatorname{Coker}\phi$.*

Proof. By construction, $\operatorname{Coker}\phi$ has finite length. Hence ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_C \twoheadrightarrow \mathcal{O}_C[\ell]$ by Lemma 2.2.4. Therefore $\operatorname{Ker}\widetilde{\phi}$ defines a CM double conic Z on C of type ℓ by Theorem 5.2.1. We have the exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{O}_C[\ell] \to 0. \tag{69}$$

Notice $H^1_*\mathcal{I}_C = 0$, since C is a complete intersection. Therefore taking the long exact cohomology sequence on (69) we have

$$0 \to H^0_*\mathcal{I}_Z \to H^0_*\mathcal{I}_C \xrightarrow{H^0_*\varphi} H^0_*\mathcal{O}_C[\ell] \to H^1_*\mathcal{I}_Z \to 0.$$

Since $H^0_*\mathcal{I}_C = I_C, H^0_*\mathcal{O}_C[\ell] = S_C[\ell]$ and $H^0_*\varphi = \phi$, we have $I_Z = H^0_*\mathcal{I}_Z = \operatorname{Ker}\phi$ and $H^1_*\mathcal{I}_Z = \operatorname{Coker}\phi$. Notice $(\operatorname{Ker}\phi)/I_C^2 \cong \operatorname{Ker}\psi \cong j^{-1}\operatorname{Ker}\tau$. Therefore $I_Z/I_C^2 = j^{-1}\operatorname{Ker}\tau$, i.e., $I_Z = I_C^2 + (\pi \circ j)^{-1}\operatorname{Ker}\tau$. □

5.3.A. Double conics of odd genus

In this subsection we describe the invariants of double conics on C of odd genus, i.e., of type $2a$, where $a \geq -2$. By Theorem 5.3.1, such a double conic arises from a map $\tau : T(-2) \oplus T(-4) \to T(2a)$ given by $\tau = (f, g)$, where f and g are homogeneous polynomials in T with $\deg f = 2a+2$ and $\deg g = 2a+4$, having no common zeros. Let F and G be homogeneous polynomials in S such that $\theta(F) = f$ and $\theta(G) = g$. Then $\deg F = a+1$ and $\deg G = a+2$. Let $\psi = (F, G)$. Notice F and G have no common zeros along C and hence $\operatorname{Coker}\psi$ has finite length by Lemma 2.1.9. Define $\phi = \psi \circ \pi$, where $\pi : I_C \twoheadrightarrow I_C/I_C^2$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccc} I_C & \xrightarrow{\phi} & S_C(a) \\ \downarrow{\scriptstyle\pi} & & \| \\ I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2) & \xrightarrow{\psi} & S_C(a). \end{array} \qquad (70)$$

Proposition 5.3.2. In the setting of Diagram (70), ϕ defines a CM double conic Z on C of type $2a$. Furthermore:

(a) $I_Z = (I_C^2, Fq - Gx)$.

(b) $H^1_* \mathcal{I}_Z \cong (S/(x, q, F, G))(a)$.

(c) If $\{F', G'\}$ defines another double conic Z' on C of type $2a$, then $Z' = Z$ if and only if $F' = \alpha F \bmod I_C$ and $G' = \alpha G \bmod I_C$ for some $\alpha \in k^*$.

Proof. By Theorem 5.3.1, ϕ defines a CM double conic Z on C of type $2a$. Moreover, $I_Z = I_C^2 + \pi^{-1} \operatorname{Ker} \psi$. Since F and G have no common zeros along C, $\operatorname{Ker} \psi$ is generated by the Koszul relation $\bar{F} e_2 - \bar{G} e_1$, where e_1, e_2 are the generators of $S_C(-1) \oplus S_C(-2)$ and $\bar{F}, \bar{G}$ are the images of F, G in S_C respectively. Since $I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$, we can identify $\bar{x}$ with e_1 and $\bar{q}$ with e_2, where $\bar{x}, \bar{q}$ are the images of x, q in I_C/I_C^2. Therefore $\operatorname{Ker} \psi$ is generated by $\overline{Fq - Gx}$. Hence $I_Z = (I_C^2, Fq - Gx)$. Also $H^1_* \mathcal{I}_Z = \operatorname{Coker} \phi$ by Theorem 5.3.1. Notice $\operatorname{Coker} \phi \cong \operatorname{Coker} \psi$. Since F and G have no common zeros along C, we have $\operatorname{Coker} \psi = (S/(x, q, F, G))(a)$. Therefore $H^1_* \mathcal{I}_Z \cong (S/(x, q, F, G))(a)$.

Finally, let Z' be another double conic on C of type $2a$ defined by the map $\psi' = (F', G')$. Then $Z' = Z \Leftrightarrow I_{Z'} = I_Z \Leftrightarrow I_{Z'}/I_C^2 = I_Z/I_C^2$. Notice I_Z/I_C^2 can be considered as a submodule of $S_C(-1) \oplus S_C(-2)$ via the inclusion $I_Z/I_C^2 \subset I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$. Since $I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$ and since I_C/I_C^2 is generated by $\bar{x}$ and $\bar{q}$, where $\bar{x}$ and $\bar{q}$ are the images of x and q in S/I_C^2, we can identify $\{\bar{x}, \bar{q}\}$ as a basis of $S_C(-1) \oplus S_C(-2)$. Therefore I_Z/I_C^2 is generated by the vector $(F, -G)$ as a submodule of $S_C(-1) \oplus S_C(-2)$. Similarly, $I_{Z'}/I_C^2$ is generated by the vector $(F', -G')$ as a submodule of $S_C(-1) \oplus S_C(-2)$. Hence

$I_{Z'}/I_C^2 = I_Z/I_C^2 \Leftrightarrow$ there exists an element $\alpha \in k^*$ such that $(F', -G') = \alpha(F, G) \bmod I_C$. Therefore $Z' = Z \Leftrightarrow F' = \alpha F \bmod I_C$ and $G' = \alpha G \bmod I_C$ for some $\alpha \in k^*$. □

Corollary 5.3.3. Let Z be a double conic on C of type -4. Then $I_Z = (x, q^2)$ and $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$.

Proof. In this case $\deg F = -1$ and $\deg G = 0$. Hence $F = 0$ and G is a unit. Therefore $I_Z = (x^2, xq, q^2, x) = (x, q^2)$ by Proposition 5.3.2, i.e., Z is a complete intersection. Hence

$$0 \to S(-5) \xrightarrow{\begin{pmatrix} -q^2 \\ x \end{pmatrix}} S(-1) \oplus S(-4) \xrightarrow{\begin{pmatrix} x & q^2 \end{pmatrix}} I_Z \to 0 \tag{71}$$

is a minimal S-resolution I_Z by Proposition 3.1.1. Tensoring (71) with S_C we get $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$. □

Corollary 5.3.4. Let Z be a double conic on C of type -2. Then $I_Z = (x^2, q - Gx)$, where $G \in S$ is some linear form. Moreover, $I_Z/I_C I_Z \cong S_C(-2)^2$.

Proof. In this case $\deg F = 0$ and $\deg G = 1$. So we may assume that $F = 1$. Hence by Proposition 5.3.2, we have $I_Z = (I_C^2, q - Gx) = (x^2, q - Gx)$, where $G \in S$ is some linear form. Therefore Z is a complete intersection and hence by Proposition 3.1.1

$$0 \to S(-4) \xrightarrow{\begin{pmatrix} -q + Gx \\ x^2 \end{pmatrix}} S(-2)^2 \xrightarrow{\begin{pmatrix} x^2 & q - Gx \end{pmatrix}} I_Z \to 0 \tag{72}$$

is a minimal S-resolution of I_Z. Tensoring (72) with S_C we get $I_Z/I_C I_Z \cong S_C(-2)^2$. □

Remark 5.3.5. Notice in Corollary 5.3.4, if $G \in I_C$ then $G = \beta x$ for some $\beta \in k^*$, since $\deg G = 1$, hence $I_Z = (x^2, q - \beta x^2) = (x^2, q)$.

Proposition 5.3.6. Let Z be a double conic on C of type $2a$, where $a \geq 0$, with total ideal $I_Z = (I_C^2, Fq - Gx)$. Then I_Z has minimal S-resolution

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} I_Z \to 0 \tag{73}$$

where $N_1 = S(-2) \oplus S(-3) \oplus S(-4) \oplus S(-a-3)$, $N_2 = S(-4) \oplus S(-5) \oplus S(-a-4) \oplus S(-a-5)$, $N_3 = S(-a-6)$ and φ_i's are S-module homomorphisms given by the matrices

$$\varphi_1 = \begin{pmatrix} x^2 & xq & q^2 & Fq - Gx \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} q & 0 & G & 0 \\ -x & q & -F & G \\ 0 & -x & 0 & -F \\ 0 & 0 & x & q \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} G \\ -F \\ -q \\ x \end{pmatrix}.$$

Moreover, $I_Z/I_C I_Z \cong S_C(-a-3) \oplus (F,G)^2(2a)$ and $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C(-a-3) \oplus \mathcal{O}_C(2a)$.

Proof. By an easy calculation we see that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_3$ are zero maps. Hence (73) is a complex. Now (73) is exact if and only if the complex

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \tag{74}$$

is exact, where $N_0 = S$. We use Buchsbaum-Eisenbud criterion 2.1.20 to prove that (74) is exact. Notice $\operatorname{rank}\varphi_3 = \operatorname{rank}\varphi_1 = 1$, since x and x^2 are nonzero elements in S. Now

$$\det\varphi_2 = \begin{vmatrix} q & 0 & G & 0 \\ -x & q & -F & G \\ 0 & -x & 0 & -F \\ 0 & 0 & x & q \end{vmatrix} = q\begin{vmatrix} q & -F & G \\ -x & 0 & -F \\ 0 & x & q \end{vmatrix} + x\begin{vmatrix} 0 & G & 0 \\ -x & 0 & -F \\ 0 & x & q \end{vmatrix},$$

i.e., $\det\varphi_2 = q\{qFx + x(-Fq - Gx)\} + x^2qG = -x^2qG + x^2qG = 0$, hence $\operatorname{rank}\varphi_2 \leq 3$. Let M_1 and M_2 be the 3×3 submatrices of φ_2 given by

$$M_1 = \begin{pmatrix} -x & q & -F \\ 0 & -x & 0 \\ 0 & 0 & x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} q & 0 & 0 \\ -x & q & G \\ 0 & 0 & q \end{pmatrix}. \tag{75}$$

Notice $\det M_1 = x^3 \neq 0$, hence $\operatorname{rank}\varphi_2 = 3$. Thus $\operatorname{rank} N_i = \operatorname{rank}\varphi_i + \operatorname{rank}\varphi_{i+1}$, where $i = 1, 2, 3$. It remains to show that $\operatorname{depth} I(\varphi_i) \geq i$ for $i = 1, 2, 3$. We have $x^2 \in I(\varphi_1)$, which is regular in S. Hence $\operatorname{depth} I(\varphi_1) \geq 1$. From (75) we see that $x^3, q^3 \in I(\varphi_2)$ since $\det M_1 = x^3$ and $\det M_2 = q^3$. Since $\{x, q\}$ is a regular sequence in S, $\{x^3, q^3\}$ is also a regular sequence in S by [26, Theorem 16.1]. Therefore $\operatorname{depth} I(\varphi_2) \geq 2$. Finally $x, q, F \in I(\varphi_3)$. Since $\{x, q\}$ is a regular sequence in S and $\bar{F}$ is regular in S_C, $\{x, q, F\}$ is a regular sequence in S. Hence $\operatorname{depth} I(\varphi_3) \geq 3$. Therefore (74) is exact by Buchsbaum-Eisenbud criterion 2.1.20. Thus (73) is exact and hence a minimal S-resolution of I_Z.

Tensoring (73) by S_C yields the exact sequence

$$N_2 \otimes S_C \xrightarrow{\varphi_2 \otimes S_C} N_1 \otimes S_C \to I_Z/I_C I_Z \to 0,$$

where $N_1 \otimes S_C = S_C(-2) \oplus S_C(-3) \oplus S_C(-4) \oplus S_C(-a-3)$, $N_2 \otimes S_C = S_C(-4) \oplus S_C(-5) \oplus S_C(-a-4) \oplus S_C(-a-5)$ and $\varphi_2 \otimes S_C$ is given by the matrix

$$\varphi_2 \otimes S_C = \begin{pmatrix} 0 & 0 & G & 0 \\ 0 & 0 & -F & G \\ 0 & 0 & 0 & -F \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $S_C(-4) \oplus S_C(-5) \subseteq \operatorname{Ker}(\varphi_2 \otimes S_C)$ and $\operatorname{Im}(\varphi_2 \otimes S_C) \subseteq S_C(-2) \oplus S_C(-3) \oplus S_C(-4)$. Let φ_2' be the restriction of $\varphi_2 \otimes S_C$ on $S_C(-a-4) \oplus S_C(-a-5)$. Then φ_2' is given by the matrix

$$\varphi_2' = \begin{pmatrix} G & 0 \\ -F & G \\ 0 & -F \end{pmatrix}$$

and we have the exact sequence

$$0 \to S_C(-a-4) \oplus S_C(-a-5) \xrightarrow{\varphi_2'} S_C(-2) \oplus S_C(-3) \oplus S_C(-4) \xrightarrow{\mu} S_C(2a).$$

By the Hilbert-Burch theorem 2.1.22, $\operatorname{Im}\mu$ is the twist of an ideal in S_C, generated by the 2×2 minors of φ_2'. Therefore $\operatorname{Coker}\varphi_2' \cong \operatorname{Im}\mu = (F^2, FG, G^2)(2a) = (F, G)^2(2a)$

and hence

$$I_Z/I_CI_Z \cong S_C(-a-3) \oplus \operatorname{Coker} \varphi_2' \cong S_C(-a-3) \oplus (F,G)^2(2a),$$

where $\overline{x}^2, \overline{xq}$ and $\overline{q}^2$ are identified with F^2, FG and G^2 respectively. Making this identification, we have the inclusion

$$\iota : I_Z/I_CI_Z \subset S_C(-a-3) \oplus S_C(2a).$$

Then $\operatorname{Coker} \iota \cong (S_C/(F,G)^2)(2a)$ and we have the short exact sequence

$$0 \to I_Z/I_CI_Z \xrightarrow{\iota} S_C(-a-3) \oplus S_C(2a) \to (S_C/(F,G)^2)(2a) \to 0. \tag{76}$$

By [26, Theorem 16.1], $\{F^2, G^2\}$ is also a regular sequence in S_C, since $\{F, G\}$ is a regular sequence in S_C. Hence $S_C/(F^2, G^2)$ has finite length by Lemma 2.1.9. Therefore $S_C/(F,G)^2$ has finite length, since $S_C/(F,G)^2$ is a quotient of $S_C/(F^2, G^2)$. Hence $S_C/(F,G)^2$ sheafifies to 0 by Lemma 2.2.4. Thus sheafifying (76) we get the isomorphism

$$\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C(-a-3) \oplus \mathcal{O}_C(2a).$$

Hence $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z$ is freely generated by $Fq - Gx$ and an element e of degree $-2a$ such that $eF^2 = \overline{x}^2, eFG = \overline{xq}$ and $eG^2 = \overline{q}^2$. □

Proposition 5.3.7. If Z is a double conic of type $2a$, where $a \geq 0$, then $h^1\mathcal{I}_C\mathcal{I}_Z(a+5) = 0$ and $h^1\mathcal{I}_C\mathcal{I}_Z(a+4) \leq 1$.

Proof. We have $I_C I_Z = (x^3, x^2q, xq^2, q^3, x(Fq-Gx), q(Fq-Gx))$ and hence the complex

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} I_C I_Z \to 0, \tag{77}$$

where

$$\begin{aligned} N_1 &= S(-3) \oplus S(-4) \oplus S(-5) \oplus S(-6) \oplus S(-a-4) \oplus S(-a-5), \\ N_2 &= S(-5) \oplus S(-6) \oplus S(-7) \oplus S(-a-5) \oplus S(-a-6)^2 \oplus S(-a-7), \\ N_3 &= S(-a-7) \oplus S(-a-8) \end{aligned}$$

and $\varphi_1, \varphi_2, \varphi_3$ are given by the matrices

$$\begin{aligned} \varphi_1 &= \begin{pmatrix} x^3 & x^2q & xq^2 & q^3 & x(Fq-Gx) & q(Fq-Gx) \end{pmatrix}, \\ \varphi_2 &= \begin{pmatrix} q & 0 & 0 & G & 0 & 0 & 0 \\ -x & q & 0 & -F & G & 0 & 0 \\ 0 & -x & q & 0 & -F & G & 0 \\ 0 & 0 & -x & 0 & 0 & -F & 0 \\ 0 & 0 & 0 & x & q & 0 & q \\ 0 & 0 & 0 & 0 & 0 & q & -x \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} -G & 0 \\ F & -G \\ 0 & F \\ q & 0 \\ -x & q \\ 0 & -x \\ 0 & -q \end{pmatrix}. \end{aligned}$$

Notice (77) is exact if and only if

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \to N_0 \to 0 \tag{78}$$

is exact, where $N_0 = S$. We use Buchsbaum-Eisenbud criterion 2.1.20 to prove that (74) is exact. Notice $\operatorname{rank} \varphi_1 = 1$ and $\operatorname{rank} \varphi_3 = 2$. By an easy calculation we can show that all the 6×6 minors of φ_2 are zero. Let M_1 and M_2 be the 5×5 submatrices of φ_2 given by

$$M_1 = \begin{pmatrix} -x & q & 0 & -F & 0 \\ 0 & -x & q & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & x & q \\ 0 & 0 & 0 & 0 & -x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ -x & q & 0 & G & 0 \\ 0 & -x & q & -F & G \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}. \tag{79}$$

Then $\det M_1 = x^5 \neq 0$ and hence $\operatorname{rank} \varphi_2 = 5$. Therefore $\operatorname{rank} N_i = \operatorname{rank} \varphi_1 + \operatorname{rank} \varphi_{i+1}$ for $i = 1, 2, 3$. Now $x^3 \in I(\varphi_1)$, which is regular in S. Hence $\operatorname{depth} I(\varphi_1) \geq 1$. From (79) we see that $x^5, q^5 \in I(\varphi_2)$, since $\det M_1 = x^5$ and $\det M_2 = q^5$. Since $\{x, q\}$ is a regular sequence in S, $\{x^5, q^5\}$ is also a regular sequence in S by [26, Theorem 16.1]. Therefore $\operatorname{depth} I(\varphi_2) \geq 2$. Finally $x^2, q^2, F^2 \in I(\varphi_3)$. Since $\{x, q\}$ is a regular sequence and F is regular in S_C, $\{x, q, F\}$ is a regular sequence in S. Therefore $\{x^2, q^2, F^2\}$ is also a regular sequence in S by [26, Theorem 16.1]. Hence $\operatorname{depth} I(\varphi_3) \geq 3$. Therefore (78) is exact by Buchsbaum-Eisenbud criterion 2.1.20. Thus (77) is exact, hence an S-resolution of $I_C I_Z$. Let E be the kernel of φ_1. Then we have the short exact sequences

$$0 \to E \to N_1 \to I_C I_Z \to 0 \tag{80}$$

and

$$0 \to N_3 \to N_2 \to E \to 0. \tag{81}$$

Sheafifying (80) and (81) we get the short exact sequences

$$0 \to \mathcal{E} \to \mathcal{N}_1 \to \mathcal{I}_C\mathcal{I}_Z \to 0 \tag{82}$$

and

$$0 \to \mathcal{N}_3 \to \mathcal{N}_2 \to \mathcal{E} \to 0. \tag{83}$$

From (82) we have the long exact cohomology sequence

$$\cdots \to H^1_*\mathcal{N}_1 \to H^1_*\mathcal{I}_C\mathcal{I}_Z \to H^2_*\mathcal{E} \to H^2_*\mathcal{N}_1 \cdots .$$

Since $\mathcal{N}_1$ is a direct sum of line bundles on $\mathbb{P}^3$, we have $H^1_*\mathcal{N}_1 = H^2_*\mathcal{N}_1 = 0$ and hence $H^1_*\mathcal{I}_C\mathcal{I}_Z \cong H^2_*\mathcal{E}$. Taking the long exact cohomology sequence on (83) we get

$$\cdots \to H^2_*\mathcal{N}_2 \to H^2_*\mathcal{E} \to H^3_*\mathcal{N}_3 \to \cdots .$$

Again since $\mathcal{N}_2$ is a direct sum of line bundles in $\mathbb{P}^3$, $H^2_*\mathcal{N}_2 = 0$ and so we have the inclusion $H^1_*\mathcal{I}_C\mathcal{I}_Z \cong H^2_*\mathcal{E} \hookrightarrow H^3_*\mathcal{N}_3$. Now

$$H^3\mathcal{N}_3(a+5) = H^3\mathcal{O}_{\mathbb{P}^3}(-2) \oplus H^3\mathcal{O}_{\mathbb{P}^3}(-3) \perp H^0\mathcal{O}_{\mathbb{P}^3}(-2) \oplus H^0\mathcal{O}_{\mathbb{P}^3}(-1) = 0$$

and

$$H^3\mathcal{N}_3(a+4) = H^3\mathcal{O}_{\mathbb{P}^3}(-3) \oplus H^3\mathcal{O}_{\mathbb{P}^3}(-4) \perp H^0\mathcal{O}_{\mathbb{P}^3}(-1) \oplus H^0\mathcal{O}_{\mathbb{P}^3} \cong k.$$

Therefore $h^1\mathcal{I}_C\mathcal{I}_Z(a+5) = 0$ and $h^1\mathcal{I}_C\mathcal{I}_Z(a+4) \leq h^3\mathcal{N}_3(a+4) = 1$. □

5.3.B. Double conics of even genus

In this subsection we describe the invariants of double conics on C of even genus, i.e., of type $\ell = 2a+1$, where $a \geq -1$. By Theorem 5.3.1, such a double conic arises from a map $\tau : T(-2) \oplus T(-4) \to T(2a+1)$ given by $\tau = (f, g)$, where f and g are homogeneous polynomials in T with $\deg f = 2a+3$ and $\deg g = 2a+5$, having no common zeros. But there do not exist $F, G \in S$ such that $\theta(F) = f$ and $\theta(G) = g$, since $\deg f$ and $\deg g$ are odd. To circumvent this issue we introduce the notion of admissible pair of sequences.

Definition 5.3.8. Let C be the conic as in (5.1.7). Let F_1, G_1, F_2, G_2 be homogeneous polynomials in S such that $F_1G_2 = F_2G_1 \bmod I_C$. Then $\{F_1, G_1\}$, $\{F_2, G_2\}$ is said to be an *admissible pair of sequences* on C if there exist two distinct points $P, Q \in C$ such that

1. $\widetilde{F_1} \cap \widetilde{G_1} = P$,
2. $\widetilde{F_2} \cap \widetilde{G_2} = Q$,
3. $\widetilde{F_1} + Q = \widetilde{F_2} + P$,
4. $\widetilde{G_1} + Q = \widetilde{G_2} + P$.

Here $\widetilde{F_i}, \widetilde{G_i}$ denote the effective divisors on C induced by F_i, G_i for $i = 1, 2$.

Definition 5.3.9. Let $P, Q \in C$ be distinct points and let $\mathcal{M}_{P,Q}$ be the set of equivalence classes of admissible pairs of sequences $\{F_1, G_1\}, \{F_2, G_2\}$ on C corresponding to P and Q, under the equivalence relation given by $\{F_1, G_1\}, \{F_2, G_2\} \sim \{\lambda F_1, \lambda G_1\}, \{\mu F_2, \mu G_2\}$, for all $\lambda, \mu \in k^*$. Let $\mathcal{N}$ be the set of equivalence classes of regular sequences $\{f, g\}$ in T with $\deg f$ and $\deg g$ odd, under the equivalence relation given by $\{f, g\} \sim \{\alpha f, \alpha g\}, \forall \alpha \in k^*$.

Proposition 5.3.10. Let $\mathcal{M}_{P,Q}$ and $\mathcal{N}$ be the sets as defined in (5.3.9). Then there exists a bijection between $\mathcal{M}_{P,Q}$ and $\mathcal{N}$.

Proof. Let $i : \mathbb{P}^1 \to \mathbb{P}^3$ be the embedding of C in $\mathbb{P}^3$ as in (5.1.7). Let $\hat{p} = i^*P$ and $\hat{q} = i^*Q$. Notice $\hat{p}$ and $\hat{q}$ are distinct points in $\mathbb{P}^1$, since P and Q are distinct. Let $\hat{p} = (a, b)$ and $\hat{q} = (c, d)$. Let $l_1 = bs - at$ and $l_2 = ds - ct$. Then $(l_1)_0 = \hat{p}$ and $(l_2)_0 = \hat{q}$, where $(l_i)_0$ denotes the effective divisor on $\mathbb{P}^1$ induced by l_i.

Let $\{F_1, G_1\}$, $\{F_2, G_2\}$ be an element of $\mathcal{M}_{P,Q}$. Let $\theta(F_1) = f_1$ and $\theta(G_1) = g_1$, where θ is the map as in (5.1.7). By definition 5.3.8, there exist $P_i, Q_j \in C$ and $a_i, b_j \in \mathbb{Z}_{\geq 0}$ such that $\widetilde{F_1} = \sum a_i P_i + P$ and $\widetilde{G_1} = \sum b_j Q_j + P$. Let $p_i = i^*P_i$ and $q_j = i^*Q_j$. Therefore $(f_1)_0 = \sum a_i p_i + \hat{p}$ and $(g_1)_0 = \sum b_j q_j + \hat{q}$, where $(f_1)_0$ and $(g_1)_0$ denote the effective divisors on $\mathbb{P}^1$ corresponding to f_1 and g_1. Notice $\sum a_i p_i$ and $\sum b_j q_j$ are effective divisors on $\mathbb{P}^1$. Hence there exist homogeneous polynomials $f, g \in T$ such that $(f)_0 = \sum a_i p_i$ and $(g)_0 = \sum b_j q_j$. Therefore

$$(f_1)_0 = (f)_0 + (l_1)_0 = (l_1 f)_0.$$

Hence $f_1 = \beta l_1 f$ for some $\beta \in k^*$. Similarly, $g_1 = \gamma l_1 g$ for some $\gamma \in k^*$. Since $\widetilde{F}_1 \cap \widetilde{G}_1 = P$, $(f_1)_0 \cap (g_1)_0 = \hat{p} = (l_1)_0$. Therefore $(f)_0 \cap (g)_0 = 0$, i.e., $Z(f, g) = \varnothing$ and hence $\{f, g\}$ is a regular sequence in T by Lemma 2.1.9. Notice $\{f, g\} \in \mathcal{N}$, since $\deg f$ and $\deg g$ are odd. Finally, if $\{F_1, G_1\}$, $\{F_2, G_2\} \sim \{\lambda F_1, \lambda G_1\}, \{\mu F_2, \mu G_2\}$ then up to the equivalence relation on $\mathcal{N}$, we get the same regular sequence $\{f, g\}$ in T.

Conversely, let $\{f, g\}$ be a regular sequence in T such that $\deg f$ and $\deg g$ are odd. Then $(l_1 f)_0$ and $(l_1 g)_0$ are effective divisors on $\mathbb{P}^1$ and hence on C. Since $\deg l_1 f$ and $\deg l_1 g$ are even, there exist homogeneous polynomials $F_1, G_1 \in S$ such that $\widetilde{F_1} = (l_1 f)_0$ and $\widetilde{G_1} = (l_1 g)_0$. Similarly, we can choose $F_2, G_2 \in S$ such that $\widetilde{F_2} = (l_2 f)_0$ and $\widetilde{G_2} = (l_2 g)_0$. Notice, $\theta(F_1) = \beta l_1 f$ and $\theta(G_1) = \gamma l_1 g$ for some $\beta, \gamma \in k^*$. Also notice, we can choose $F_2, G_2 \in S$ such that $\theta(F_2) = \beta l_2 f$ and $\theta(G_2) = \gamma l_2 g$. Therefore $\theta(F_1 G_2 - F_2 G_1) = 0$ and hence $F_1 G_2 = F_2 G_1 \bmod I_C$. Also $\widetilde{F_1} \cap \widetilde{G_1} = (l_1)_0 = P, \widetilde{F_2} \cap \widetilde{G_2} = (l_2)_0 = Q, \widetilde{F_1} + Q = \widetilde{F_2} + P$ and $\widetilde{G_1} + Q = \widetilde{G_2} + P$. Thus $\{F_1, G_1\}$, $\{F_2, G_2\}$ is an element of $\mathcal{M}_{P,Q}$. Finally, if $\{f, g\} \sim \{\alpha f, \alpha g\}$ then up to the equivalence relation on $\mathcal{M}_{P,Q}$, we get the same admissible pair of sequences $\{F_1, G_1\}$, $\{F_2, G_2\}$ on C. $\square$

Example 5.3.11. Let $F_1 = y, G_1 = zw, F_2 = w$ and $G_2 = z^2$. Then $\widetilde{F_1} \cap \widetilde{G_1} = P$ and $\widetilde{F_2} \cap \widetilde{G_2} = Q$, where $P = (0, 0, 1, 0)$ and $Q = (0, 1, 0, 0)$. Notice $\widetilde{F_1} = 2P$, since $I_{F_1 \cap C} = I_{F_1} + I_C = (y) + (x, yz - w^2) = (x, y, w^2)$. On the other hand, $\widetilde{F_2} = P + Q$. Thus $\widetilde{F_1} + Q = \widetilde{F_2} + P$. Similarly we can show that $\widetilde{G_1} = P + 3Q, \widetilde{G_2} = 4Q$, and hence $\widetilde{G_1} + Q = \widetilde{G_2} + P$. Therefore $\{F_1, G_1\}$, $\{F_2, G_2\}$ is an admissible pair of sequences on C and it yields the regular sequence $\{s, t^3\}$ in T.

Let $a \geq -1$ be an integer and let $\tau : T(-2) \oplus T(-4) \to T(2a + 1)$ be a map given by $\tau = (f, g)$, where $\{f, g\}$ is a regular sequence in T with $\deg f = 2a+3$ and $\deg g = 2a+5$. Let $\psi : S_C(-1) \oplus S_C(-2) \to S_C[2a + 1]$ be the map corresponding to τ as in Lemma 5.1.10. Define $\phi = \psi \circ \pi$, where $\pi : I_C \twoheadrightarrow I_C/I_C^2$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ker}\phi & \hookrightarrow & I_C & \xrightarrow{\phi} & S_C[2a+1] \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \pi & & \| \\
\operatorname{Ker}\psi & \hookrightarrow & I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2) & \xrightarrow{\psi} & S_C[2a+1] \\
\downarrow & & \downarrow j & & \downarrow \\
\operatorname{Ker}\tau & \hookrightarrow & T(-2) \oplus T(-4) & \xrightarrow{(f,g)} & T(2a+1),
\end{array} \tag{84}$$

where j is the inclusion $S_C(-1) \oplus S_C(-2) \hookrightarrow T(-2) \oplus T(-4)$ as in (5.1.7).

Proposition 5.3.12. In the setting of Diagram (84), ϕ defines a CM double conic Z on C of type $2a+1$ with $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$, where $\{F_1, G_1\}, \{F_2, G_2\}$ is an admissible pair of sequences on C corresponding to $\{f, g\}$. Moreover, if $\{F_1', G_1'\}, \{F_2', G_2'\}$ is an admissible pair of sequences on C that defines some double conic Z' on C of type $2a+1$, then $Z' = Z \Leftrightarrow$ there exists an $M \in GL(2,k)$ such that

$$\begin{pmatrix} F_1' & F_1' \\ G_1' & G_2' \end{pmatrix} = M \begin{pmatrix} F_1 & F_2 \\ G_1 & G_2 \end{pmatrix} \bmod I_C.$$

Proof. By construction, $\operatorname{Coker}\phi$ has finite length and hence ϕ defines a CM double conic Z on C of type $2a+1$ with total ideal $I_Z = I_C^2 + (\pi \circ j)^{-1} \operatorname{Ker}\tau$ by Theorem 5.3.1. Let $\hat{e}_1$ and $\hat{e}_2$ be the generators of $T(-2) \oplus T(-4)$. Since $\{f, g\}$ is a regular sequence in T, $\operatorname{Ker}\tau$ is generated by the Koszul relation $\eta = f\hat{e}_2 - g\hat{e}_1$. But $j^{-1}(\eta) = \varnothing$ since $\deg \eta = 2a+7$, which is odd. Notice $j^{-1}(s\eta), j^{-1}(t\eta) \in S_C(-1) \oplus S_C(-2)$, since $\deg s\eta$ and $\deg t\eta$ are even. Hence $(j^{-1}(s\eta), j^{-1}(t\eta)) \subseteq j^{-1} \operatorname{Ker}\tau$. Conversely, let $u \in j^{-1}\operatorname{Ker}\tau$. Then $j(u) \in (\eta)$ and hence there exists $\lambda \in T$ such that $j(u) = \lambda\eta$. Notice $\deg \lambda$

is odd, since $\deg j(u)$ is even and $\deg \eta$ is odd. Hence there exist $\mu, \nu \in T$ such that $\lambda = \mu s + \nu t$. Therefore $j(u) = \mu s\eta + \nu t\eta$ and hence $u \in \theta^{-1}(\mu)j^{-1}(s\eta) + \theta^{-1}(\nu)j^{-1}(t\eta)$. Thus $j^{-1}\operatorname{Ker}\tau \subseteq (j^{-1}(s\eta), j^{-1}(t\eta))$ and hence $j^{-1}\operatorname{Ker}\tau = (j^{-1}(s\eta), j^{-1}(t\eta))$. Therefore $j^{-1}\operatorname{Ker}\tau$ is generated by $j^{-1}(s\eta)$ and $j^{-1}(t\eta)$.

Let $\{F_1, G_1\}, \{F_2, G_2\}$ be an admissible pair of sequences on C corresponding to $\{f, g\}$. Since $s\eta = sf\hat{e}_2 - sg\hat{e}_1, \theta(F_1) = sf$ and $\theta(G_1) = sg$ we have $j^{-1}(s\eta) = \overline{F_1q - G_1x}$. Similarly, $j^{-1}(t\eta) = \overline{F_2q - G_2x}$. Therefore $(\pi \circ j)^{-1}\operatorname{Ker}\tau$ is generated by $F_1q - G_1x$ and $F_2q - G_2x \bmod I_C^2$. Hence $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$.

Let $\{F_1', G_1'\}, \{F_2', G_2'\}$ be another admissible pair of sequences on C that defines a CM double conic Z' on C of type $2a+1$. Then $Z' = Z \Leftrightarrow I_{Z'} = I_Z \Leftrightarrow I_{Z'}/I_C^2 = I_Z/I_C^2$. Notice I_Z/I_C^2 can be considered as a submodule of $S_C(-1) \oplus S_C(-2)$ via the inclusion $I_Z/I_C^2 \subset I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$. Since $I_C/I_C^2 \cong S_C(-1) \oplus S_C(-2)$ and since I_C/I_C^2 is generated by $\bar{x}$ and $\bar{q}$, where $\bar{x}$ and $\bar{q}$ are the images of x and q in S/I_C^2, we can identify $\{\bar{x}, \bar{q}\}$ as a basis of $S_C(-1) \oplus S_C(-2)$. Therefore I_Z/I_C^2 is generated by the vectors $(F_1, -G_1)$ and $(F_2, -G_2)$ as a submodule of $S_C(-1) \oplus S_C(-2)$. Similarly, $I_{Z'}/I_C^2$ is generated by the vectors $(F_1', -G_1')$ and $(F_2', -G_2')$ as a submodule of $S_C(-1) \oplus S_C(-2)$. Therefore $I_{Z'}/I_C^2 = I_Z/I_C^2 \Leftrightarrow$ there exists an $N \in GL(2, k)$ such that

$$\begin{pmatrix} F_1' & F_1' \\ -G_1' & -G_2' \end{pmatrix} = N \begin{pmatrix} F_1 & F_2 \\ -G_1 & -G_2 \end{pmatrix} \bmod I_C.$$

Let $N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $M = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$. Then $\det M = \det N$. Hence $N \in GL(2, k) \Leftrightarrow$

$M \in GL(2,k)$. Moreover,

$$\begin{pmatrix} F_1' & F_1' \\ -G_1' & -G_2' \end{pmatrix} = N \begin{pmatrix} F_1 & F_2 \\ -G_1 & -G_2 \end{pmatrix} \mod I_C \Leftrightarrow \begin{pmatrix} F_1' & F_1' \\ G_1' & G_2' \end{pmatrix} = M \begin{pmatrix} F_1 & F_2 \\ G_1 & G_2 \end{pmatrix} \mod I_C.$$

Therefore $Z' = Z \Leftrightarrow I_{Z'}/I_C^2 = I_Z/I_C^2 \Leftrightarrow$ there exists an $M \in GL(2,k)$ such that

$$\begin{pmatrix} F_1' & F_1' \\ G_1' & G_2' \end{pmatrix} = M \begin{pmatrix} F_1 & F_2 \\ G_1 & G_2 \end{pmatrix} \mod I_C.$$

□

Remark 5.3.13. Let Z be a double conic on C of type $2a+1$, where $a \geq -1$, given by the regular sequence $\{f, g\}$ in T. Then $\deg f = 2a+3$ and $\deg g = 2a+5$. Notice $\deg f \geq 1$ and $\deg g \geq 3$, since $a \geq -1$. So there exist $f_1, f_2, g_1, g_2 \in T$ such that $f = sf_1 + tf_2$ and $g = sg_1 + tg_2$. Notice $\deg f_i$ and $\deg g_i$ are even. Let $F_{11}, F_{12}, G_{11}, G_{12} \in S$ such that $\theta(F_{11}) = f_1, \theta(F_{12}) = f_2, \theta(G_{11}) = g_1$ and $\theta(G_{12}) = g_2$. Let $\{F_1, G_1\}, \{F_2, G_2\}$ be an admissible pair of sequences on C corresponding to $\{f, g\}$ such that $\theta(F_1) = sf$, $\theta(G_1) = sg, \theta(F_2) = tf, \theta(G_2) = tg$. Then $F_1 = yF_{11} + wF_{12}$, since $sf = s^2 f_1 + stf_2$. Similarly, we have $F_2 = wF_{11} + zF_{12}, G_1 = yG_{11} + wG_{12}$ and $G_1 = wG_{11} + zG_{12}$. Notice we can choose $F_{11}, F_{12}, G_{11}, G_{12}$ from S_C. We can express these relations in a matrix form as follows:

$$\begin{pmatrix} F_1 & G_1 \\ F_2 & G_2 \end{pmatrix} = \begin{pmatrix} y & w \\ w & z \end{pmatrix} \begin{pmatrix} F_{11} & G_{11} \\ F_{12} & G_{12} \end{pmatrix}. \tag{85}$$

Proposition 5.3.14. Let Z be a double conic on C of type $2a+1$, where $a \geq -1$, with total ideal $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$. Let $F_{11}, F_{12}, G_{11}, G_{12} \in S_C$ be homogeneous polynomials that satisfy the relations in (85). Then I_Z has minimal S-resolution

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} I_Z \to 0, \tag{86}$$

where $N_1 = S(-2) \oplus S(-3) \oplus S(-4) \oplus S(-a-4)^2, N_2 = S(-4) \oplus S(-5) \oplus S(-a-5)^4$, $N_3 = S(-a-6)^2$ and $\varphi_1 = (x^2, xq, q^2, F_1q - G_1x, F_2q - G_2x)$,

$$\varphi_2 = \begin{pmatrix} q & 0 & G_1 & G_2 & 0 & 0 \\ -x & q & -F_1 & -F_2 & G_{11} & G_{12} \\ 0 & -x & 0 & 0 & -F_{11} & -F_{12} \\ 0 & 0 & x & 0 & z & -w \\ 0 & 0 & 0 & x & -w & y \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} -G_{11} & G_{12} \\ F_{11} & -F_{12} \\ z & w \\ -w & -y \\ -x & 0 \\ 0 & x \end{pmatrix}.$$

Proof. From (85) we get the relations:

$$F_1 = yF_{11} + wF_{12}, G_1 = yG_{11} + wG_{12}, F_2 = wF_{11} + zF_{12}, G_2 = wG_{11} + zG_{12}. \tag{87}$$

Let C_i denote the i^{th} column of φ_2. Using the relations in (87) we can show that $\varphi_1 \cdot C_i = 0$ for all i. For example,

$$
\begin{aligned}
\varphi_1 \cdot C_5 &= G_{11}xq - F_{11}q^2 + z(F_1q - G_1x) - w(F_2q - G_2x) \\
&= G_{11}xq - F_{11}q^2 + zF_1q - zG_1x - wF_2q + wG_2x \\
&= G_{11}xq - F_{11}q^2 + z(yF_{11} + wF_{12})q - z(yG_{11} + wG_{12})x - w(wF_{11} + zF_{12})q \\
&\quad +w(wG_{11} + zG_{12})x \\
&= G_{11}xq - F_{11}q^2 + (yz - w^2)F_{11}q - (yz - w^2)G_{11}x \\
&= G_{11}xq - F_{11}q^2 + F_{11}q^2 - G_{11}xq, \text{ since } q = yz - w^2. \\
&= 0.
\end{aligned}
$$

Hence $\varphi_2 \circ \varphi_1$ is the zero map. Similarly, $\varphi_3 \circ \varphi_2$ is also the zero map. Thus (86) is a complex. Notice (86) is exact if and only if the complex

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \tag{88}$$

is exact, where $N_0 = S$. We use Buchsbaum-Eisenbud criterion 2.1.20 to prove that (88) is exact. Notice $\operatorname{rank} \varphi_1 = 1$, since $x^2 \neq 0$ in S. Let M be the 5×5 submatrix of φ_2 obtained by deleting its first column, i.e.,

$$M = \begin{pmatrix} 0 & G_1 & G_2 & 0 & 0 \\ q & -F_1 & -F_2 & G_{11} & G_{12} \\ -x & 0 & 0 & -F_{11} & -F_{12} \\ 0 & x & 0 & z & -w \\ 0 & 0 & x & -w & y \end{pmatrix}.$$

Then using the relations in (87) we see that

$$\begin{aligned}
\det M &= -q \begin{vmatrix} G_1 & G_2 & 0 & 0 \\ 0 & 0 & -F_{11} & -F_{12} \\ x & 0 & z & -w \\ 0 & x & -w & y \end{vmatrix} - x \begin{vmatrix} G_1 & G_2 & 0 & 0 \\ -F_1 & -F_2 & G_{11} & G_{12} \\ x & 0 & z & -w \\ 0 & x & -w & y \end{vmatrix} \\
&= -xqG_1(wF_{11} + zF_{12}) + xqG_2(yF_{11} + wF_{12}) \\
&\quad -xG_1 \begin{vmatrix} -F_2 & G_{11} & G_{12} \\ 0 & z & -w \\ x & -w & y \end{vmatrix} + xG_2 \begin{vmatrix} -F_1 & G_{11} & G_{12} \\ x & z & -w \\ 0 & -w & y \end{vmatrix} \\
&= -xqF_2G_1 + xqF_1G_2 + xqF_2G_1 + x^2G_1(wG_{11} + zG_{12}) - xqF_1G_2 \\
&\quad -x^2G_2(yG_{11} + zG_{12}) \\
&= x^2G_1G_2 - x^2G_1G_2 \\
&= 0.
\end{aligned}$$

Similarly, we can show that all the 5×5 minors of φ_2 are zero. Hence $\operatorname{rank}\varphi_2 \leq 4$. Let M_1 and M_2 be the 4×4 submatrices of φ_2 given by

$$M_1 = \begin{pmatrix} -x & q & -F_1 & -F_2 \\ 0 & -x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} q & 0 & 0 & 0 \\ -x & q & G_{11} & G_{12} \\ 0 & 0 & z & -w \\ 0 & 0 & -w & y \end{pmatrix} \tag{89}$$

then $\det M_1 = x^4$ and $\det M_2 = q^3$. Thus φ_2 has some nonzero 4×4 minors and hence $\operatorname{rank}\varphi_2 = 4$. Similarly we can show that $\operatorname{rank}\varphi_3 = 2$. Thus $\operatorname{rank} N_i = \operatorname{rank}\varphi_i + \operatorname{rank}\varphi_{i+1}$ for $i = 1, 2, 3$. It remains to show that $\operatorname{depth} I(\varphi_i) \geq i$. We have $x^2 \in I(\varphi_1)$, which is regular in S. Hence $\operatorname{depth} I(\varphi_1) \geq 1$. From (89) we see that $x^4, q^3 \in I(\varphi_2)$, since $\det M_1 = x^4$ and $\det M_2 = q^3$. Since $\{x, q\}$ is a regular sequence in S, $\{x^4, q^3\}$ is also a regular sequence in S by [26, Theorem 16.1]. Hence $\operatorname{depth} I(\varphi_2) \geq 2$. Finally, let T_1, T_2 and T_3 be the 2×2 submatrices of φ_3 given by

$$T_1 = \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix}, \quad T_2 = \begin{pmatrix} z & w \\ -w & -y \end{pmatrix} \text{ and } T_3 = \begin{pmatrix} F_{11} & -F_{12} \\ -w & -y \end{pmatrix}.$$

Then $x^2, q, F_1 \in I(\varphi_3)$, since $\det T_1 = -x^2, \det T_2 = -q$ and $\det T_3 = -F_1$. By construction $F_1 \in \theta^{-1}(sf)$, where $f \in T$ is some regular element. If $F_1 \in (x, q)$ then $\theta(F_1) = sf = 0$, which contradicts the regularity of f. Hence $F_1 \notin (x, q)$. Suppose $UF_1 \in (x, q)$ for some $U \in S$. Set $u := \theta(U)$. Then $\theta(UF_1) = usf = 0$ in T. Since sf is regular in T we must have $u = 0$. Thus $U \in (x, q)$ and F_1 is regular in $S/(x, q)$. Hence

$\{x, q, F_1\}$ is a regular sequence in S. Therefore $\{x^2, q, F_1\}$ is also a regular sequence in S by [26, Theorem 16.1]. Hence depth $I(\varphi_3) \geq 3$ and therefore (88) is exact by Buchsbaum-Eisenbud criterion 2.1.20. Thus (86) is exact and hence a minimal S-resolution of I_Z. $\square$

Proposition 5.3.15. If Z is a double conic on C of type $2a+1$, where $a \geq -1$, then $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C(2a+1) \oplus \mathcal{O}_C[-2a-7]$.

Proof. Let Z be given by the regular sequence $\{f, g\}$ in T. Then $\deg f = 2a+3$ and $\deg g = 2a+5$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_Z/\mathcal{I}_C^2 & \longrightarrow & \mathcal{I}_C/\mathcal{I}_C^2 & \longrightarrow & i_*\mathcal{O}_{\mathbb{P}^1}(2a+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-2a-7) & \xrightarrow{\begin{pmatrix} -g \\ f \end{pmatrix}} & \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-4) & \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} & \mathcal{O}_{\mathbb{P}^1}(2a+1) & \longrightarrow & 0.
\end{array}$$

Notice, the last two vertical maps are isomorphisms. Therefore $\mathcal{I}_Z/\mathcal{I}_C^2 \cong i_*\mathcal{O}_{\mathbb{P}^1}(-2a-7)$ by the snake lemma. On the other hand, $\mathcal{I}_C^2/\mathcal{I}_C\mathcal{I}_Z \cong (\mathcal{I}_C/\mathcal{I}_Z)^{\otimes 2}$ by Corollary 4.2.8 (b). Since $\mathcal{I}_C/\mathcal{I}_Z \cong i_*\mathcal{O}_{\mathbb{P}^1}(2a+1)$, we therefore have $\mathcal{I}_C^2/\mathcal{I}_C\mathcal{I}_Z \cong i_*\mathcal{O}_{\mathbb{P}^1}(4a+2)$. Since $\mathcal{I}_Z \subset \mathcal{I}_C$, we have $\mathcal{I}_C\mathcal{I}_Z \subset \mathcal{I}_C^2$ and hence the commutative diagram

$$0 \to \mathcal{I}_C^2/\mathcal{I}_C\mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_C^2 \to 0. \tag{90}$$

Notice

$$\begin{aligned}
\operatorname{Ext}^1_{\mathcal{O}_C}(\mathcal{I}_Z/\mathcal{I}_C^2, \mathcal{I}_C^2/\mathcal{I}_C\mathcal{I}_Z) &\cong \operatorname{Ext}^1_{\mathcal{O}_C}(i_*\mathcal{O}_{\mathbb{P}^1}(-2a-7), i_*\mathcal{O}_{\mathbb{P}^1}(4a+2)) \\
&\cong \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(-2a-7), \mathcal{O}_{\mathbb{P}^1}(4a+2)) \\
&\cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6a+9))
\end{aligned}$$

by [18, III, Proposition 6.3 (c)]. Now $h^1\mathcal{O}_{\mathbb{P}^1}(6a+9) = h^0\mathcal{O}_{\mathbb{P}^1}(-6a-11)$ by [18, III, Theorem 5.1 (d)]. Since $a \geq -1$, $-6a-11 \leq -5$ and hence $h^0\mathcal{O}_{\mathbb{P}^1}(-6a-11) = 0$. Therefore $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{I}_Z/\mathcal{I}_C^2, \mathcal{I}_C^2/\mathcal{I}_C\mathcal{I}_Z) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6a+9)) = 0$ and hence (90) is split exact. Thus $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong i_*\mathcal{O}_{\mathbb{P}^1}(4a+2) \oplus i_*\mathcal{O}_{\mathbb{P}^1}(-2a-7) \cong \mathcal{O}_C(2a+1) \oplus \mathcal{O}_C[-2a-7]$. □

Corollary 5.3.16. If Z is a double conic on C of type ℓ, then $\operatorname{proj\,dim} S_Z = 3 \Leftrightarrow \ell \geq -1$. In particular, Z is not ACM $\Leftrightarrow \ell \geq -1$.

Proof. By Corollaries 5.3.3, 5.3.4 and Propositions 5.3.6, 5.3.14 we have $\operatorname{proj\,dim} S_Z = 3$ if and only if $\ell \geq -1$. Therefore Z is not ACM $\Leftrightarrow \ell \geq -1$. □

5.4 Linkage of double conics

In this section we give criteria for double conics of the same support to be linked by a complete intersection. In particular, we give a criterion for double conics to be self-linked.

Definition 5.4.1. Let Y, Y' and X be curves $\mathbb{P}^3$ such that X is a complete intersection curve with $I_X \subseteq I_Y \cap I_{Y'}$. Then Y is (algebraically) directly linked to Y' by X if and only if $[I_X : I_Y] = I_{Y'}$ and $[I_X : I_{Y'}] = I_Y$. If Y is linked to Y' by X, we write $Y \sim Y'$ by X. If Y is linked to itself by X, we say Y is self-linked by X.

Proposition 5.4.2. Let $Y, Y' \subset \mathbb{P}^3$ be CM curves. If $Y \sim Y'$ by a complete intersection X with $I_X = (F, G)$, then

(a) $\deg Y + \deg Y' = \deg X$ and

(b) $p_a(Y) - p_a(Y') = \frac{1}{2}(\deg F + \deg G - 4)(\deg Y - \deg Y')$.

Proof. [25, III, Proposition 1.2] or [28, Corollaries 5.2.13 and 5.2.14]. □

Corollary 5.4.3. Let Z and Z' be double conics on C of types ℓ and ℓ' respectively. If $Z \sim Z'$ then $\ell = \ell'$.

Proof. Since $\deg Z = \deg Z'$, we have $p_a(Z) = p_a(Z')$ by Proposition 5.4.2 (b). Also $p_a(Z) = -1 - \ell$ and $p_a(Z') = -1 - \ell'$ by Theorem 5.3.1. Hence $\ell = \ell'$. □

Lemma 5.4.4. Let Z and Z' be double conics on C of type $\ell \geq -1$. If $Z \sim Z'$ by a complete intersection X then $I_X = (x^2, \alpha x q + q^2)$ for some linear form $\alpha \in S$.

Proof. By Proposition 5.4.2 (a), $\deg X = \deg Z + \deg Z' = 8$. Let $I_X = (A, B)$. Then A, B are homogeneous polynomials in I_Z such that $\deg A \cdot \deg B = \deg X = 8$. By Propositions 5.3.2 and 5.3.12, I_Z has no linear term and the only qudratic term in I_Z is x^2. Hence $\deg A$ and $\deg B$ can be either 2 or 4. Let $\deg A = 2$ and $\deg B = 4$. Since the only quadratic form in I_Z is x^2, we may assume that $A = x^2$. It remains to show that $B = \alpha x q + q^2$ for some linear form $\alpha \in S$.

First suppose $\ell = 2a$, where $a \geq 0$. Then $I_Z = (I_C^2, Fq - Gx)$ by Proposition 5.3.2. Notice $\deg(Fq - Gx) = a + 3$. Thus if $a \geq 2$ then $\deg(Fq - Gx) \geq 5$ and hence $B \in I_C^2$. So we can take $B = \alpha x q + \beta q^2$ for some linear form $\alpha \in S$ and $\beta \in k$. Now suppose $0 \leq a \leq 1$. Then $B = \alpha x q + \beta q^2 + \gamma(Fq - Gx)$, where $\deg \gamma = 4 - (a + 3) = 1 - a \leq 1$. Since $Z \sim Z'$ by X, we must have $\operatorname{Supp} X = C$, i.e., $Z(A, B) = Z(x, q)$ where $Z(A, B)$

means the common zero locus of A and B. Since

$$\begin{aligned} Z(A,B) &= Z(x^2, \alpha xq + \beta q^2 + \gamma(Fq - Gx)) \\ &= Z(x, \alpha xq + \beta q^2 + \gamma(Fq - Gx)) \\ &= Z(x, (\beta q + \gamma F)q) \\ &= Z(x,q) \cup Z(x, \beta q + \gamma F), \end{aligned}$$

we must have $Z(x, \beta q + \gamma F) = Z(x,q)$, hence $\sqrt{(x, \beta q + \gamma F)} = \sqrt{(x,q)} = (x,q)$. Therefore $(x, \beta q + \gamma F) \subseteq (x,q)$, hence $\beta q + \gamma F \in (x,q)$, i.e., $\gamma F \in (x,q)$. Since F is regular in $S_C = S/(x,q)$, we must have $\gamma \in (x,q)$. Now if $a = 1$, i.e., $\deg \gamma = 0$ then we have $\gamma = 0$ and $B = \alpha xq + \beta q^2$. If $a = 0$, i.e., $\deg \gamma = 1$ then $\gamma = \nu x$ for some $\nu \in k^*$. Replacing α by $\alpha + \nu F$ we see that $B = \alpha xq + \beta q^2 - \nu G x^2$. Hence $I_X = (x^2, \alpha xq + \beta q^2)$. Therefore we can take $B = \alpha xq + \beta q^2$ whenever $\ell = 2a \geq 0$. Notice if $\beta = 0$ then $\{A, B\}$ is not a regular sequence in S and hence X fails to be a complete intersection. Thus we must have $\beta \neq 0$. Hence we can assume that $\beta = 1$. Therefore $I_X = (x^2, \alpha xq + q^2)$.

Now suppose $\ell = 2a+1$, where $a \geq -1$. Then $I_Z = (I_C^2, F_1 q - G_1 x, F_2 q - G_2 x)$ by Proposition 5.3.12. Notice $\deg(F_i q - G_i x) = a + 4$. Hence if $a \geq 1$ then $\deg(F_i q - G_i x) \geq 5$ and hence $B \in I_C^2$. So we can take $B = \alpha xq + \beta q^2$ for some linear form $\alpha \in S$ and $\beta \in k$. Now suppose $-1 \leq a \leq 0$. Then $B = \alpha xq + \beta q^2 + \gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x)$, where $\deg \gamma = \deg \delta = 4 - (a+4) = -a \leq 1$. Since $Z(A,B) = Z(x,q)$ and since

$$
\begin{aligned}
Z(A,B) &= Z(x^2, \alpha xq + \beta q^2 + \gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x)) \\
&= Z(x, \alpha xq + \beta q^2 + \gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x)) \\
&= Z(x, (\beta q + \gamma F_1 + \delta F_2) q) \\
&= Z(x,q) \cup Z(x, \beta q + \gamma F_1 + \delta F_2),
\end{aligned}
$$

we must have $Z(x, \beta q + \gamma F_1 + \delta F_2) = Z(x,q)$, hence $\sqrt{(x, \beta q + \gamma F_1 + \delta F_2)} = (x,q)$. Therefore $(x, \beta q + \gamma F_1 + \delta F_2) \subseteq (x,q)$, hence $\beta q + \gamma F_1 + \delta F_2 \in (x,q)$, i.e., $\gamma F_1 + \delta F_2 \in (x,q)$. Hence $(\bar{\gamma} s + \bar{\delta} t) f = 0$ in T, where $\bar{\gamma}, \bar{\delta}$ are the images of γ, δ in T under the map θ as in (5.1.7) and $f \in T$ is a regular element such that $\theta(F_1) = sf, \theta(F_2) = tf$. Therefore

$$\bar{\gamma} s + \bar{\delta} t = 0. \tag{91}$$

First suppose $a = 0$. Then $\deg \gamma = \deg \delta = 0$, hence $\deg \bar{\gamma} = \deg \bar{\delta} = 0$ Therefore from (91) we see that $\bar{\gamma} = \bar{\delta} = 0$. Hence $\gamma = \delta = 0$ and $B = \alpha xq + \beta q^2$. Finally, let $a = -1$. Then $\deg \gamma = \deg \delta = 1$. Hence $\deg \bar{\gamma} = \deg \bar{\delta} = 2$. From (91) we see that $\bar{\gamma} s = -\bar{\delta} t$. Hence $s \mid \bar{\delta}$ and $t \mid \bar{\gamma}$. So there exist $a, b, c, d \in k$ such that $\bar{\gamma} = ast + bt^2$ and $\bar{\delta} = cs^2 + dst$. Using (91) we have $st[(a+c)s + (b+d)t] = 0$. Hence $(a+c)s + (b+d)t = 0$ since st is a nonzerodivisor in T. Thus $a + c = b + d = 0$, i.e., $c = -a$ and $d = -b$. Hence

$\bar{\gamma} = ast + bt^2, \bar{\delta} = -(as^2 + bst)$ and therefore $\gamma = aw + bz, \delta = -(ay + bw)$, i.e.,

$$\begin{pmatrix} \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} w & -y \\ z & -w \end{pmatrix}. \tag{92}$$

Notice

$$\gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x) = \begin{pmatrix} \gamma & \delta \end{pmatrix} \begin{pmatrix} F_1 q - G_1 x \\ F_2 q - G_2 x \end{pmatrix} = \begin{pmatrix} \gamma & \delta \end{pmatrix} \begin{pmatrix} F_1 & G_1 \\ F_2 & G_2 \end{pmatrix} \begin{pmatrix} q \\ -x \end{pmatrix}. \tag{93}$$

From (85) we have

$$\begin{pmatrix} F_1 & G_1 \\ F_2 & G_2 \end{pmatrix} = \begin{pmatrix} y & w \\ w & z \end{pmatrix} \begin{pmatrix} F_{11} & G_{11} \\ F_{12} & G_{12} \end{pmatrix}. \tag{94}$$

Combining (92), (93) and (94) we get

$$\begin{aligned} \gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x) &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} w & -y \\ z & -w \end{pmatrix} \begin{pmatrix} y & w \\ w & z \end{pmatrix} \begin{pmatrix} F_{11} & G_{11} \\ F_{12} & G_{12} \end{pmatrix} \begin{pmatrix} q \\ -x \end{pmatrix} \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \begin{pmatrix} F_{11} & G_{11} \\ F_{12} & G_{12} \end{pmatrix} \begin{pmatrix} q \\ -x \end{pmatrix} \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} -F_{12} & -G_{12} \\ F_{11} & G_{11} \end{pmatrix} \begin{pmatrix} q^2 \\ -xq \end{pmatrix} \\ &= (aG_{12} - bG_{11})xq + (bF_{11} - aF_{12})q^2. \end{aligned}$$

Therefore

$$\begin{aligned} B &= \alpha x q + \beta q^2 + \gamma(F_1 q - G_1 x) + \delta(F_2 q - G_2 x) \\ &= (\alpha + aG_{12} - bG_{11})xq + (\beta + bF_{11} - aF_{12})q^2. \end{aligned}$$

Replacing $\alpha + aG_{12} - bG_{11}$ by α and $\beta + bF_{11} - aF_{12}$ by β we get $B = \alpha xq + \beta q^2$. Notice if $\beta = 0$ then $\{A, B\}$ is not a regular sequence in S, hence X fails to be a complete intersection. Hence $\beta \neq 0$. We can assume that $\beta = 1$. Therefore $I_X = (x^2, \alpha xq + q^2)$. $\square$

Proposition 5.4.5. Let Z, Z' be a double conics on C of type $\ell = 2a$, where $a \geq 0$. Let $I_Z = (I_C^2, Fq - Gx)$ and let X be the complete intersection with $I_X = (x^2, \alpha xq + q^2)$, where $\alpha \in S$ is a linear form. Then $Z \sim Z'$ by $X \iff I_{Z'} = (I_C^2, Fq + (G + \alpha F)x)$.

Proof. Let $Y \subset \mathbb{P}^3$ be the closed subscheme with total ideal $I_Y = (I_C^2, Fq + (G + \alpha F)x)$. Since $\{F, G\}$ is a regular sequence in S_C, $\{F, -(G+\alpha F)\}$ is also a regular sequence in S_C by Lemma 2.1.8. Therefore Y is a double conic on C of type $2a$ by Proposition 5.3.2. So it suffices to prove that $Z \sim Z'$ by $X \Leftrightarrow Z' = Y$. Notice, $(Fq + (G + \alpha F)x)(Fq - Gx) = -(G^2 + FG\alpha)x^2 + F^2(\alpha xq + q^2) \in I_X$. Similarly, we can show that $uv \in I_X$, for all $u \in I_Y$ and $v \in I_Z$. Hence $I_Y I_Z \subseteq I_X$, i.e., $I_Y \subseteq [I_X : I_Z]$. Now if $Z \sim Z'$ by X then $I_{Z'} = [I_X : I_Z]$. Hence $I_Y \subseteq I_{Z'}$, i.e., $Z' \subseteq Y$. Notice Z' and Y are double conics on C of type $2a$ and hence they have the same Hilbert polynomial by Theorem 5.2.1. Therefore $Z' = Y$ by Lemma 3.1.2. In particular, $Z \sim Y$ by X. Hence $Z \sim Z'$ by $X \Leftrightarrow Z' = Y$. $\square$

Proposition 5.4.6. Let Z and Z' be double conics on C of type $\ell = 2a + 1$, where $a \geq -1$. Let $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$ and let X be the complete intersection with $I_X = (x^2, \alpha xq + q^2)$, where $\alpha \in S$ is a linear form. Then

$$Z \sim Z' \text{ by } X \iff I_{Z'} = (I_C^2, F_1q + (G_1 + F_1\alpha)x, F_2q + (G_2 + F_2\alpha)x).$$

Proof. Let $\{f, g\}$ be a regular sequence in T induced by the admissible pair of sequences $\{F_1, G_1\}$, $\{F_2, G_2\}$. Then $\theta(F_1) = sf, \theta(G_1) = sg, \theta(F_2) = tf$ and $\theta(G_2) = tg$. Hence $\theta(-(G_1 + \alpha F_1)) = -s(g + \bar{\alpha}f)$ and $\theta(-(G_2 + \alpha F_2)) = -t(g + \bar{\alpha}f)$, where $\bar{\alpha} = \theta(\alpha)$. By Lemma 2.1.8, $\{f, -(g + \bar{\alpha}f)\}$ is a regular sequence in T for all $\bar{\alpha} \in T$ and hence $\{F_1, -(G_1 + \alpha F_1)\}, \{F_2, -(G_2 + \alpha F_2)\}$ is an admissible pair of sequences on C for all linear forms $\alpha \in S$. Let $Y \subseteq \mathbb{P}^3$ be the closed subscheme defined by the total ideal $I_Y = (I_C^2, F_1q + (G_1 + F_1\alpha)x, F_2q + (G_2 + F_2\alpha)x)$. By Proposition 5.3.12, Y is a double conic on C of type $2a + 1$. So it suffices to show that $Z \sim Z'$ by $X \Leftrightarrow Z' = Y$. Notice, $(F_1q + (G_1 + F_1\alpha)x)(F_1q - G_1x) = -(F_1G_1\alpha + G_1^2)x^2 + F_1^2(\alpha xq + q^2) \in I_X$. Similarly, we can show that $uv \in I_X$, for all $u \in I_Y$ and $v \in I_Z$. Hence $I_YI_Z \subseteq I_X$, i.e., $I_Y \subseteq [I_X : I_Z]$. Now if $Z \sim Z'$ by X then $I_{Z'} = [I_X : I_Z]$. Hence $I_Y \subseteq I_{Z'}$, i.e., $Z' \subseteq Y$. Notice Z' and Y are double conics on C of type $2a + 1$ and hence they have the same Hilbert polynomial by Theorem 5.2.1. Therefore $Z' = Y$ by Lemma 3.1.2. In particular, $Z \sim Y$ by X. Hence $Z \sim Z'$ by $X \Leftrightarrow Z' = Y$. ∎

Corollary 5.4.7. Let Z be a double conic on C of type $\ell \geq -1$. Then Z is self-linked if and only if char $k = 2$.

Proof. First suppose ℓ is even. Let $Z \sim Z$ by a complete intersection X. Then by Lemma 5.4.4, $I_X = (x^2, \alpha xq + q^2)$ for some linear form $\alpha \in S$. Hence by Proposition 5.4.5, $G + \alpha F = -G \bmod I_C \Rightarrow \alpha F = -2G$ in $S_C \Rightarrow \alpha F = 0$ in $S_C/(G) \Rightarrow \alpha = 0$ in S_C, since $\{F, G\}$ is a regular sequence in S_C and α is a linear form. Hence $Z \sim Z \Rightarrow G = -G$ in $S_C \Rightarrow 2G = 0$ in $S_C \Rightarrow 2 = 0$, since G is nonzero in $S_C \Rightarrow$ char $k = 2$. Conversely, let char $k = 2$. Then $I_Z = (I_C^2, Fq - Gx) = (I_C^2, Fq + Gx)$. Therefore by Proposition 5.4.5, $Z \sim Z$ by the complete intersection X', where $I_{X'} = (x^2, q^2)$.

Now suppose ℓ is odd. Let $\{f, g\}$ be a regular sequence in T corresponding to Z. Then by Proposition 5.4.6, $Z \sim Z \Rightarrow G_1 + \alpha F_1 = -G_1$ in $S_C \Rightarrow 2G_1 + \alpha F_1 = 0$ in S_C $\Rightarrow s(2g + \bar{\alpha} f) = 0$ in T, where $\bar{\alpha}$ is the image of α in T under θ. Since s is regular in T, $s(2g + \bar{\alpha} f) = 0$ in $T \Rightarrow 2g + \bar{\alpha} f = 0$ in $T \Rightarrow \bar{\alpha} f = -2g$ in $T \Rightarrow \bar{\alpha} = 0$ in T, since $\{f, g\}$ is a regular sequence in $T \Rightarrow 2g = 0$ in $T \Rightarrow 2 = 0$, since g is nonzero in $T \Rightarrow$ char $k = 2$. Conversely, let char $k = 2$. Then $I_Z = (I_C^2, F_1 q - G_1 x, F_2 q - G_2 x) = (I_C^2, F_1 q + G_1 x, F_2 q + G_2 x)$. Therefore by Proposition 5.4.6, $Z \sim Z$ by X', where $I_{X'} = (x^2, q^2)$. □

Remark 5.4.8. Double conics of types -4 and -2 are self-linked over any algebraically closed field k, since they are complete intersections by Corollaries 5.3.3 and 5.3.4.

Remark 5.4.9. Corollary 5.4.7 extends a well-known theorem of Juan Migliore [27, Theorem 4.4] which says that a double line of arithmetic genus less than -1 is self-linked if and only if char $k = 2$. Luis Aguirre [1, Corollary 4.3.2] also extended Migliore's theorem to extremal p-lines in $\mathbb{P}^3$, where p is a prime.

Proposition 5.4.10. Let Z be a double conic on C of type $2a+1$, where $a \geq -1$, with total ideal $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$. Let M_Z be the Rao module of Z and let $F_{11}, F_{12}, G_{11}, G_{12} \in S$ be homogeneous polynomials that satisfy the relations (85). Then M_Z has S-presentation

$$S(-2) \oplus S(-1) \oplus S(a-1)^4 \xrightarrow{\sigma} S(a)^2 \to M_Z \to 0, \tag{95}$$

where

$$\sigma = \begin{pmatrix} G_{11} & F_{11} & z & -w & -x & 0 \\ -G_{12} & -F_{12} & w & -y & 0 & x \end{pmatrix}.$$

Proof. Let Z' be a double conic on C of type $2a+1$ such that $Z \sim Z'$ by the complete intersection X, where $I_X = (x^2, q^2)$. Let $M_{Z'}$ be the Rao module of Z' and let

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \to L_2 \to L_1 \to L_0 \to M_{Z'} \to 0 \tag{96}$$

be a minimal free resolution of $M_{Z'}$. Dualizing (96) we get the exact sequence

$$L_3^\vee \xrightarrow{\sigma_4^\vee} L_4^\vee \to \operatorname{Ext}_S^4(M_{Z'}, S) \to 0 \tag{97}$$

by [25, II, § 2]. Since $Z \sim Z'$ by X, we have $M_Z \cong \operatorname{Ext}_S^4(M_{Z'}, S)(-6)$ by [?, 2.1]. Hence we get the exact sequence

$$L_3^\vee(-6) \xrightarrow{\sigma_4^\vee} L_4^\vee(-6) \to M_Z \to 0. \tag{98}$$

By Proposition 5.4.6, we have $I_{Z'} = (I_C^2, F_1q + G_1x, F_2q + G_2x)$, i.e., Z' is given by the admissible pair of sequences $\{F_1, -G_1\}, \{F_2, -G_2\}$. Let $F'_{11}, F'_{12}, G'_{11}, G'_{12} \in S$ be homogeneous polynomials that satisfy the relations (85) for $\{F_1, -G_1\}, \{F_2, -G_2\}$. Then $F'_{11} = F_{11}, F'_{12} = F_{12}, G'_{11} = -G_{11}, G'_{12} = -G_{12}$. Therefore by Proposition 5.3.14, $I_{Z'}$ has minimal S-resolution

$$0 \to N_3 \xrightarrow{\varphi_3} N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} I_{Z'} \to 0, \tag{99}$$

where $N_1 = S(-2) \oplus S(-3) \oplus S(-4) \oplus S(-a-4)^2, N_2 = S(-4) \oplus S(-5) \oplus S(-a-5)^4$, $N_3 = S(-a-6)^2$ and $\varphi_1 = (x^2, xq, q^2, F_1q + G_1x, F_2q + G_2x)$,

$$\varphi_2 = \begin{pmatrix} q & 0 & -G_1 & -G_2 & 0 & 0 \\ -x & q & -F_1 & -F_2 & -G_{11} & -G_{12} \\ 0 & -x & 0 & 0 & -F_{11} & -F_{12} \\ 0 & 0 & x & 0 & z & -w \\ 0 & 0 & 0 & x & -w & y \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} G_{11} & -G_{12} \\ F_{11} & -F_{12} \\ z & w \\ -w & -y \\ -x & 0 \\ 0 & x \end{pmatrix}.$$

On the other hand, applying Rao's theorem [?, Theorem 2.5] to (96) we see that $I_{Z'}$ has a minimal resolution of the form

$$0 \to L_4 \xrightarrow{(\sigma_4, 0)} L_3 \oplus (\oplus_{i=1}^r S(-l_i)) \to \oplus_{i=1}^m S(-e_i) \to I_{Z'} \to 0. \tag{100}$$

Compairing (99) and (100) we see that $L_4 = N_3, L_3 = N_2$ and $\sigma_4 = \phi_3$. Let $\sigma = \phi_3^T$. Then (98) gives the S-presentation (95) of M_Z. □

5.5 Surfaces containing double conics

In this section we prove some properties of general surfaces containing a double conic. In particular, we show that such a surface is normal and the number of its singular points is determined by its degree and the genus of the double conic contained in it.

Proposition 5.5.1. Let Z be a double conic on C of type ℓ. If Z is contained in a nonsingular surface E of degree $d > 0$, then $\ell = 2d - 6$.

Proof. Let $\mathcal{I}_{C|E}$ and $\mathcal{I}_{Z|E}$ be the ideal sheaves of C and Z in E respectively. Then we have the exact sequence

$$0 \to \mathcal{I}_{Z|E} \to \mathcal{O}_E \to \mathcal{O}_Z \to 0. \tag{101}$$

Applying Euler characteristics to the sheaves in (101) we get

$$\chi\mathcal{O}_Z = \chi\mathcal{O}_E - \chi\mathcal{I}_{Z|E}. \tag{102}$$

Since E is nonsingular, C is an effective divisor on E. Hence $\mathcal{I}_{C|E}$ is an invertible $\mathcal{O}_E$-module and $\mathcal{I}_{Z|E} = \mathcal{I}^2_{C|E}$. Thus we have the isomorphism $\mathcal{N}^\vee_{C|E} \cong \mathcal{I}_{C|E}/\mathcal{I}_{Z|E}$, where $\mathcal{N}^\vee_{C|E} = \mathcal{I}_{C|E}/\mathcal{I}^2_{C|E}$ is the conormal bundle of C in E. Hence we have the exact sequence

$$0 \to \mathcal{I}_{Z|E} \to \mathcal{I}_{C|E} \to \mathcal{N}^\vee_{C|E} \to 0. \tag{103}$$

Applying Euler characteristics to the sheaves in (103) we get

$$\chi\mathcal{I}_{Z|E} = \chi\mathcal{I}_{C|E} - \chi\mathcal{N}^\vee_{C|E}. \tag{104}$$

We also have the exact sequence

$$0 \to \mathcal{I}_{C|E} \to \mathcal{O}_E \to \mathcal{O}_C \to 0. \tag{105}$$

Applying Euler characteristics to the sheaves in (105) we get

$$\chi\mathcal{O}_E = \chi\mathcal{I}_{C|E} + \chi\mathcal{O}_C. \tag{106}$$

Combining (102), (104), (106) and using the fact that $\chi\mathcal{O}_C = 1$, we get

$$\chi\mathcal{O}_Z = 1 + \chi\mathcal{N}^\vee_{C|E}. \tag{107}$$

Since E is nonsingular and C is a nonsingular closed subscheme of E, we have the exact sequence

$$0 \to \mathcal{N}^\vee_{C|E} \to \Omega_E \otimes \mathcal{O}_C \to \Omega_C \to 0 \tag{108}$$

by [18, II, Theorem 8.17]. Taking the highest exterior powers of the sheaves in (108) we get

$$\wedge^2 (\Omega_E \otimes \mathcal{O}_C) \cong \wedge^1\mathcal{N}^\vee_{C|E} \otimes \wedge^1\Omega_C = \mathcal{N}^\vee_{C|E} \otimes \Omega_C \tag{109}$$

by [18, II, Exercise 5.16 (d)]. Also by [18, II, Example 8.20.3], we have $\Omega_C = \omega_C \cong \mathcal{O}_C(-1)$ and $\omega_E \cong \mathcal{O}_E(d-4)$, where $d = \deg E$. Thus $\wedge^2(\Omega_E \otimes \mathcal{O}_C) = \omega_E \otimes \mathcal{O}_C \cong \mathcal{O}_C(d-4)$ and hence $\mathcal{N}^\vee_{C|E} \cong \mathcal{O}_C(d-4) \otimes \mathcal{O}_C(1) \cong \mathcal{O}_C(d-3)$ by (109). Therefore $\chi\mathcal{N}^\vee_{C|E} = 2(d-3)+1$ and hence $\chi\mathcal{O}_Z = 2(d-3)+2 = 2d-4$ by (107). On the other hand, $\chi\mathcal{O}_Z = 1-p_a(Z) = \ell+2$ by Theorem 5.2.1. Thus $\ell+2 = 2d-4$, i.e., $\ell = 2d-6$. □

Theorem 5.5.2. Let Z be a double conic on C of type $\ell \geq 4$. Then Z is contained in a nonsingular surface if and only if ℓ is even.

Proof. Let E be a nonsingular surface containing Z with $\deg E = d$. Then $\ell = 2d - 6$, i.e., ℓ is even, by Proposition 5.5.1. Conversely, let $\ell = 2a$, where $a \geq -2$. Then by Proposition 5.3.2, $I_Z = (I_C^2, Fq - Gx)$, where $\deg F = a+1, \deg G = a+2$ and F, G have no common zeros along C. Set $E := Fq - Gx$ and $d := \deg E$. Notice $d = a + 3$. Let J_E denote the Jacobian of E, i.e., $J_E = \begin{pmatrix} E_x & E_y & E_z & E_w \end{pmatrix}$, where E_x, E_y, E_z, E_w are the partial derivatives of E with respect to x, y, z, w respectively. Then

$$J_E = \begin{pmatrix} F_x q - G_x x - G & F_y q + Fz - G_y x & F_z q + Fy - G_z x & F_w q - 2Fw - G_w x \end{pmatrix}.$$

Let $P \in C$ be a closed point. Then we get

$$J_E(P) = \begin{pmatrix} -G(P) & F(P)z(P) & F(P)y(P) & -2F(P)w(P) \end{pmatrix}.$$

If $G(P) \neq 0$ then $\operatorname{rank} J_E(P) = 1$ and $Z(E)$ is nonsingular at P. On the other hand, if $G(P) = 0$ then $F(P) \neq 0$, since F and G have no common zeros along C. Therefore $J_E(P) = 0 \Rightarrow y(P) = z(P) = 0$. But then $[w(P)]^2 = y(P)z(P) = 0$, i.e., $w(P) = 0$. Thus $J_E(P) = 0 \Leftrightarrow P = (1, 0, 0, 0)$, which is a contradiction since $(1, 0, 0, 0) \notin C$. Therefore $\operatorname{Sing} Z(E) \cap C = \varnothing$, where $Z(E)$ is the surface $\{E = 0\}$.

Let $U_1 = \{G \in H^0\mathcal{I}_Z(d) | Z(G) \text{ is nonsingular along } C\}$. Then U_1 is an open subset of $\mathbb{P}H^0\mathcal{I}_Z(d)$ by an application of Elimination Theory [18, I, Theorem 5.7A]. Notice $E \in U_1$, since $Z(E) \cap C = \varnothing$. Hence U_1 is a nonempty open subset of $\mathbb{P}H^0\mathcal{I}_Z(d)$. Let $\delta \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$

be the incomplete linear system corresponding to the vector subspace $V = H^0\mathcal{I}_C^2(d)$ and let $D \in \delta$ be general. Notice $d = a+3 \geq 5$, since $\ell = 2a \geq 4$. Hence $\mathcal{I}_C^2(d-1)$ is generated by global sections and therefore $\operatorname{Sing} D \subseteq C$ by Lemma 4.5.3. Let U_2 be a nonempty open subset of $\{D \in \delta | \operatorname{Sing} Z(D) \subseteq C\}$. Then $U_1 \cap U_2 \subset \mathbb{P}H^0\mathcal{I}_Z(d)$ is a nonempty open dense set. Hence $Z(D)$ is a nonsingular surface containing Z, for all $D \in U_1 \cap U_2$. □

Proposition 5.5.3. Let Z be a double conic on C of type ℓ and let $Z(E)$ be a general surface containing Z of degree $d \geq \lceil \frac{\ell+8}{2} \rceil$. Then $Z(E)$ is normal. Moreover,

(a) $|\operatorname{Sing} Z(E)| = 2d - \ell - 6$.

(b) If $\operatorname{char} k = 0$ and E is very general in the linear system $|\mathcal{I}_Z(d)|$ then $\operatorname{Cl} Z(E)$ is freely generated by $\mathcal{O}_{Z(E)}(1)$ and C.

Proof. Since $d \geq \lceil \frac{\ell+8}{2} \rceil$, $\mathcal{I}_Z(d-1)$ is generated by global sections. Hence E is normal by Proposition 4.5.4 (a). We have $I_{Z(E)} = (E)$. Let $J_{Z(E)|C}$ be the Jacobain of E restricted on C. First suppose $\ell = 2a$. By Proposition 5.3.2, we have $I_Z = (I_C^2, Fq - Gx)$, where $\deg F = a+1, \deg G = a+2$ and F, G have no common zeros along C. Since $E \in I_Z$, there exist $\alpha, \beta, \gamma, A \in S$ such that $E = \alpha x^2 + \beta xq + \gamma q^2 + A(Fq - Gx)$. Therefore

$$J_{Z(E)|C} = \begin{pmatrix} -Ag & AFz & AFy & -2AFw \end{pmatrix}.$$

Let $P \in \operatorname{Sing} Z(E) \cap C$. Then $J_{Z(E)|C}(P) = 0$. Notice, if $A(P) \neq 0$ then we must have $G(P) = 0$ and hence $F(P) \neq 0$. Thus $y(P) = z(P) = 0$. But then $P = (1, 0, 0, 0) \notin C$. Therefore we must have $A(P) = 0$. Thus $\operatorname{Sing} Z(E) = Z(A) \cap C$. Since $\deg A = d - a - 3$, we have $|\operatorname{Sing} Z(E)| = |A \cap C| = 2(d - a - 3) = 2d - \ell - 6$.

Now suppose $\ell = 2a+1$. Then by Proposition 5.3.12, $I_Z = (I_C^2, F_1q - G_1x, F_2q - G_2x)$, where $\deg F_i = a+2, \deg G_i = a+3$ and $\{F_1, G_1\}, \{F_2, G_2\}$ is an admissible pair of sequences on C. Then $E = \alpha x^2 + \beta xq + \gamma q^2 + A(F_1q - G_1x) + B(F_2q - G_2x)$ for some $\alpha, \beta, \gamma, A, B \in S$. Hence

$$J_{Z(E)|C} = \begin{pmatrix} -(AG_1 + BG_2) & (AF_1 + BF_2)z & (AF_1 + BF_2)y & -2(AF_1 + BF_2)w \end{pmatrix}.$$

Let $J^T_{Z(E)|C}$ denote the representation of $J_{Z(E)|C}$ in T and let $\theta(A) = a, \theta(B) = b$. Therefore

$$J^T_{Z(E)|C} = \begin{pmatrix} -(as+bt)g & (as+bt)ft^2 & (as+bt)fs^2 & (as+bt)fst \end{pmatrix},$$

where $\{f, g\}$ is a regular sequence in T induced by the admissible pair of sequences $\{F_1, G_1\}, \{F_2, G_2\}$. Let $P \in \operatorname{Sing} Z(E)$ and $p = i^*(P)$. Then $J^T_{Z(E)|C}(p) = 0$. Notice if $(as+bt)(p) \neq 0$ then we must have $g(p) = 0$, and hence $f(p) \neq 0$. But then $p = (0,0) \notin \mathbb{P}^1$. Therefore $\operatorname{Sing} Z(E)$ is in one-to-one correspondence with the set of zeros of $as+bt$. Since k is algebraically closed, we have $|\operatorname{Sing} Z(E)| = \deg(as+bt) = 2(d-a-4)+1 = 2d - \ell - 6$.

Finally, if $\operatorname{char} k = 0$ and E is very general in the linear system $|\mathcal{I}_Z(d)|$ then by Proposition 4.5.4 (c), $\operatorname{Cl} Z(E)$ is freely generated by $\mathcal{O}_{Z(E)}(1)$ and C. $\square$

6 Triple conics in $\mathbb{P}^3$

In this final chapter we describe triple conics in $\mathbb{P}^3$. In Section 6.1 we prove a theorem regarding the existence and constructions of triple conics. In Sections 6.2, 6.3 and 6.4 we give total ideal descriptions of triple conics whose underlying double conics are planar, complete intersections of two quadrics, and have negative odd genus respectively.

6.1 Classification of triple conics

In this section we prove the classification theorem of triple conics in $\mathbb{P}^3$. In particular, we give the range of (ℓ, c) for which there exists a quasi-primitive triple conic of type (ℓ, c).

Proposition 6.1.1. Let Z be a double conic on C of type ℓ. Then

$$\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C[-\ell-6] \oplus \mathcal{O}_C[2\ell]. \tag{110}$$

Proof. By Corollaries 5.3.3, 5.3.4 and Propositions 5.3.6, 5.3.15 we have

$$\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \begin{cases} \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-4), & \text{if } \ell = -4 \\ \mathcal{O}_C^2(-2), & \text{if } \ell = -2 \\ \mathcal{O}_C(-a-3) \oplus \mathcal{O}_C(2a), & \text{if } \ell = 2a \geq 0 \\ \mathcal{O}_C[-2a-7] \oplus \mathcal{O}_C(2a+1), & \text{if } \ell = 2a+1 \geq -1 \end{cases} \tag{111}$$

and hence (110). □

Corollary 6.1.2. Let Z be the double conic on C of type -4. Then there exists a surjection $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[-8+c]$ if and only if $c = 0$ or $c \geq 6$.

Proof. Let $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[-8+c]$ be a surjection, where $c \geq 0$. By Proposition 6.1.1, we have $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C[-2] \oplus \mathcal{O}_C[-8]$ and hence the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_C[-2] \oplus \mathcal{O}_C[-8] & \xrightarrow{\psi} & \mathcal{O}_C[-8+c] \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-8) & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^1}(-8+c), \end{array} \tag{112}$$

where $\tau : \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-8) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-8+c)$ is the map corresponding to ψ as in Lemma 5.1.10. Then $\tau = (p, r)$, where p and r are homogeneous polynomials in T with $\deg p = c - 6$ and $\deg r = c$. Since τ is a surjection, p and r have no common zeros. Now $\deg p < 0 \Leftrightarrow c < 6$. Hence $p = 0 \Leftrightarrow c \leq 5$. But if $p = 0$ then r must be a constant, otherwise p and r will have some common zeros. Therefore we must have $\deg r = c = 0$. Thus for $1 \leq c \leq 5$ there does not exist any surjection τ and hence ψ. Conversely, let $c = 0$ or $c \geq 6$. If $c = 0$ then $\tau = (0, 1)$ defines a surjection. Now suppose $c \geq 6$. Let $\tau : T(-2) \oplus T(-8) \to T(-8+c)$ be the map given by $\tau = (p, r)$, where $p = s^{c-6}$ and $r = t^c$. Notice $\deg p \geq 0$, since $c \geq 6$. Therefore p and r have no common zeros by construction. Hence $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Therefore τ sheafifies to a surjection $\widetilde{\tau} : \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-8) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-8+c)$ by Lemma 2.2.4. Let $\psi : \mathcal{O}_C[-2] \oplus \mathcal{O}_C[-8] \twoheadrightarrow \mathcal{O}_C[-8+c]$ be the map corresponding to $\widetilde{\tau}$ as in Lemma 5.1.10. Then $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[-8+c]$ is a surjection. □

Corollary 6.1.3. Let Z be a double conic on C of type $\ell \geq -2$. Then there exists a surjection $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell + c]$ for all $c \geq 0$.

Proof. By Proposition 6.1.1, $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \cong \mathcal{O}_C[-\ell-6] \oplus \mathcal{O}_C[2\ell]$. Let $c \geq 0$ be an integer and let $\tau : T(-\ell-6) \oplus T(2\ell) \to T(2\ell + c)$ be the map given by $\tau = (p, r)$, where $p = s^{3\ell+c+6}$ and $r = t^c$. Notice, $3\ell + c + 6 \geq c \geq 0$, since $\ell \geq -2$. Hence $\deg p \geq 0$. Therefore p and r have no common zeros by construction. Hence $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Therefore τ sheafifies to a surjection $\widetilde{\tau} : \mathcal{O}_{\mathbb{P}^1}(-\ell-6) \oplus \mathcal{O}_{\mathbb{P}^1}(2\ell) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(2\ell + c)$ by Lemma 2.2.4. Let $\psi : \mathcal{O}_C[-\ell-6] \oplus \mathcal{O}_C[2\ell] \twoheadrightarrow \mathcal{O}_C[2\ell + c]$ be the map corresponding to $\widetilde{\tau}$ as in Lemma 5.1.10. Then $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell + c]$ is a surjection. □

Proposition 6.1.4. Let W be the thick triple conic on C, i.e., $\mathcal{I}_W = \mathcal{I}_C^2$. Then $I_W = I_C^2$. Moreover, W can be obtained from either of the following maps.

(a) I_W is the kernel of the map $I_Z \to I_Z/I_CI_Z \cong S_C(-1) \oplus S_C(-4) \xrightarrow{\psi} S_C(-1)$, where Z is the double conic of type -4 and $\psi = (1, 0)$.

(b) I_W is the kernel of the map $I_Z \to I_Z/I_CI_Z \cong S_C(-2)^2 \xrightarrow{\xi} S_C(-2)$, where Z is a double conic of type -2 and $\xi = (0, 1)$.

Proof. We have the complex

$$0 \to S(-4) \oplus S(-5) \xrightarrow{\varphi} S(-2) \oplus S(-3) \oplus S(-4) \to I_C^2 \to 0, \tag{113}$$

where φ is given by the matrix

$$\varphi = \begin{pmatrix} q & 0 \\ -x & q \\ 0 & -x \end{pmatrix}.$$

Notice rank $\varphi = 2$ and $I(\varphi) = I_C^2$. Since $\{x, q\}$ is a regular sequence in S, $\{x^2, q^2\}$ is also a regular sequence in S by [26, Theorem 16.1]. Hence depth $I(\varphi) \geq 2$. Therefore (113) is exact and hence an S-resolution of I_C^2 by the Hilbert-Burch theorem 2.1.22. Sheafifying (113) we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-5) \xrightarrow{\tilde{\varphi}} \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{I}_C^2 = \mathcal{I}_W \to 0. \quad (114)$$

Applying H^0_* to (114) we get the exact sequence

$$0 \to S(-4) \oplus S(-5) \xrightarrow{\varphi} S(-2) \oplus S(-3) \oplus S(-4) \to I_W \to 0, \quad (115)$$

since $H^1_*(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-5)) = 0$ by [18, III, Theorem 5.1]. Compairing the exact sequences (113) and (115) we see that $I_W = I_C^2$.

Let Z be the double conic of type -4. Then $I_Z = (x, q^2)$ and $I_Z/I_CI_Z \cong S_C(-1) \oplus S_C(-2)$ by Corollary 5.3.3. Let $\phi : I_Z \to S_C(-1)$ be the map $\phi = \psi \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_CI_Z$

is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & I_Z & \xrightarrow{\phi} & S_C(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \pi & & \| & & \\
0 & \longrightarrow & \operatorname{Ker}\phi/I_CI_Z & \longrightarrow & I_Z/I_CI_Z \cong S_C(-1)\oplus S_C(-4) & \xrightarrow{\psi} & S_C(-1) & \longrightarrow & 0.
\end{array}$$

Let e_1, e_2 be the generators of $S_C(-1) \oplus S_C(-4)$. Since $I_Z/I_CI_Z \cong S_C(-1) \oplus S_C(-4)$, we can identify $\bar{x}$ with e_1 and $\bar{q}^2$ with e_2. Notice, $\operatorname{Ker}\psi$ is generated by $\bar{q}^2$. Therefore $\operatorname{Ker}\phi = (I_CI_Z, q^2) = (x^2, xq, xq^2, q^3, q^2) = (x^2, xq, q^2) = I_W$.

Now suppose Z is a double conic of type -2. Then $I_Z = (x^2, q - gx)$, where $g \in S$ is a linear form, and $I_Z/I_CI_Z \cong S_C(-2)^2$ by Corollary 5.3.4. Let $\phi : I_Z \to S_C(-2)$ be the map $\phi = \xi \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_CI_Z$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\phi & \longrightarrow & I_Z & \xrightarrow{\phi} & S_C(-2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \pi & & \| & & \\
0 & \longrightarrow & \operatorname{Ker}\phi/I_CI_Z & \longrightarrow & I_Z/I_CI_Z \cong S_C(-2)^2 & \xrightarrow{\xi} & S_C(-2) & \longrightarrow & 0.
\end{array}$$

Let e_1, e_2 be the generators of $S_C(-2)^2$. Since $I_Z/I_CI_Z \cong S_C(-2)^2$, we can identify $\bar{x}^2$ with e_1 and $\overline{q - gx}$ with e_2. Notice, $\operatorname{Ker}\psi$ is generated by $\bar{x}^2$. Therefore

$$\operatorname{Ker}\phi = (I_CI_Z, x^2) = (x^3, xq - gx^2, x^2q, q^2 - gxq, x^2) = (x^2, xq, q^2) = I_W.$$

$\square$

Theorem 6.1.5. Let Z be a CM double conic on C of type ℓ, where $\ell \geq -4$ is an integer such that $\ell \neq -3$. Let $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \to \mathcal{O}_C[2\ell + c]$ be a surjection, where $c \geq 0$ is an integer. Then ψ defines a CM triple conic W on C with Hilbert polynomial $P_W(n) = 6n + 3\ell + c + 3$ by $\mathcal{I}_W = \operatorname{Ker} \psi \circ \pi$, where $\pi : \mathcal{I}_Z \twoheadrightarrow \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z$ is the canonical surjection. Conversely, every CM triple conic W on C arises from this construction.

Proof. Let $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \to \mathcal{O}_C[2\ell + c]$ be a surjection, where $c \geq 0$ is an integer. Let $\varphi : \mathcal{I}_Z \to \mathcal{O}_C[2\ell + c]$ be the surjection $\varphi = \psi \circ \pi$. Then $\operatorname{Ker} \varphi$ has the form $\mathcal{I}_W/\mathcal{I}_C\mathcal{I}_Z$, where $W \subset \mathbb{P}^3$ is a closed subscheme. We get the exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{O}_C[2\ell + c] \to 0. \tag{116}$$

By Lemma 3.3.5, W is a CM multiplicity structure on C. From (116) we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_W & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_W & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0. \\
 & & \big\downarrow & & & & & & \\
 & & \mathcal{O}_C[2\ell + c] & & & & & &
\end{array} \tag{117}$$

Applying the snake lemma to (117) we get the exact sequence

$$0 \to \mathcal{O}_C[2\ell + c] \to \mathcal{O}_W \to \mathcal{O}_Z \to 0. \tag{118}$$

Twisting by n and taking the Euler characteristics of the sheaves in (118) we get

$$\begin{aligned} P_W(n) = \chi\mathcal{O}_W(n) &= \chi\mathcal{O}_Z(n) + \chi\mathcal{O}_C[2\ell + c](n) \\ &= \chi\mathcal{O}_Z(n) + \chi\mathcal{O}_{\mathbb{P}^1}(2\ell + c + 2n) \\ &= 6n + 3\ell + c + 3. \end{aligned}$$

Hence $\deg W = 6$ and therefore W is a triple conic on C.

Conversely, let W be a CM triple conic on C. If W is a thick extension then by Proposition 6.1.4, W arises by this construction. Now suppose W is a quasi-primitive extension. Let Z be the 2^{nd} CM filtrant of W. Set $\mathcal{L} := \mathcal{I}_C/\mathcal{I}_Z$ and $\mathcal{L}_2 := \mathcal{I}_Z/\mathcal{I}_W$. Notice, $\mathcal{L}$ is a line bundle on C by Proposition 4.3.2. Hence $\mathcal{L} \cong \mathcal{O}_C[\ell]$ for some $\ell \in \mathbb{Z}$. Moreover $\mathcal{L}_2 = \mathcal{L}^2(D_2)$ for some effective divisor D_2 on C by Proposition 4.3.5. Therefore $\mathcal{L}_2 = \mathcal{O}_C[2\ell + c]$ where $c = \deg D_2 \geq 0$. Moreover the map $\mathcal{I}_Z \to \mathcal{O}_C[2\ell + c]$ factors through $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z$, so W arises from this construction. □

Corollary 6.1.6. Let W be the thick triple conic on C. If W arises from a surjection $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell + c]$ as in Theorem 6.1.5, then $(\ell, c) = (-4, 6)$ or $(-2, 0)$.

Proof. We have $\mathcal{I}_W = \mathcal{I}_C^2$. Since $C \subset W$ is the CM filtration of W, we get the exact sequence

$$0 \to \mathcal{I}_C/\mathcal{I}_W = \mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2) \to \mathcal{O}_W \to \mathcal{O}_C \to 0 \qquad (119)$$

by Proposition 4.2.5 (3). Twisting by n and taking the Euler characteristics of the sheaves in (119) we get $P_W(n) = \chi\mathcal{O}_W(n) = \chi\mathcal{O}_C(n) + \chi\mathcal{O}_C(n-1) + \chi\mathcal{O}_C(n-2) = 6n - 3$.

Let $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell + c]$ be a surjection that defines W. Then by Theorem 6.1.5, $P_W(n) = 6n + 3\ell + c + 3$. Therefore ψ defines the thick triple conic on C if and only if $6n+3\ell+c+3 = 6n-3$, i.e., $3\ell+c+6 = 0$. Notice, if $\ell \geq -1$ then $3\ell+c+6 > 0$. Therefore we must have $\ell = -4$ or -2, since $\ell \geq -4$ and $\ell \neq -3$. Hence the only solutions to the equation $3\ell + c + 6 = 0$ are $(-4, 6)$ and $(-2, 0)$. □

Proposition 6.1.7. Let Z be a double conic on C of type $2a$, where $a \geq -2$. Then there exists a canonical inclusion

$$\iota : I_Z \otimes S_C \subseteq S_C(-a-3) \oplus S_C(2a) \tag{120}$$

such that $\operatorname{Coker} \iota$ has finite length.

Proof. From Corollaries 5.3.3, 5.3.4 and Proposition 5.3.6 we have

$$I_Z \otimes S_C \cong \begin{cases} S_C(-1) \oplus S_C(-4), & \text{if } a = -2 \\ S_C(-2)^2, & \text{if } a = -1 \\ S_C(-a-3) \oplus (f,g)^2(2a), & \text{if } a \geq 0. \end{cases} \tag{121}$$

Let ι be the isomorphisms in (121) if $a = -2, -1$ and the inclusion $S_C(-a-3) \oplus (f,g)^2(2a) \subseteq S_C(2a)$ if $a \geq 0$. Notice $\operatorname{Coker} \iota = 0$ if $a = -2, -1$. Finally, if $a \geq 0$ then $\operatorname{Coker} \iota$ has finite length by Lemma 2.1.9, since the images of f and g form a regular sequence in S_C. Therefore we get (120). □

Let Z be a double conic on C of type $2a$, where $a \geq -2$. Let $\pi : I_Z \twoheadrightarrow I_Z \otimes S_C$ be the canonical surjection and let $\iota : I_Z \otimes S_C \subseteq S_C(-a-3) \oplus S_C(2a)$ be the canonical inclusion as in Proposition 6.1.7. Let $c \geq 0$ be an integer and let $\psi : S_C(-a-3) \oplus S_C(2a) \to S_C[2\ell + c]$ be a map such that $\operatorname{Coker} \psi$ has finite length. Define $\phi = \psi \circ \iota \circ \pi$. Let τ be the map corresponding to ψ as in Lemma 5.1.10. Then we have the commutative diagram

$$\begin{array}{ccccc}
I_Z & & \xrightarrow{\phi} & & S_C[4a+c] \\
\downarrow \pi & & & & \| \\
I_Z \otimes S_C & \xhookrightarrow{\iota} & S_C(-a-3) \oplus S_C(2a) & \xrightarrow{\psi} & S_C[4a+c] \\
 & & \downarrow j & & \downarrow \\
 & & T(-2a-6) \oplus T(4a) & \xrightarrow{\tau} & T(4a+c)
\end{array} \tag{122}$$

where j is the inclusion as in (5.1.7).

Theorem 6.1.8. In the setting of diagram (122), $\operatorname{Ker} \phi$ is the total ideal of a CM triple conic W on C. Moreover, $I_W = I_C I_Z + (j \circ \iota \circ \pi)^{-1} \operatorname{Ker}(\tau)$.

Proof. By construction, $\operatorname{Coker} \phi$ has finite length. Hence by Lemma 2.2.4, ϕ sheafifies to the surjection $\varphi : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[4a+c]$, where $\varphi = \widetilde{\phi}$. Therefore by Theorem 6.1.5, $\operatorname{Ker} \varphi$ is the ideal sheaf of a CM triple conic W on C. We have the exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{O}_C[4a+c] \to 0. \tag{123}$$

Applying H^0_* to (123) we get the exact sequence

$$0 \to I_W \to I_Z \xrightarrow{H^0_* \varphi} S_C[4a+c] \to H^1_* W \to H^1_* Z.$$

Since $H^0_*\varphi = \phi$, we have $I_W = \operatorname{Ker}\phi$. Therefore $\operatorname{Ker}\phi$ is saturated. Finally let $\tau = (p, r)$, where p and r are homogeneous polynomials in T having no common zeros such that $\deg p = 6a + c + 6$ and $\deg r = c$. Then $\operatorname{Ker}\tau$ is generated by the Koszul relation $\eta = pe_2 - re_1$, where e_1 and e_2 are the generators of $T(-2a-6) \oplus T(4a)$. Therefore $\operatorname{Ker}\phi = I_C I_Z + ((j \circ \iota \circ \pi)^{-1}(\eta)) = I_C I_Z + (j \circ \iota \circ \pi)^{-1}\operatorname{Ker}(\tau)$. □

Remark 6.1.9. Let W be a triple conic as in Theorem 6.1.8. If W is a quasi-primitive extension of C then it is of type (ℓ, c) and has Z as its 2^{nd} CM filtrant.

6.2 Triple conics arising from planar double conics

In this section we describe triple conics that arise from planar double conics. Let Z be a planar double conic on C. Then Z is of type -4. Also $I_Z = (x, q^2)$ and $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$ by Corollary 5.3.3.

Proposition 6.2.1. Let Z be the planar double conic on C. Let W be a triple conic on C defined by a surjection $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(b-4)$, where $b \geq 0$.

(a) If $b = 0$ then $I_W = (x, q^3)$, i.e., W is a complete intersection.

(b) If $b \geq 3$ then $I_W = (I_C I_Z, Pq^2 - Rx)$, where P and R are homogeneous polynomials in S of degrees $b-3$ and b respectively, having no common zeros along C.

Moreover, W is a thick triple conic $\Leftrightarrow b = 3$ and $R \in I_C$.

Proof. We have $I_Z = (x, q^2)$ and $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$ by Corollary 5.3.3. By Corollary 6.1.2, there does not exist any triple conic if $b = 1, 2$. Let $b = 0$ or $b \geq 3$ and let $\psi : I_Z/I_C I_Z \to S_C(b-4)$ be a map such that $\operatorname{Coker}\psi$ has finite length. Then

$\psi = (\bar{P}, \bar{R})$, where $\bar{P}, \bar{R}$ are the images of homogeneous polynomials $P, R \in S$ in S_C with $\deg P = b - 3$ and $\deg R = b$, having no common zeros along C. Let $\phi : I_Z \to S_C(b-4)$ be the map defined as follows

$$\begin{array}{ccc} I_Z & \xrightarrow{\quad\quad\phi\quad\quad} & S_C(b-4) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\| \\ I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4) & \xrightarrow{\psi} & S_C(b-4) \end{array}$$

where π is the canonical surjection. Then by Theorem 6.1.8, ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(b-4)$ and defines a CM triple conic W on C with $I_W = \operatorname{Ker}\phi = I_C I_Z + \pi^{-1}\operatorname{Ker}\psi$. Since P and R have no common zeros along C, $\operatorname{Ker}\psi$ is generated by the Koszul relation $Pe_2 - Re_1$, where e_1 and e_2 are the generators of $S_C(-1) \oplus S_C(-4)$. Since $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$, we can identify e_1 with $\bar{x}$ and and e_2 with $\bar{q}^2$, where $\bar{x}, \bar{q}$ are the images of x, q in S_C respectively. Therefore $\operatorname{Ker}\psi$ is generated by $P\bar{q}^2 - R\bar{x}$ and hence by Theorem 6.1.8, $I_W = (I_C I_Z, Pq^2 - Rx)$. This proves part (b) of the proposition. Now suppose $b = 0$. Then $\deg P = -3$ and $\deg R = 0$. Hence $P = 0$ and R is a unit. Hence $I_W = (I_C I_Z, -Rx) = (x^2, xq, xq^2, q^3, -Rx) = (x, q^3)$, since R is a unit. Hence W is a complete intersection. This proves part (a) of the proposition.

Finally, let W be a thick triple conic. Then $b \neq 0$ by part (a) above. Hence $b \geq 3$. By Proposition 6.1.4, $I_W = I_C^2$. Hence $q^2 \in I_W = (I_C I_Z, Pq^2 - Rx)$ and therefore $Rx \in I_W = I_C^2$. Let $Rx = \alpha x^2 + \beta xq + \gamma q^2$, where $\alpha, \beta, \gamma \in S$. Then $x \mid \gamma$. Let $\gamma = \gamma' x$. Thus $R = \alpha x + \beta q + \gamma' q^2 \in (x, q) = I_C$, i.e., $\bar{R} = 0$. Since P and R have no common zeros along C, P must be a constant. Therefore $\deg P = b - 3 = 0$, i.e., $b = 3$. Conversely, let $b = 3$ and $R \in I_C$. Then $\deg P = 0$ and hence P is a unit. Replacing the map

ψ by $P^{-1}\psi$ we get the same triple conic. Therefore we can assume that $P = 1$. Thus $I_W = (I_C I_Z, q^2 - Rx) = (x^2, xq, q^3, q^2 - Rx)$. Since $R \in I_C$ there exist $\alpha, \beta \in S$ such that $R = \alpha x + \beta q$. Hence $Rx = \alpha x^2 + \beta xq$ and therefore $I_W = (x^2, xq, q^2) = I_C^2$, i.e., W is a thick triple conic on C. $\square$

Let Z be the double conic on C of type -4. By Corollary 5.3.3, we have $I_Z = (x, q^2)$ and $I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4)$. Let $b \geq 3$ be an integer and let $\tau : T(-2) \oplus T(-8) \to T(2b-7)$ be the map given by $\tau = (p, r)$, where $\{p, r\}$ is a regular sequence in T with $\deg p = 2b - 5$ and $\deg r = 2b + 1$. Let $\psi : S_C(-1) \oplus S_C(-4) \to S_C[2b-7]$ be the map corresponding to τ as in Lemma 5.1.10. Define $\phi = \psi \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_C I_Z$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccc} I_Z & \xrightarrow{\quad\phi\quad} & S_C[2b-7] \\ \downarrow{\scriptstyle \pi} & & \| \\ I_Z/I_C I_Z \cong S_C(-1) \oplus S_C(-4) & \xrightarrow{\psi} & S_C[2b-7] \\ \downarrow{\scriptstyle j} & & \downarrow \\ T(-2) \oplus T(-8) & \xrightarrow{\quad\tau\quad} & T(2b-7) \end{array} \tag{124}$$

where j is the inclusion as in (5.1.7). Notice, $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Therefore $\operatorname{Coker} \psi$ and hence $\operatorname{Coker} \phi$ have finite lengths. Also notice, since $\deg p$ and $\deg r$ are odd, $\{p, r\}$ does not lift to a regular sequence in S_C, rather it lifts to an admissible pair of sequences on C.

Proposition 6.2.2. In the setting of diagram (124), τ defines a triple conic W on C of type $(-4, 2b+1)$, having Z as the 2^{nd} CM filtrant. Moreover,

$$I_W = (I_C I_Z, P_1 q^2 - R_1 x, P_2 q^2 - R_2 x),$$

where $\{P_1, R_1\}, \{P_2, R_2\}$ is an admissible pair of sequences on C corresponding to $\{p, r\}$.

Proof. By Theorem 6.1.8, ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2b-7]$ and defines a CM triple conic W on C with $I_W = I_C I_Z + (j \circ \pi)^{-1} \operatorname{Ker} \tau$. Since $\{p, r\}$ is a regular sequence in T, $\operatorname{Ker} \tau$ is generated by the Koszul relation $\eta = p\hat{e}_2 - r\hat{e}_1$, where $\hat{e}_1$ and $\hat{e}_2$ are the generators of $T(-2) \oplus T(-8)$. Notice $j^{-1}(\eta) \notin S_C(-1) \oplus S_C(-4)$, since $\deg \eta = 2b+3$, i.e., $\deg \eta$ is odd. Hence $j^{-1} \operatorname{Ker} \tau$ is generated by $j^{-1}(s\eta)$ and $j^{-1}(t\eta)$. Now $s\eta = sp\hat{e}_2 - sr\hat{e}_1$ and $t\eta = tp\hat{e}_2 - tr\hat{e}_1$. Therefore $j^{-1}(s\eta) = \bar{P}_1 e_2 - \bar{R}_1 e_1$ and $j^{-1}(t\eta) = \bar{P}_2 e_2 - \bar{R}_2 e_1$, where e_1 and e_2 are the generators of $S_C(-1) \oplus S_C(-4)$. Since $I_Z / I_C I_Z \cong S_C(-1) \oplus S_C(-4)$, we can identify e_1 and e_2 with $\bar{x}$ and $\bar{q}^2$ respectively, where $\bar{x}$ and $\bar{q}$ are the images of x and q in S_C respectively. Therefore $(j \circ \pi)^{-1} \operatorname{Ker} \tau = (P_1 q^2 - R_1 x, P_2 q^2 - R_2 x)$ and hence $I_W = (I_C I_Z, P_1 q^2 - R_1 x, P_2 q^2 - R_2 x)$. Since $2b+1$ is odd, W is not a thick extension by Corollary 6.1.6. Hence W is a triple conic on C of type $(-4, 2b+1)$, having Z as the 2^{nd} CM filtrant. □

6.3 Triple conics arising from a complete intersection of quadrics

In this section we describe triple conics that arise from double conics which are complete intersections of two quadrics.

Proposition 6.3.1. Let Z be a double conic on C of type -2 with $I_Z = (x^2, q - gx)$, where $g \in S$ is a linear form. If W is a triple conic on C defined by a map $\psi : I_Z/I_C I_Z \to S_C(-2)$, then $\psi = (\lambda, \delta)$ for some $\lambda, \delta \in k$ such that $(\lambda, \delta) \neq (0, 0)$. Moreover,

(a) If $\lambda \neq 0$ then $I_W = (x^3, q - gx - \delta x^2)$, i.e., W is a complete intersection.

(b) If $\lambda = 0$ then W is the thick triple conic.

Proof. We have $I_Z/I_C I_Z \cong S_C(-2)^2$ by Corollary 5.3.4. Hence $I_Z/I_C I_Z \cong S_C(-2)^2$. Therefore $\psi = (\lambda, \delta)$, where $\lambda, \delta \in k$. Notice $(\lambda, \delta) \neq (0, 0)$, for otherwise ψ is the zero map and hence does not define any triple conic on C. Let $\phi : I_Z \to S_C(-2)$ be the map defined as follows

$$\phi : I_Z \xrightarrow{\pi} I_Z/I_C I_Z \cong S_C(-2)^2 \xrightarrow{\psi} S_C(-2),$$

where π is the canonical surjection. Then ϕ sheafifies to the surjection $\varphi : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(-2)$, where $\varphi = \widetilde{\phi}$, and $\operatorname{Ker} \varphi = \mathcal{I}_W$. By Theorem 6.1.8, $I_W = I_C I_Z + \pi^{-1} \operatorname{Ker} \psi$. Since λ and δ have no common zeros along C, $\operatorname{Ker} \psi$ is generated by the Koszul relation $\lambda e_2 - \delta e_1$, where e_1 and e_2 are the generators of $S_C(-2)^2$. Since $I_Z/I_C I_Z \cong S_C(-2)^2$, we can identify e_1 and e_2 with $\bar{x}^2$ and $\overline{q - gx}$ respectively; where $\bar{x}, \overline{q - gx}$ are the images of $x, q - gx$ in S_C respectively. Therefore $\operatorname{Ker} \psi$ is generated by $\lambda \overline{(q - gx)} - \delta \bar{x}^2$ and hence $I_W = (I_C I_Z, \lambda(q - gx) - \delta x^2)$.

Now if $\lambda \neq 0$ then replacing ψ by $\lambda^{-1}\psi$ we get the same triple conic. Hence we can assume that $\lambda = 1$. Therefore $I_W = (I_C I_Z, q - gx - \delta x^2) = (x^3, x(q-gx), x^2q, q(q-gx), q - gx - \delta x^2)$. Notice $\delta x^3 + x(q - gx - \delta x^2) = x(q - gx)$ and $\delta x^2 q + q(q - gx - \delta x^2) = q(q - gx)$. Hence $I_W = (x^3, x^2q, q - gx - \delta x^2)$. Finally, $(g + \delta x)x^3 + x^2(q - gx - \delta x^2) = x^2q$. Therefore $I_W = (x^3, q - gx - \delta x^2)$ and hence W is a complete intersection.

Now suppose $\lambda = 0$. Then $\delta \neq 0$. Replacing ψ by $\delta^{-1}\psi$ we get the same triple conic. Hence we can assume that $\delta = 1$. Therefore

$$I_W = (I_C I_Z, x^2) = (x^3, x(q - gx), x^2q, q(q - gx), x^2) = (x^2, xq, x^2) = I_C^2$$

and hence W is the thick triple conic. □

Corollary 6.3.2. Let W be the thick triple conic on C. Then W arises from a surjection $\psi : \mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2\ell+c]$ as in Theorem 6.1.5 if and only if $(\ell, c) = (-4, 6)$ and $\psi = (1, 0)$ or $(\ell, c) = (-2, 0)$ and $\psi = (0, 1)$.

Proof. By Corollary 6.1.6, we have $(\ell, c) = (-4, 6)$ or $(-2, 0)$. Now if $(\ell, c) = (-4, 6)$ then ψ is the surjection $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(-1)$, where Z is the double conic on C of type -4. Therefore by Proposition 6.2.1, ψ defines a thick triple conic if and only if $\psi = (1, 0)$. On the other hand, if $(\ell, c) = (-2, 0)$ then ψ is the surjection $\mathcal{I}_Z/\mathcal{I}_C\mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(-2)$, where Z is a double conic on C of type -2. Therefore by Proposition 6.3.1, ψ defines a thick triple conic if and only if $\psi = (0, 1)$. □

Corollary 6.3.3. Let $\mathfrak{S} = \{(-4, 0)\} \cup \{(-4, c) | c \geq 6\} \cup \{(\ell, c) | \ell \geq -2 \text{ and } c \geq 0\} \subseteq \mathbb{Z} \times \mathbb{Z}$. Then there exists a quasi-primitive triple conic W on C of type $(\ell, c) \Leftrightarrow (\ell, c) \in \mathfrak{S}$.

Proof. This follows from Corollaries 6.1.2, 6.1.3, 6.3.2 and Propositions 6.2.1, 6.3.1. □

Proposition 6.3.4. Let Z be a double conic on C of type -2. Let $b \geq 1$ be an integer and let P, R be homogeneous polynomials in S of degree b, having no common zeros along C. Let $\psi = (\bar{P}, \bar{R})$, where $\bar{P}, \bar{R}$ are the images of P, R in S_C respectively. Then ψ defines a CM triple conic W on C of type $(-2, 2b)$, having Z as the 2^{nd} CM filtrant. Moreover, $I_W = (I_C I_Z, P(q - gx) - Rx^2)$.

Proof. We have $I_Z = (x^2, q - gx)$ and $I_Z/I_C I_Z \cong S_C(-2)^2$ by Corollary 5.3.4. Since P and R have no common zeros along C, $\operatorname{Coker} \psi$ has finite length by Lemma 2.1.9. Let $\phi : I_Z \to S_C(b-2)$ be the map defined as follows

$$\phi : I_Z \xrightarrow{\pi} I_Z/I_C I_Z \cong S_C(-2)^2 \xrightarrow{\psi} S_C(b-2),$$

where π is the canonical surjection. By Theorem 6.1.8, ϕ sheafifies to the surjection $\varphi : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(b-2)$, where $\varphi = \widetilde{\phi}$. Moreover, $\operatorname{Ker} \varphi$ is the ideal sheaf of a CM triple conic W on C with $I_W = I_C I_Z + \pi^{-1} \operatorname{Ker} \psi$. Notice, $\{\bar{P}, \bar{R}\}$ is a regular sequence in S_C. Hence $\operatorname{Ker} \psi$ is generated by the Koszul relation $Pe_2 - Re_1$, where e_1 and e_2 are the generators of $S_C(-2)^2$. Since $I_Z/I_C I_Z \cong S_C(-2)^2$, we can identify e_1 and e_2 with $\bar{x}^2$ and $\overline{q - gx}$ respectively, where $\bar{x}$ and $\overline{q - gx}$ are the images of x and $q - gx$ in S_C respectively. Therefore $\operatorname{Ker} \psi$ is generated by $P\overline{(q - gx)} - R\bar{x}^2$ and hence $I_W = (I_C I_Z, P(q - gx) - Rx^2)$. By Corollary 6.3.2, W is not a thick extension, since $b \geq 1$. Therefore W is of type $(-2, 2b)$ and Z is the 2^{nd} CM filtrant of W. □

Let Z be a double conic on C of type -2. Then $I_Z/I_C I_Z \cong S_C(-2)^2$ by Corollary 5.3.4. Let $b \in \mathbb{Z}_{\geq 0}$ and let $\tau : T(-4)^2 \to T(2b-3)$ be the map given by $\tau = (p, r)$, where $\{p, r\}$ is a regular sequence in T with $\deg p = \deg r = 2b+1$. Let $\psi : S_C(-2)^2 \to S_C[2b-3]$ be the map corresponding to τ as in Lemma 5.1.10. Define $\phi = \psi \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_C I_Z$ is the canonical surjection. Then we have the commutative diagram

$$\begin{array}{ccc} I_Z & \xrightarrow{\quad\phi\quad} & S_C[2b-3] \\ \Big\downarrow{\scriptstyle \pi} & & \Big\| \\ I_Z/I_C I_Z \cong S_C(-2)^2 & \xrightarrow{\quad\psi\quad} & S_C[2b-3] \\ \Big\downarrow{\scriptstyle j} & & \Big\downarrow \\ T(-4)^2 & \xrightarrow{\quad\tau\quad} & T(2b-3) \end{array} \tag{125}$$

where j is the inclusion as in (5.1.7). Notice, $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Therefore $\operatorname{Coker} \psi$ and hence $\operatorname{Coker} \phi$ have finite lengths. Also notice, since $\deg p$ and $\deg r$ are odd, $\{p, r\}$ does not lift to a regular sequence in S_C, rather it lifts to an admissible pair of sequences on C.

Proposition 6.3.5. In the setting of diagram (125), τ defines a triple conic W on C of type $(-2, 2b+1)$, having Z as the 2^{nd} CM filtrant. Moreover,

$$I_W = (I_C I_Z, P_1(q - gx) - R_1 x^2, P_2(q - gx) - R_2 x^2),$$

where $\{P_1, R_1\}, \{P_2, R_2\}$ is an admissible pair of sequences on C corresponding to $\{p, r\}$.

Proof. By Theorem 6.1.8, ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[2b-3]$ and defines a CM triple conic W on C with $I_W = \operatorname{Ker} \phi = I_C I_Z + (j \circ \pi)^{-1} \operatorname{Ker} \tau$. Since $\{p, r\}$ is a regular sequence in T, $\operatorname{Ker} \tau$ is generated by the Koszul relation $\eta = p\hat{e}_2 - r\hat{e}_1$, where $\hat{e}_1$ and $\hat{e}_2$ are the generators of $T(-4)^2$. Then $\operatorname{Ker} \tau$ is generated by η. Notice $j^{-1}(\eta) \notin S_C(-2)^2$, since

$\deg \eta = 2b + 5$, i.e., $\deg \eta$ is odd. Hence $j^{-1} \operatorname{Ker} \tau$ is generated by $j^{-1}(s\eta)$ and $j^{-1}(t\eta)$. Now $s\eta = sp\hat{e}_2 - sr\hat{e}_1$ and $t\eta = tp\hat{e}_2 - tr\hat{e}_1$. Therefore $j^{-1}(s\eta) = \bar{P}_1 e_2 - \bar{R}_1 e_1$ and $j^{-1}(t\eta) = \bar{P}_2 e_2 - \bar{R}_2 e_1$, where e_1 and e_2 are the generators of $S_C(-2)^2$. Since $I_Z/I_C I_Z \cong S_C(-2)^2$, we can identify e_1 and e_2 with $\bar{x}^2$ and $\overline{q - gx}$ respectively, where $\bar{x}$ and $\overline{q - gx}$ are the images of x and $q - gx$ in S_C respectively. Therefore $(j \circ \pi)^{-1} \operatorname{Ker} \tau = (P_1(q - gx) - R_1 x^2, P_2(q - gx) - R_2 x^2)$ and hence $I_W = (I_C I_Z, P_1(q - gx) - R_1 x^2, P_2(q - gx) - R_2 x^2)$. Since $2b + 1$ is odd, W is not a thick extension by Corollary 6.3.2. Hence W is a triple conic on C of type $(-2, 2b + 1)$, having Z as the 2nd CM filtrant. □

6.4 Triple conics arising from double conics of negative odd genus

Let Z be a double conic on C of type $2a$, where $a \geq 0$. Then $I_Z = (I_C^2, fq - gx)$ by Proposition 5.3.2, and $I_Z/I_C I_Z \cong S_C(-a - 3) \oplus (f, g)^2(2a)$ by Proposition 5.3.6. Let $b \geq 0$ be an integer. Let P and R be homogeneous polynomials in S of degrees $3a + b + 3$ and b respectively, having no common zeros along C. Let $\psi : S_C(-a - 3) \oplus S_C(2a) \to S_C(2a + b)$ be the map given by $\psi = (\bar{P}, \bar{R})$, where $\bar{P}$ and $\bar{R}$ are the images of P and R in S_C respectively. Then $\operatorname{Coker} \psi$ has finite length by Lemma 2.1.9. Let $\iota : S_C(-a - 3) \oplus (f, g)^2(2a) \to S_C(-a - 3) \oplus S_C(2a)$ be the canonical inclusion as in Proposition 6.1.7. Define $\phi = \psi \circ \iota \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_C I_Z$ is the canonical surjection.

Then we have the commutative diagram

$$\begin{array}{ccccc} I_Z & \xrightarrow{\phi} & & & S_C(2a+b) \\ \downarrow\!\!\downarrow \pi & & & & \| \\ I_Z/I_CI_Z & \xhookrightarrow{\iota} & S_C(-a-3)\oplus S_C(2a) & \xrightarrow{\psi} & S_C(2a+b). \end{array} \tag{126}$$

Proposition 6.4.1. In the setting of diagram (126), ϕ defines a triple conic W on C of type $(2a, 2b)$ with $I_W = \operatorname{Ker}\phi = I_CI_Z + (\iota\circ\pi)^{-1}\operatorname{Ker}\psi$, having Z as the 2[nd] CM filtrant. Moreover, I_W/I_CI_Z is cyclic $\Leftrightarrow P \in (f, g)^2 \bmod I_C$. In particular, if $P \in (f, g)^2 \bmod I_C$ then there exist $\alpha, \beta, \gamma \in S$ such that $I_W = (I_CI_Z, \alpha x^2 + \beta xq + \gamma q^2 - R(fq - gx))$.

Proof. By Theorem 6.1.8, ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C(2a+b)$ and defines a CM triple conic W on C with $I_W = \operatorname{Ker}\phi = I_CI_Z + (\iota\circ\pi)^{-1}\operatorname{Ker}\psi$. Since $a \geq 0$, W is a quasi-primitive triple conic on C of type $(2a, 2b)$ by Corollary 6.3.2. Hence Z as the 2[nd] CM filtrant of W. Since $\bar{P}$ and $\bar{R}$ have not common zeros in S_C, $\operatorname{Ker}\psi$ is generated by the Koszul relation $\eta = \bar{P}e_2 - \bar{R}e_1$, where e_1 and e_2 are the generators of $S_C(-a-3)\oplus S_C(2a)$. Since $I_Z/I_CI_Z \cong S_C(-a-3) \oplus (f, g)^2(2a)$, we can identify $\bar{x}^2, \bar{x}\bar{q}, \bar{q}^2$ and $\overline{fq - gx}$ with f^2, fg, g^2 and e_1 respectively. Then $e_2f^2 = \bar{x}^2, e_2fg = \bar{x}\bar{q}$ and $e_2g^2 = \bar{q}^2$. Since $fq - gx \in I_Z$, $\iota^{-1}(\eta) \in I_Z/I_CI_Z$ if and only if $P \in (f, g)^2 \bmod I_C$. Hence $I_W/I_CI_Z \cong (\iota\circ\pi)^{-1}\operatorname{Ker}\psi$ is cyclic $\Leftrightarrow P \in (f, g)^2 \bmod I_C$. Finally, let $P \in (f, g)^2 \bmod I_C$. Then there exist $\alpha, \beta, \gamma \in S$ such that $P = \bar{\alpha}f^2 + \bar{\beta}fg + \bar{\gamma}g^2$, where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are the images of α, β, γ in S_C. Thus $\eta = (\bar{\alpha}f^2 + \bar{\beta}fg + \bar{\gamma}g^2)e_2 - \bar{R}e_1$ and hence $\iota^{-1}(\eta) = \overline{\alpha x^2 + \beta xq + \gamma q^2 - R(fq - gx)}$. Therefore $I_W = (I_CI_Z, \alpha x^2 + \beta xq + \gamma q^2 - R(fq - gx))$. $\square$

Proposition 6.4.2. If $C \subset \mathbb{P}^3$ is a conic then $\dim S_C(l)_n = \begin{cases} 2(n+l)+1, & \text{if } n \geq -l \\ 0, & \text{otherwise} \end{cases}$ where $l, n \in \mathbb{Z}$.

Proof. Since $S_C(l)_n = S_C(n+l) = H^0(\mathbb{P}^3, \mathcal{O}_C(n+l)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2n+2l))$, we have $\dim S_C(l)_n = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2n+2l)) = \begin{cases} 2(n+l)+1, & \text{if } n \geq -l \\ 0, & \text{otherwise.} \end{cases}$ □

Proposition 6.4.3. Let $C \subset \mathbb{P}^3$ be a conic and let $f, g \in S$ be homogeneous polynomials with $\deg f = d, \deg g = e$. If the images of f and g in S_C form a regular sequence, then $\dim(S_C/(f,g))_n = 0$, whenever $n \geq d + e$.

Proof. The sequence

$$0 \to S_C(-d-e) \xrightarrow{\begin{pmatrix} -g \\ f \end{pmatrix}} S_C(-d) \oplus S_C(-e) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} S_C \to S_C/(f,g) \to 0$$

is exact, since the images of f and g in S_C form a regular sequence. Hence by Proposition 6.4.2 and some dimension counting, we have $\dim(S_C/(f,g))_n = 0$, whenever $n \geq d+e$. □

Proposition 6.4.4. Let $C \subset \mathbb{P}^3$ be a conic and let $f, g \in S$ be homogeneous polynomials with $\deg f = d \leq \deg g = e$. If $\{f, g\}$ is a regular sequence in S_C, then the sequence

$$S_C(-2d-e) \oplus S_C(-2e-d) \overset{\varphi_2}{\hookrightarrow} S_C(-2d) \oplus S_C(-d-e) \oplus S_C(-2e) \xrightarrow{\varphi_1} S_C \quad (127)$$

with $\varphi_1 = (f,g)^2$ and $\varphi_2 = \begin{pmatrix} g & 0 \\ -f & g \\ 0 & -f \end{pmatrix}$ is exact, hence right exact in degrees $\geq 2e+d$.

Proof. Notice φ_1 is generated by the 2×2 minors of φ_2. Since $\{f,g\}$ is a regular sequence in S_C, $\{f^2, g^2\}$ is also a regular sequence in S_C by [26, Theorem 16.1]. Thus depth $I_2(\varphi_2) \geq 2$. Therefore by the Hilbert-Burch theorem 2.1.22 the sequence

$$S_C(-2d-e)\oplus S_C(-2e-d) \xhookrightarrow{\varphi_2} S_C(-2d)\oplus S_C(-d-e)\oplus S_C(-2e) \xrightarrow{\varphi_1} (f,g)^2 \tag{128}$$

is exact. Thus (127) is exact. Now from (128) we have $\dim(f,g)^2_n = \dim S_C(n-2d) + \dim S_C(n-d-e) + \dim S_C(n-2e) - \dim S_C(n-2d-e) - \dim S_C(n-2e-d)$. Notice if $n \geq 2e+d$ then $n \geq 2d, d+e, 2e, 2d+e$, since by hypothesis $d \leq e$. After a brief calculation using Proposition 6.4.2 we see that $\dim(f,g)^2_n = 2n+1 = \dim(S_C)_n$, whenever $n \geq 2e+d$. Hence (127) is right exact in degrees $\geq 2e+d$. □

Corollary 6.4.5. Let $C \subset \mathbb{P}^3$ be a conic and let $f, g \in S$ be homogeneous polynomials with $\deg f = a+1, \deg g = a+2$; where $a \geq 0$. Let $M = S_C/(f,g)^2$. If the images of f and g in S_C form a regular sequence then

$$\dim M_n = \begin{cases} 0, & \text{if } n \geq 3a+5 \\ 1, & \text{if } n = 3a+4 \\ 3, & \text{if } n = 3a+3 \text{ and } a = 0 \\ 4, & \text{if } n = 3a+3 \text{ and } a \geq 1. \end{cases}$$

Proof. Noitce M is the cokernel of the map

$$S_C(-2a-2) \oplus S_C(-2a-3) \oplus S_C(-2a-4) \xrightarrow{(f,g)^2} S_C.$$

Hence if the images of f and g in S_C form a regular sequence then by Proposition 6.4.4, the map $(f,g)^2$ is surjective in degrees $\geq 2\deg g + \deg f = 3a+5$. Hence $\dim M_n = 0$, if $n \geq 3a+5$. Using (128) and Proposition 6.4.2 we have $\dim(f,g)^2_{3a+4} = 6a+8$ and hence $\dim M_{3a+4} = 1$. Similarly we have

$$\dim M_{3a+3} = \dim(S_C)_{3a+3} - \dim(f,g)^2_{3a+3} = 6a+7 - \begin{cases} 4, \text{ if } a = 0, \\ 6a+3, \text{ if } a \geq 1 \end{cases} = \begin{cases} 3, \text{ if } a = 0, \\ 4, \text{ if } a \geq 1. \end{cases}$$

□

Corollary 6.4.6. Let W be a triple conic on C of type $(2a, 2b)$, where $a \geq 0$ and $b \geq 2$, given by the map $\psi = (\bar{P}, \bar{R})$ as in Proposition 6.4.1. Then there exist $\alpha, \beta, \gamma \in S_C$ such that $I_W = (I_C I_Z, \alpha x^2 + \beta xq + \gamma q^2 - R(fq - gx))$.

Proof. Notice in these cases $\deg P = 3a+b+3 \geq 3a+5$. Hence by Corollary 6.4.5, $P \in (f,g)^2$. Therefore by Proposition 6.4.1 I_W takes the form above. □

Therefore for triple conics of type $(2a, 2b)$ it remains to consider the cases $a \geq 0$ and $0 \leq b \leq 1$, but $P \notin (f,g)^2 \bmod I_C$. To deal with these cases we introduce two invariants of homogeneous polynomials $P \in S_C/(f,g)^2$.

Definition 6.4.7. Let M be the module $S_C/(f,g)^2$ as defined in Corollary 6.4.5 and let $P \in S$ be a homogeneous polynomial. Set $N_P := \operatorname{Ker}(M \xrightarrow{\cdot P} M)$. Notice $(N_P)_1$ is a vector subspace of $(S_C)_1$. We define ν_P to be the dimension of $(N_P)_1$ as a k-vector space, i.e., $\nu_P = \dim \operatorname{Ker}(M_1 \xrightarrow{\cdot P} M_{1+\deg P})$. We define σ_P to be the length of a maximal S_C-sequence contained in $(N_P)_1$.

Remark 6.4.8. Notice $\nu_P \leq 3$ and $\sigma_P \leq 2$ by construction.

Lemma 6.4.9. *Let $P \in S$ be a homogeneous polynomial of degree d.*

1. *If $d \geq 3a+4$ then $\nu_P = 3$.*

2. *If $d = 3a+3$ then $\nu_P \geq 2$.*

Proof. Notice $\dim M_1 = 3$. By Corollary 6.4.5, we have $\dim M_n = \begin{cases} 0, & \text{if } n \geq 3a+5 \\ 1, & \text{if } n = 3a+4. \end{cases}$
Hence $d \geq 3a+4 \Rightarrow \nu_P = 3$ and $d = 3a+3 \Rightarrow \nu_P \geq 2$. ☐

Remark 6.4.10. Notice $P \in (f,g)^2 \bmod I_C \Rightarrow \nu_P = 3$. But the converse is not true in general. For example, if $f = y, g = z^2$ and $P = yzw$ then $lP \in (f,g)^2 \bmod I_C$ for all linear forms $l \in S$, hence $\nu_P = 3$ and yet $P \notin (f,g)^2 \bmod I_C$.

Proposition 6.4.11. *Let W be a triple conic on C of type $(2a, 2b)$, where $a \geq 0$ and $0 \leq b \leq 1$. If W is given by a map $\psi = (\bar{P}, \bar{R})$ such that $P \notin (f,g)^2 \bmod I_C$ but $\nu_P = 3$, then there exist $H_1, H_2, H_3 \in I_C^2$ such that*

$$I_W = (I_C I_Z, H_1 - yR(fq - gx), H_2 - zR(fq - gx), H_3 - wR(fq - gx)).$$

Proof. Since $P \notin (f,g)^2$, by Proposition 6.4.1 we have I_W/I_CI_Z is not cyclic, and hence $\iota^{-1}(\eta) \notin I_Z/I_CI_Z$, where η is the Koszul relation $\bar{P}e_2 - \bar{R}e_1$. But since $\nu_P = 3$, $lP \in (f,g)^2$ for all linear forms $l \in S$. Therefore I_W/I_CI_Z is generated by $\iota^{-1}(\bar{y}\eta), \iota^{-1}(\bar{z}\eta)$ and $\iota^{-1}(\bar{z}\eta)$. Now since $yP \in (f,g)^2 \bmod I_C$, there exist $\alpha_1, \beta_1, \gamma_1 \in S$ such that $yP = \alpha_1 f^2 + \beta_1 fg + \gamma_1 g^2 \bmod I_C$. Hence $\iota^{-1}(\bar{y}\eta) = \bar{\alpha}_1\bar{x}^2 + \bar{\beta}_1\bar{x}\bar{q} + \bar{\gamma}_1\bar{q}^2 - \bar{y}\bar{R}\overline{(fq - gx)}$. Let $H_1 = \alpha_1 x^2 + \beta_1 xq + \gamma_1 q^2$. Then $H_1 \in I_C^2$ and $\iota^{-1}(\bar{y}\eta) = \bar{H}_1 - \bar{y}\bar{R}\overline{(fq - gx)}$. Similarly $\iota^{-1}(\bar{z}\eta) = \bar{H}_2 - \bar{z}\bar{R}\overline{(fq - gx)}$ and $\iota^{-1}(\bar{w}\eta) = \bar{H}_3 - \bar{w}\bar{R}\overline{(fq - gx)}$ for some $H_2, H_3 \in I_C^2$. Hence I_W takes the form above. □

Corollary 6.4.12. If W is a triple conic on C of type $(2a, 2)$, where $a \geq 0$, given by a map $\psi = (\bar{P}, \bar{R})$ such that $P \notin (f,g)^2 \bmod I_C$, then there exist $H_1, H_2, H_3 \in I_C^2$ such that

$$I_W = (I_CI_Z, H_1 - yR(fq - gx), H_2 - zR(fq - gx), H_3 - wR(fq - gx)).$$

Proof. In this case $\deg P = 3a + 4$ and hence by Lemma 6.4.9, $\nu_P = 3$. Therefore by Proposition 6.4.11 I_W takes the form above. □

Therefore for triple conics of type $(2a, 2b)$ it remains to consider the cases $(2a, 0)$, where $a \geq 0$ and $\nu_P = 2$.

Proposition 6.4.13. Let W be a triple conic on C of type $(2a, 0)$, where $a \geq 0$, given by a map $\psi = (\bar{P}, 1)$ such that $\nu_P = 2$.

1. If $\sigma_P = 2$ then there exist $H_1, H_2 \in I_C^2$ such that

$$I_W = (I_CI_Z, H_1 - l_1R(fq - gx), H_2 - l_2R(fq - gx)),$$

where $\{l_1, l_2\} \subseteq (N_P)_1$ is a regular sequence in S_C.

2. If $\sigma_P = 1$ then there exist $H_1, H_2, H_3 \in I_C^2$ such that

$$I_W = (I_C I_Z, H_1 - l_1 R(fq - gx), H_2 - l_2 R(fq - gx), H_3 - l_3^2 R(fq - gx)),$$

where $\{l_1, l_2\}$ is a basis of $(N_P)_1$ and l_3 spans $(M/(l_1, l_2))_1$.

Proof. First suppose $\sigma_P = 2$. Then there exist linear forms $l_1, l_2 \in (N_P)_1$ such that $\{l_1, l_2\}$ is a regular sequence in S_C. Hence l_1 and l_2 are linearly independent. Since $\nu_P = 2$, $\{l_1, l_2\}$ is a basis of $(N_P)_1$. On the other hand, $\dim(S_C/(l_1, l_2))_n = 0, \forall n \geq 2$ by Proposition 6.4.3. Therefore $\{l_1, l_2\}$ generates N_P and hence $I_W/I_C I_Z$ is generated by $\iota^{-1}(l_1\eta)$ and $\iota^{-1}(l_2\eta)$. Thus I_W takes the form (1) above.

Now suppose $\sigma_P = 1$. Let $\{l_1, l_2\}$ be a basis of $(N_P)_1$. Since $\dim(S_C)_1 = 3$, we can extend $\{l_1, l_2\}$ to a basis $\{l_1, l_2, l_3\}$ of $(S_C)_1$. Notice if $n \geq 2$ and $h \in (S_C)_n$ then we have $\deg(hP) = 3a + n + 3 \geq 3a + 5$ and hence $hP \in (f, g)^2$ by Corollary 6.4.5. Therefore $h \in (N_P)_n$ for all $h \in (S_C)_n$, whenever $n \geq 2$. Thus $(N_P)_2$ is spanned by $\{l_1^2, l_2^2, l_3^2, l_1 l_2, l_1 l_3, l_2 l_3\}$. Notice l_3^2 is not in the span of $\{l_1, l_2\}$, hence $l_3^2\eta$ cannot be generated by $l_1\eta$ and $l_2\eta$. Therefore $I_W/I_C I_Z$ is generated by $\iota^{-1}(l_1\eta), \iota^{-1}(l_2\eta)$ and $\iota^{-1}(l_3^2\eta)$. Thus I_W takes the form (2) above. □

Example 6.4.14. Let Z be the double conic on C with total ideal $I_Z = (I_C^2, fq - gx)$, where $f = y$ and $g = z^2$. Let W be a triple conic on C of type $(0, 0)$, having Z as the 2^{nd} CM filtrant. Then W is given by a map $\psi = (\bar{P}, 1)$, where P is a homogeneous polynomial in S of degree $3a + 3$ and $\bar{P}$ is the image of P in S_C.

1. If $P = y^3$ then $P \in (f,g)^2$. Since $P = y^3 = y \cdot y^2 = y \cdot f^2$, we can identify $\bar{P}$ with $\bar{y}\bar{x}^2$. Therefore $I_W = (I_C I_Z, yx^2 - (yq - z^2x))$ by Proposition 6.4.1.

2. If $P = yzw$ then $P \notin (f,g)^2$. Since $yP, zP, wP \in (f,g)^2$, we have $\nu_P = 3$. Notice $yP = zw \cdot y^2 = zw \cdot f^2$. Hence we can identify $\bar{y}\bar{P}$ with $\overline{zw}\bar{x}^2$. Similarly, we can identify $\bar{z}\bar{P}$ with $\bar{w}\overline{xq}$ and $\bar{w}\bar{P}$ with $\bar{z}^2\bar{x}^2$. Therefore by Proposition 6.4.11 we have

$$I_W = (I_C I_Z, zwx^2 - y(yq - z^2x), wxq - z(yq - z^2x), z^2x^2 - w(yq - z^2x)).$$

3. If $P = z^3$ then $P \notin (f,g)^2$. Notice $\nu_P = 2$ since $yP, zP \in (f,g)^2$ but $wP \notin (f,g)^2$. Since $yP = yz^3 = z \cdot yz^2 = z \cdot fg$, we can identify $\bar{y}\bar{P}$ with $\overline{zxq}$. Similarly, we can identify $\bar{z}\bar{P}$ with $\bar{q}^2$. Notice $\sigma_P = 2$, since $\{y, z\}$ is a regular sequence in S_C. Hence $I_W = (I_C I_Z, zxq - y(yq - z^2x), q^2 - z(yq - z^2x))$ by Proposition 6.4.13 (1).

4. If $P = z^2w$ then $P \notin (f,g)^2$. Also $\nu_P = 2$ since $yP, wP \in (f,g)^2$ but $zP \notin (f,g)^2$. Notice $\sigma_P = 1$, since $\{y, w\}$ is not a regular sequence in S_C. Since $yP = yz^2w = w \cdot yz^2 = w \cdot fg$, we can identify $\bar{y}\bar{P}$ with $\overline{wxq}$. Similarly, we can identify $\bar{w}\bar{P}$ with $\overline{zxq}$ and $\bar{z}^2\bar{P}$ with $\overline{wq}$. Therefore by Proposition 6.4.13 (2) we have

$$I_W = (I_C I_Z, wxq - y(yq - z^2x), zxq - w(yq - z^2x), wq - z^2(yq - z^2x)).$$

Let Z be a double conic on C of type $2a$, where $a \geq 0$. Then $I_Z = (I_C^2, fq - gx)$ by Proposition 5.3.2, and $I_Z/I_CI_Z \cong S_C(-a-3) \oplus (f,g)^2(2a)$ by Proposition 5.3.6. Let $b \geq 0$ be an integer. Let $\tau : T(-a-6) \oplus T(4a) \to T(4a+2b+1)$ be the map given by $\tau = (p, r)$, where $\{p, r\}$ is a regular sequence in T with $\deg p = 6a + 2b + 7$ and $\deg r = 2b + 1$. Let $\psi : S_C(-a-3) \oplus S_C(2a) \to S_C[4a+2b+1]$ be the map corresponding to τ as in Lemma 5.1.10. Let $\iota : S_C(-a-3) \oplus (f,g)^2(2a) \to S_C(-a-3) \oplus S_C(2a)$ be the canonical inclusion as in Proposition 6.1.7. Define $\phi = \psi \circ \iota \circ \pi$, where $\pi : I_Z \twoheadrightarrow I_Z/I_CI_Z$ is the canonical surjection and $\iota : I_Z/I_CI_Z \cong S_C(-a-3) \oplus (f,g)^2(2a) \to S_C(-a-3) \oplus S_C(2a)$ is the canonical inclusion as in Proposition 6.1.7. Then we have the commutative diagram

$$
\begin{array}{ccccc}
I_Z & & \xrightarrow{\quad\quad\quad\phi\quad\quad\quad} & & S_C[4a+2b+1] \\
\big\downarrow \pi & & & & \big\| \\
I_Z/I_CI_Z & \xhookrightarrow{\iota} & S_C(-a-3) \oplus S_C(2a) & \xrightarrow{\psi} & S_C[4a+2b+1] \\
& & \big\downarrow j & & \big\downarrow \\
& & T(-2a-6) \oplus T(4a) & \xrightarrow{\tau} & T(4a+2b+1)
\end{array}
\tag{129}
$$

Since $\{p, r\}$ is a regular sequence in T, $\operatorname{Coker} \tau$ has finite length by Lemma 2.1.9. Therefore $\operatorname{Coker} \psi$ and hence $\operatorname{Coker} \phi$ have finite lengths.

Proposition 6.4.15. In the setting of diagram (129), ϕ defines a triple conic W on C of type $(2a, 2b+1)$ with $I_W = \operatorname{Ker} \phi = I_CI_Z + (j \circ \iota \circ \pi)^{-1} \operatorname{Ker} \tau$, having Z as the 2^{nd} CM filtrant. Moreover, if $\{P_1, R_1\}, \{P_2, R_2\}$ is an admissible pair of sequences on C corresponding to the regular sequence $\{p, r\}$, then there exist $H_1, H_2 \in I_C^2$ such that $I_W = (I_CI_Z, H_1 - R_1(fq - gx), H_2 - R_2(fq - gx))$ if and only if $P_1, P_2 \in (f, g)^2$.

Proof. By Theorem 6.1.8, ϕ sheafifies to the surjection $\widetilde{\phi} : \mathcal{I}_Z \twoheadrightarrow \mathcal{O}_C[4a+2b+1]$ and hence defines a CM triple conic W on C with $I_W = \operatorname{Ker}\phi = I_C I_Z + (j\circ\iota\circ\pi)^{-1}\operatorname{Ker}\tau$. Since $a \geq 0$, W is a quasi-primitive triple conic on C of type $(2a, 2b+1)$ by Corollary 6.3.2. Hence Z is the 2[nd] CM filtrant of W. Since $\{p, r\}$ is a regular sequence in T, $\operatorname{Ker}\tau$ is generated by the Koszul relation $\eta = p\hat{e}_2 - r\hat{e}_1$, where $\hat{e}_1$ and $\hat{e}_2$ are the generators of $T(-2a-6)\oplus T(4a)$. Notice $j^{-1}(\eta) \notin S_C(-a-3)\oplus S_C(2a)$, since $\deg\eta = 2a+2b+7$, i.e., $\deg\eta$ is odd. Hence $j^{-1}\operatorname{Ker}\tau$ is generated by $j^{-1}(s\eta)$ and $j^{-1}(t\eta)$. Now $s\eta = sp\hat{e}_2 - sr\hat{e}_1$ and $t\eta = tp\hat{e}_2 - tr\hat{e}_1$. Therefore $j^{-1}(s\eta) = \bar{P}_1 e_2 - \bar{R}_1 e_1$ and $j^{-1}(t\eta) = \bar{P}_2 e_2 - \bar{R}_2 e_1$, where e_1 and e_2 are the generators of $S_C(-a-3)\oplus S_C(2a)$. Since $I_Z/I_C I_Z \cong S_C(-a-3)\oplus(f,g)^2(2a)$, we can identify $\bar{x}^2, \bar{x}\bar{q}, \bar{q}^2$ and $\overline{fq-gx}$ with f^2, fg, g^2 and e_1 respectively. Then $e_2 f^2 = \bar{x}^2$, $e_2 fg = \bar{x}\bar{q}$ and $e_2 g^2 = \bar{q}^2$. Since $\overline{fq-gx} \in I_Z/I_C I_Z$, $(j\circ\iota)^{-1}(s\eta) = \iota^{-1}(j^{-1}(s\eta)) \in I_Z/I_C I_Z$ if and only if $P_1 \in (f,g)^2$. Similarly, $(j\circ\iota)^{-1}(t\eta)) \in I_Z/I_C I_Z$ if and only if $P_2 \in (f,g)^2$. Now if $P_1 \in (f,g)^2$ then there exist $\alpha, \beta, \gamma \in S$ such that $\bar{P}_1 = \bar{\alpha}f^2 + \bar{\beta}fg + \bar{\gamma}g^2$, where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are the images of α, β, γ in S_C. Thus $j^{-1}(s\eta) = (\bar{\alpha}f^2 + \bar{\beta}fg + \bar{\gamma}g^2)e_2 - \bar{R}e_1$, hence $(j\circ\iota)^{-1}(s\eta) = \overline{\alpha x^2 + \beta xq + \gamma q^2 - R_1(fq-gx)}$. Let $H_1 = \alpha_1 x^2 + \beta_1 xq + \gamma_1 q^2$. Then $H_1 \in I_C^2$ and $(j\circ\iota)^{-1}(s\eta) = \overline{H_1 - R_1(fq-gx)}$. Similarly, if $P_2 \in (f,g)^2$ then there exists $H_2 \in I_C^2$ such that $(j\circ\iota)^{-1}(t\eta) = \overline{H_2 - R_2(fq-gx)}$. Therefore we have $I_W = (I_C I_Z, H_1 - R_1(fq-gx), H_2 - R_2(fq-gx))$ if and only if $P_1, P_2 \in (f,g)^2$. □

Corollary 6.4.16. Let W be a triple conic on C of type $(2a, 2b+1)$ as in Proposition 6.4.15. If $b \geq 1$ then there exist $H_1, H_2 \in I_C^2$ such that

$$I_W = (I_C I_Z, H_1 - R_1(fq-gx), H_2 - R_2(fq-gx)).$$

Proof. Notice $\deg P_i = 3a+b+4 \geq 3a+5$, since $b \geq 1$. Hence by Corollary 6.4.5, $P_i \in (f,g)^2$ for $i=1,2$. Therefore by Proposition 6.4.15, I_W takes the form above. □

Therefore for triple conics of type $(2a, 2b+1)$ it remains to consider the cases $(2a,1)$, where $a \geq 0$ and $P_i \notin (f,g)^2$ for at least one i.

Proposition 6.4.17. Let W be a triple conic on C of type $(2a,1)$ as in Proposition 6.4.15. If $P_i \notin (f,g)^2 \bmod I_C$ for some i, then up to a choice of admissible pair of sequences corresponding to $\{p,r\}$, I_W has a unique form. More precisely,

$$I_W = (I_C I_Z, H_1 - R_1(fq-gx), H_2 - zR_2(fq-gx))$$

for some $H_1, H_2 \in I_C^2$.

Proof. First suppose $P_1 \in (f,g)^2 \bmod I_C$ but $P_2 \notin (f,g)^2 \bmod I_C$. Then according to the proof of Proposition 6.4.15 $(j \circ \iota)^{-1}(s\eta) \in I_Z/I_C I_Z$ but $(j\circ\iota)^{-1}(t\eta) \notin I_Z/I_C I_Z$. Notice $j^{-1} \operatorname{Ker}\tau$ consists of all $j^{-1}(l\eta)$, where $l \in T$ has odd degree. Hence $(j\circ\iota)^{-1}\operatorname{Ker}\tau$ is generated by $j^{-1}(s\eta)$ and $j^{-1}(t^d\eta)$, where $d \geq 3$ is some odd integer. Notice $t^3\eta = t^3 p\hat{e}_2 - t^3 r\hat{e}_1$, hence $j^{-1}(t^3\eta) = \bar{z}\bar{P}_2 e_2 - \bar{z}\bar{R}_2 e_1$. Since $\deg(zP_2) = 3a+5$, $zP_2 \in (f,g)^2 \bmod I_C$ by Corollary 6.4.5. Therefore $(j\circ\iota)^{-1}\operatorname{Ker}\tau$ is generated by $(j\circ\iota)^{-1}(s\eta)$ and $(j\circ\iota)^{-1}(t^3\eta)$. Hence there exists $H_2 \in I_C^2$ such that $(j\circ\iota)^{-1}(t^3\eta) = \overline{H_2 - zR_2(fq-gx)}$. Hence I_W takes the form above. Now suppose $P_2 \in (f,g)^2 \bmod I_C$ but $P_1 \notin (f,g)^2 \bmod I_C$. Then $(j\circ\iota)^{-1}(t\eta) \in I_Z/I_C I_Z$ but $(j\circ\iota)^{-1}(s\eta) \notin I_Z/I_C I_Z$. Interchanging the roles of s and t we can get back to the previous case. Finally suppose $P_1, P_2 \notin (f,g)^2 \bmod I_C$. Then P_1, P_2 are nonzero elements of M_{3a+4}, where $M = S_C/(f,g)^2$. But then $P_1 = \lambda P_2 \bmod (f,g)^2$

for some $\lambda \in k^*$, since $\dim M_{3a+4} = 1$ by Corollary 6.4.5. Therefore $P_1 - \lambda P_2 \in (f,g)^2$. Thus $(j \circ \iota)^{-1}((s-\lambda t)\eta) \in I_Z/I_C I_Z$ but $(j \circ \iota)^{-1}(t\eta) \notin I_Z/I_C I_Z$. Let $\{P_1', R_1'\}, \{P_2', R_2'\}$ be an admissible pair of sequences corresponding to $\{p, r\}$ such that $\theta(P_1') = (s - \lambda t)p$. Then $P_1' = P_1 - \lambda P_2 \in (f,g)^2$ and we get back to the original case. □

This completes the total ideal description of triple conics whose 2nd CM filtrant is a double conic of odd genus.

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ABSTRACT

MULTIPLICITY STRUCTURES ON CONICS

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Let $C \subset \mathbb{P}^3$ be a conic. A multiplicity structure on C is a closed subscheme $Z \subset \mathbb{P}^3$ such that $\operatorname{Supp} Z = \operatorname{Supp} C$. The multiplicity of Z is defined by the ratio $\deg Z / \deg C$, which we prove to be an integer. In this dissertation we give complete classification of double conics on C. This classification includes descriptions of their of total ideals, minimal free resolutions of their total ideals, their Rao modules, descriptions of general surfaces containing such structures and the criterion for two double conics on C to be linked by a complete intersection, which extends a well-known theorem of Migliore on self-linkage of double lines to double conics. We also give a partial classification of triple conics on C, which is complicated by new behaviors such as the jumping of cohomology groups and the non-splitting nature of the restriction of total ideals of the second Cohen-Macaulay filtrant of odd genera.