

Article

Neutrino Flavor Transitions as Mass State Transitions

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Received: 16 May 2019; Accepted: 19 July 2019; Published: 24 July 2019



Abstract: Experiments have shown that transitions occur between electron neutrino, muon neutrino, and tau neutrino flavors. Some experiments indicate the possible existence of a fourth neutrino known as the sterile neutrino. The question arises: do all neutrino flavors participate in transitions between flavors? These transitions are viewed as mass state transitions in parametrized relativistic dynamics (PRD). PRD frameworks have been developed for neutrino flavor transitions associated with the mixing of two mass states or the mixing of three mass states. This paper presents an extension of the framework to neutrino flavor transitions associated with the mixing of four mass states.

Keywords: neutrino oscillations; mass state transitions; flavor transitions; sterile neutrino

1. Introduction

Flavor mixing between two or three neutrino flavors composed of up to three neutrino mass states [1] has been observed in experiments with solar, atmospheric, reactor, and accelerator neutrinos. The existence of a fourth neutrino has been proposed as a result of observations at the Liquid Scintillator Neutrino Detector (LSND) at Los Alamos National Laboratory, New Mexico and the Mini Booster Neutrino Experiment (MiniBooNE) at the Fermi National Accelerator Laboratory, Batavia, Illinois (see Section 14 in [1]). The fourth neutrino is referred to as the sterile neutrino because it is possible that the sterile neutrino only interacts with other matter through the gravitational interaction, while the electron, muon, and tau neutrinos are able to interact with matter through the weak and gravitational interactions. The MiniBooNE and LSND results are based on single detector studies of electron neutrino appearance in muon neutrino beams.

Aguilar-Arevalo, et al. [2] analyzed MiniBooNE and LSND data and found that analysis of the data by a two-neutrino oscillation model “would require at least four neutrino types and indicate physics beyond the three-neutrino paradigm.” By contrast, the MINOS+ collaboration [3] used long baseline neutrino detectors to search for sterile neutrinos in muon disappearance experiments. The MINOS+ collaboration did not find evidence of sterile neutrino mixing—consequently, the MINOS+ collaboration plans to improve the sensitivity of future analyses to set more stringent limits on mixing with sterile neutrinos.

Another way to analyze flavor mixing experiments has been proposed in the context of parametrized relativistic dynamics (PRD). For example, Fanchi presented PRD formalisms for studying neutrino flavor transitions associated with the mixing of two mass states [4] and the mixing of three mass states [5]. Neutrino oscillations are an example of two-state flavor mixing. Introductions to PRD are presented by Fanchi [6,7], Pavšič [8], and Horwitz [9]. The purpose of this paper is to present an extension of the PRD framework to neutrino flavor transitions associated with the mixing of four mass states.

2. Four-State Transitions

It is possible to use more than one PRD approach to study transitions between flavor states. For example, we can study flavor transitions by assuming that they are induced when a system with

discrete mass states interacts with a perturbation [10,11]. This approach is outlined in Section 2.1 to provide context for the more conventional technique presented in Section 2.2.

2.1. Perturbation Induced Transitions between Flavor States

The scalar field equation for a perturbation analysis of transitions between flavor states due to a perturbing potential V_I is

$$i\hbar \frac{\partial \Phi(x_1, s)}{\partial s} = [K_0 + V_I] \Phi(x_1, s) = K \Phi(x_1, s) \quad (1)$$

for space-time coordinates x_1 and scalar evolution parameter s . The operator K is considered a mass operator. In this approach, K is composed of operator K_0 , which is independent of s , and the perturbation V_I which may depend on s . Equation (1) is equivalent to Equation (A3) with $\alpha_R = 0$ and

$$K_0 = \frac{p_1^\mu p_{1\mu}}{2m_1} = -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^\mu \partial x_{1\mu}}, \quad \Phi(x_1, s) = \psi(1, s) \quad (2)$$

The perturbing interaction V_I is required to be Hermitian so that subsequent matrix elements involving V_I are real and therefore physically meaningful.

A perturbation solution for the state Φ is obtained by writing the expansion

$$\Phi(x_1, s) = \sum_{n=1}^N a_n(s) \phi_n(x_1, s) \quad (3)$$

for N flavor states. Solutions of the unperturbed field equation

$$i\hbar \frac{\partial \phi_n(x_1, s)}{\partial s} = K_0 \phi_n(x_1, s) \quad (4)$$

are $\phi_n(x_1, s)$ for $\{n = 1, 2, \dots, N\}$. The normalization condition on Φ

$$\int \Phi^* \Phi d^4 x_1 = 1 \quad (5)$$

implies

$$\sum_{n=1}^N a_n^*(s) a_n(s) = 1 \quad (6)$$

The set of differential equations for the expansion coefficients is

$$i\hbar \frac{\partial a_m}{\partial s} = \sum_{n=1}^N V_{mn} a_n, \quad m = 1, 2, \dots, N \quad (7)$$

and the set of matrix elements $\{V_{mn}\}$ of the perturbing potential is

$$V_{mn} = \int \phi_m^* V_I \phi_n d^4 x_1, \quad \text{for } m = 1, 2, \dots, N \text{ and } n = 1, 2, \dots, N \quad (8)$$

2.2. Flavor State Transitions as Mass State Transitions

The perturbation formalism presented in Section 2.1 can be used to construct a model of neutrino oscillations for $N = 2$ flavor states. Rather than using the perturbation approach, a more conventional approach is to consider transitions between $N = 4$ neutrino flavor states $\{|v_\alpha\rangle; \alpha = e, \mu, \tau, \sigma\}$ based on

the assumption that neutrinos are composed of up to four mass states $\{|v_j\rangle; j = 1, 2, 3, 4\}$. The subscripts $\{\alpha = e, \mu, \tau, \sigma\}$ refer to electron, muon, tau, and sterile neutrinos respectively.

We begin by assuming that mass and flavor states can be written as 4-component column vectors:

$$|v_j\rangle = \begin{bmatrix} |v_1\rangle \\ |v_2\rangle \\ |v_3\rangle \\ |v_4\rangle \end{bmatrix} \quad (9)$$

and

$$|v_\alpha\rangle = \begin{bmatrix} |v_e\rangle \\ |v_\mu\rangle \\ |v_\tau\rangle \\ |v_\sigma\rangle \end{bmatrix} \quad (10)$$

We assume that the mass basis $\{|v_j\rangle; j = 1, 2, 3, 4\}$ is related to the flavor basis $\{|v_\alpha\rangle; \alpha = e, \mu, \tau, \sigma\}$ by a unitary transformation:

$$\begin{bmatrix} |v_e\rangle \\ |v_\mu\rangle \\ |v_\tau\rangle \\ |v_\sigma\rangle \end{bmatrix} = U \begin{bmatrix} |v_1\rangle \\ |v_2\rangle \\ |v_3\rangle \\ |v_4\rangle \end{bmatrix} \quad (11)$$

where U is the unitary matrix

$$U = \begin{bmatrix} u_{e1} & u_{e2} & u_{e3} & u_{e4} \\ u_{\mu1} & u_{\mu2} & u_{\mu3} & u_{\mu4} \\ u_{\tau1} & u_{\tau2} & u_{\tau3} & u_{\tau4} \\ u_{\sigma1} & u_{\sigma2} & u_{\sigma3} & u_{\sigma4} \end{bmatrix} \quad (12)$$

satisfying

$$U^{-1} = (U^*)^T \quad (13)$$

The elements of the unitary matrix are

$$u_{\alpha j}^{-1} = u_{j\alpha}^*; j = 1, 2, 3, 4 \text{ and } \alpha = e, \mu, \tau, \sigma \quad (14)$$

The expanded form of the unitary transformation is

$$\begin{aligned} |v_e\rangle &= u_{e1}|v_1\rangle + u_{e2}|v_2\rangle + u_{e3}|v_3\rangle + u_{e4}|v_4\rangle \\ |v_\mu\rangle &= u_{\mu1}|v_1\rangle + u_{\mu2}|v_2\rangle + u_{\mu3}|v_3\rangle + u_{\mu4}|v_4\rangle \\ |v_\tau\rangle &= u_{\tau1}|v_1\rangle + u_{\tau2}|v_2\rangle + u_{\tau3}|v_3\rangle + u_{\tau4}|v_4\rangle \\ |v_\sigma\rangle &= u_{\sigma1}|v_1\rangle + u_{\sigma2}|v_2\rangle + u_{\sigma3}|v_3\rangle + u_{\sigma4}|v_4\rangle \end{aligned} \quad (15)$$

A mass basis state satisfies the temporal evolution equation

$$T|v_j\rangle = i\hbar \frac{\partial}{\partial s} |v_j\rangle = T_j |v_j\rangle \quad (16)$$

where the eigenvalue of the temporal operator T is T_j , and s is the scalar evolution parameter. In the standard theory, the eigenvalue T_j of state j is the energy E_j . By contrast, the eigenvalue T_j in PRD is the eigenvalue K_j of the mass operator for mass state j . Equation (16) has the formal solution

$$\begin{bmatrix} |v_1(s)\rangle \\ |v_2(s)\rangle \\ |v_3(s)\rangle \\ |v_4(s)\rangle \end{bmatrix} = \begin{bmatrix} e^{-i\frac{T_1s}{\hbar}} & & & \\ & e^{-i\frac{T_2s}{\hbar}} & & \\ & & e^{-i\frac{T_3s}{\hbar}} & \\ & & & e^{-i\frac{T_4s}{\hbar}} \end{bmatrix} \begin{bmatrix} |v_1(0)\rangle \\ |v_2(0)\rangle \\ |v_3(0)\rangle \\ |v_4(0)\rangle \end{bmatrix} \quad (17)$$

where all off-diagonal terms are zero and $|v_j(0)\rangle$ is mass state j at $s = 0$.

3. Transitions between Flavor States: Electron Neutrino Disappearance

The formalism presented in Section 2.2 is illustrated by applying it to the disappearance of electron neutrinos. Consider a pure beam of electron neutrinos in flavor state $|v_e\rangle$. The probability of transition from v_e to v_β for $\{\beta = \mu, \tau \text{ or } \sigma\}$ is

$$P(v_e \rightarrow v_\beta) = |\langle v_\beta | v_e(s) \rangle|^2 \quad (18)$$

The matrix element is

$$\begin{aligned} \langle v_\beta | v_e(s) \rangle &= \left[u_{\beta 1}^* v_1^* + u_{\beta 2}^* v_2^* + u_{\beta 3}^* v_3^* + u_{\beta 4}^* v_4^* \right] \\ &\quad \times [u_{e1} v_1(s) + u_{e2} v_2(s) + u_{e3} v_3(s) + u_{e4} v_4(s)] \end{aligned} \quad (19)$$

or, in expanded form,

$$\begin{aligned} \langle v_\beta | v_e(s) \rangle &= \left\{ u_{\beta 1}^* [u_{e1} v_1^* v_1(s) + u_{e2} v_1^* v_2(s) + u_{e3} v_1^* v_3(s) + u_{e4} v_1^* v_4(s)] \right. \\ &\quad + u_{\beta 2}^* [u_{e1} v_2^* v_1(s) + u_{e2} v_2^* v_2(s) + u_{e3} v_2^* v_3(s) + u_{e4} v_2^* v_4(s)] \\ &\quad + u_{\beta 3}^* [u_{e1} v_3^* v_1(s) + u_{e2} v_3^* v_2(s) + u_{e3} v_3^* v_3(s) + u_{e4} v_3^* v_4(s)] \\ &\quad \left. + u_{\beta 4}^* [u_{e1} v_4^* v_1(s) + u_{e2} v_4^* v_2(s) + u_{e3} v_4^* v_3(s) + u_{e4} v_4^* v_4(s)] \right\} \end{aligned} \quad (20)$$

Equation (20) is simplified by applying the orthonormality condition $\langle v_i | v_j \rangle = \delta_{ij}$ to obtain

$$\langle v_\beta | v_e(s) \rangle = u_{\beta 1}^* u_{e1} v_1^* v_1(s) + u_{\beta 2}^* u_{e2} v_2^* v_2(s) + u_{\beta 3}^* u_{e3} v_3^* v_3(s) + u_{\beta 4}^* u_{e4} v_4^* v_4(s) \quad (21)$$

The dependence on the temporal operator T is obtained by writing the matrix element in Equation (21) at $s = 0$:

$$\begin{aligned} \langle v_\beta | v_e(s) \rangle &= u_{\beta 1}^* u_{e1} \exp\left(-i\frac{T_1s}{\hbar}\right) + u_{\beta 2}^* u_{e2} \exp\left(-i\frac{T_2s}{\hbar}\right) + u_{\beta 3}^* u_{e3} \exp\left(-i\frac{T_3s}{\hbar}\right) \\ &\quad + u_{\beta 4}^* u_{e4} \exp\left(-i\frac{T_4s}{\hbar}\right) \end{aligned} \quad (22)$$

4. Application to Neutrino Oscillation Experiments

We demonstrate the applicability of the 4-flavor state-formalism by considering the oscillation $v_e \leftrightarrow v_\mu$ in vacuum between electron neutrino and muon neutrino flavors. Transitions to tau neutrinos and sterile neutrinos are negligible in this application. The 4-state unitary matrix simplifies to

$$U(4) = \begin{bmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (23)$$

where θ_{12} is the mixing angle between mass states 1 and 2 in vacuum. The resulting transition probability amplitude is

$$\langle \nu_\mu | \nu_e(s) \rangle = u_{\mu 1}^* u_{e 1} \exp\left(-i \frac{T_1 s}{\hbar}\right) + u_{\mu 2}^* u_{e 2} \exp\left(-i \frac{T_2 s}{\hbar}\right) \quad (24)$$

where $\{T_j, j = 1, 2\}$ are the eigenvalues of the temporal evolution operator and

$$\begin{aligned} u_{\mu 1}^* &= -\sin \theta_{12} \\ u_{e 1} &= \cos \theta_{12} \\ u_{\mu 2}^* &= \cos \theta_{12} \\ u_{e 2} &= \sin \theta_{12} \end{aligned} \quad (25)$$

The evolution equation in PRD for state $|v_j\rangle$ is

$$i\hbar \frac{\partial}{\partial s} |v_j\rangle = K_j |v_j\rangle \quad (26)$$

where K_j is the eigenvalue of the mass operator for mass state j , m_j is the mass of state j , and k_j^μ is the energy-momentum of state j . The solution of Equation (26) is

$$\begin{bmatrix} |v_1(s)\rangle \\ |v_2(s)\rangle \end{bmatrix} = \begin{bmatrix} e^{-i \frac{K_1 s}{\hbar}} & 0 \\ 0 & e^{-i \frac{K_2 s}{\hbar}} \end{bmatrix} \begin{bmatrix} |v_1(0)\rangle \\ |v_2(0)\rangle \end{bmatrix} \quad (27)$$

where

$$K_j = \hbar^2 \frac{k_j^\mu k_{j\mu}}{2m_j} = \hbar^2 \frac{\left[\left(\frac{\omega_j}{c}\right)^2 - k_j \cdot k_j\right]}{2m_j} \quad (28)$$

In PRD, the components of k_j^μ are observables and the mass m_j depends on statistical values of k_j^μ . Substituting Equations (25) and (28) into Equation (24) gives

$$\begin{aligned} \langle \nu_\mu | \nu_e(s) \rangle &= -\sin \theta_{12} \cos \theta_{12} \exp\left(-i \frac{K_1 s}{\hbar}\right) + \cos \theta_{12} \sin \theta_{12} \exp\left(-i \frac{K_2 s}{\hbar}\right) \\ &= \cos \theta_{12} \sin \theta_{12} \left[\exp\left(-i \frac{K_2 s}{\hbar}\right) - \exp\left(-i \frac{K_1 s}{\hbar}\right) \right] \end{aligned} \quad (29)$$

and

$$\langle \nu_\tau | \nu_e(s) \rangle = 0 \text{ and } \langle \nu_\sigma | \nu_e(s) \rangle = 0 \quad (30)$$

The transition probability is

$$P(\nu_e \rightarrow \nu_\mu) = |\langle \nu_\mu | \nu_e(s) \rangle|^2 \quad (31)$$

In the flavor oscillation process $\nu_e \leftrightarrow \nu_\mu$, we begin with a pure beam of electron neutrinos and calculate the probability of obtaining muon neutrinos. The probability of obtaining the final state ν_μ from initial state ν_e is

$$P_{PRD}(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \sin^2 \left\{ \frac{(m_2 - m_1)c^2}{4\hbar} s \right\} \equiv \sin^2 2\theta \sin^2 \alpha_{PRD} \quad (32)$$

where s is temporal duration measured by an evolution parameter clock (see Appendix A). Dynamical factors are collected in the term α_{PRD} .

The most probable trajectory of the non-interacting s-clock particle traveling a distance $\delta x = L$ in an interval δt is Equation (A11), namely

$$s = \frac{L [1 - \beta^2]^{1/2}}{c \beta} \quad (33)$$

Substituting Equation (33) into Equation (32) gives

$$\begin{aligned} P_{PRD}(v_e \rightarrow v_\mu) &= \sin^2 2\theta \sin^2 \alpha_{PRD} \\ \alpha_{PRD} &= \frac{(m_2 - m_1)c^2}{4\hbar} \frac{L}{c} \frac{[1 - \beta^2]^{1/2}}{\beta} \end{aligned} \quad (34)$$

The result for the standard theory denoted by subscript *Std* is

$$\begin{aligned} P_{Std}(v_e \rightarrow v_\mu) &= \sin^2 2\theta \sin^2 \alpha_{Std} \\ \alpha_{Std} &= \frac{(m_2^2 - m_1^2)c^4}{4\hbar E_v} \frac{L}{c} \end{aligned} \quad (35)$$

where E_v is the energy of the ultrarelativistic incident neutrino

$$E_v = \frac{m_\nu c^2}{[1 - \beta^2]^{1/2}} \quad (36)$$

The probability of the electron neutrino surviving is

$$P_{PRD}(v_e \rightarrow v_e) = 1 - P_{PRD}(v_e \rightarrow v_\mu) \quad (37)$$

in the PRD model, and

$$P_{Std}(v_e \rightarrow v_e) = 1 - P_{Std}(v_e \rightarrow v_\mu) \quad (38)$$

in the standard Std model.

We combine Equations (34) and (36) and rearrange to simplify comparison with Equation (35):

$$\begin{aligned} P_{PRD}(v_e \rightarrow v_\mu) &= \sin^2 2\theta \sin^2 \alpha_{PRD} \\ \alpha_{PRD} &= \frac{(m_2 - m_1)c^2}{4\hbar} \frac{L}{c} \frac{m_\nu c^2}{E_v \beta} = \frac{m_\nu (m_2 - m_1)c^4}{4\hbar E_v} \frac{L}{c} \frac{1}{\beta} \end{aligned} \quad (39)$$

The ratio of the dynamical factors α_{PRD} , α_{Std} is

$$\frac{\alpha_{Std}}{\alpha_{PRD}} = \frac{(m_2^2 - m_1^2)}{m_\nu (m_2 - m_1)} \beta = \frac{m_1 + m_2}{m_\nu} \beta \quad (40)$$

and the ratio of probabilities in Equations (35) and (39) is

$$\frac{P_{Std}}{P_{PRD}} = \frac{\sin^2 \alpha_{Std}}{\sin^2 \alpha_{PRD}} \quad (41)$$

The probabilities P_{PRD} , P_{Std} have the same dependence on the flavor mixing angle θ , but the dynamical factors α_{PRD} , α_{Std} are significantly different. We can estimate the ratio of dynamical factors $\alpha_{Std}/\alpha_{PRD}$ by assuming the mass difference between neutrino mass and flavor states is very small and the neutrinos are ultrarelativistic. In this case, we obtain $\frac{m_1 + m_2}{m_\nu} \approx 2$ and $\beta \approx 1$ so that $\frac{\alpha_{Std}}{\alpha_{PRD}} \approx 2$.

Rusov and Vlasenko [12] used the difference between the standard and PRD models of two-state vacuum flavor mixing to conduct an experimental test of the two models. They estimated neutrino masses for the electron, muon, and tau neutrinos based on data for solar and atmospheric neutrinos. The neutrino masses were then used to estimate the diameter of a neutrino cloud and expressed

a preference for PRD results. They concluded that neutrino oscillations indicate physics beyond the Standard Model. Furthermore, direct experimental measurement of neutrino mass can justify reconsideration of quantum theoretical issues and a “holistic understanding of the nature of physical reality” [12].

5. Conclusions

Some experiments indicate the possible existence of a fourth neutrino, known as the sterile neutrino, in addition to the electron neutrino, muon neutrino, and tau neutrino. Frameworks have been developed previously in the context of parametrized relativistic dynamics (PRD) for neutrino flavor transitions associated with the mixing of two mass states or the mixing of three mass states. The PRD framework was extended here to neutrino flavor transitions associated with the mixing of four mass states. The framework does not tell us how many neutrino flavor states exist in nature. More work is needed to examine experimental results within the context of PRD.

Funding: The APC was funded by the TCU Open Access Fund.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. The Physical System

In this paper we assume that flavor transitions result from the mixing of discrete mass states. We quantify the scalar evolution parameter that appears in the PRD analysis of flavor transitions by first considering an N-body system that contains an evolution parameter clock with N_c objects and an experimental system with N_e objects. The evolution parameter clock provides an operational representation of the scalar invariant parameter s .

The experimental system of interest illustrated in Figure A1 should have negligible interaction with the evolution parameter clock. Rods and clocks for quantifying the space-time $\{x, t\}$ coordinates of the experiment (system 1) and the “s-clock” (system 2) are used to determine the scalar invariant evolution parameters s_1, s_2 for system 1 and system 2 respectively. Calibration of the clocks lets us write $s_1 = s_2$ so that we can use the s-clock world-line to parametrize the experimental system. This procedure is discussed in more detail in the literature [6,7].

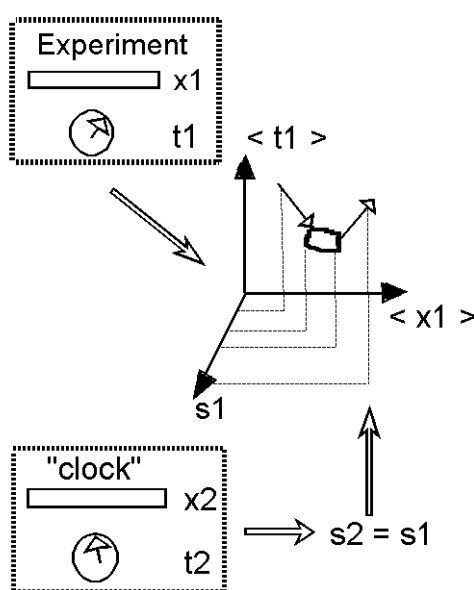


Figure A1. The Experimental System and Evolution Parameter Clock [10].

In our case here, the system of interest contains two particles. Particle 1 does not interact with particle 2 but can interact with other particles in the experimental system through an interaction

potential V_I . For simplicity, particle 2 is a free, scalar particle that serves as the evolution parameter clock (s-clock). We also make the assumption that the rest frame clocks of the two particles in our system are calibrated.

The Stueckelberg equation for the two-particle system is

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} + V_I \right] \right\} \psi(1, 2, s) = 0 \quad (\text{A1})$$

where s-dependent eigenfunction $\psi(1, 2, s)$ of the two-particle system applies to both particles 1 and 2. The eigenfunction $\psi(1, 2, s)$ has 4-space components y_1^μ, y_2^μ . Writing $\psi(1, 2, s)$ as the product of single particle eigenfunctions

$$\psi(1, 2, s) = \psi(1, s)\psi(2, s) \quad (\text{A2})$$

and substituting Equation (A2) into Equation (A1) lets us separate Equation (A1) into two equations

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_1^\mu p_{1\mu}}{2m_1} + V_I + \alpha_R \right] \right\} \psi(1, s) = 0 \quad (\text{A3})$$

and

$$\left\{ i\hbar \frac{\partial}{\partial s} - \left[\frac{p_2^\mu p_{2\mu}}{2m_2} - \alpha_R \right] \right\} \psi(2, s) = 0 \quad (\text{A4})$$

with separation constant α_R . We choose a system in which particles 1 and 2 are physically independent so that we can write $\alpha_R = 0$.

Equation (A4) with $\alpha_R = 0$ is the free particle equation

$$\hbar \frac{\partial \psi_f(2, s)}{\partial s} = \left[-\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^\mu \partial x_{2\mu}} \right] \psi_f(2, s) \quad (\text{A5})$$

with the particular solution

$$\psi_f(2, s) = \eta_f^{1/2} \exp \left[-\frac{i\hbar}{2m_2} (k_2^\mu k_{2\mu}) s + ik_{2\mu} x_2^\mu \right] \quad (\text{A6})$$

and normalization constant η_f . The most probable trajectory of particle 2 in the classical limit of negligible dispersion is

$$(\delta s)^2 = (s - s_0)^2 = \frac{1}{c^2} \delta \langle x_2^\mu \rangle \delta \langle x_{2\mu} \rangle \quad (\text{A7})$$

where s_0 is the scalar evolution parameter when the rest frame clocks of particles 1 and 2 were calibrated. We define the s-axis so that $s_0 = 0$ and Equation (A7) becomes

$$s^2 = \frac{1}{c^2} \delta \langle x_2^\mu \rangle \delta \langle x_{2\mu} \rangle \quad (\text{A8})$$

If we consider only linear motion for the free particle and neglect statistical variations, we obtain

$$s^2 = \delta t^2 - \frac{\delta x^2}{c^2} = \delta t^2 [1 - \beta^2] \quad (\text{A9})$$

where $\{x_2^0, x_2^1\} = \{t, x\}$, and

$$\beta = \frac{v}{c} \text{ with } v \equiv \frac{\delta x}{\delta t} \quad (\text{A10})$$

The distance δx traveled by particle 2 in the interval δt can be written as L so that Equation (A9) becomes

$$s = \frac{L}{v} [1 - \beta^2]^{1/2} = \frac{L}{c} \frac{[1 - \beta^2]^{1/2}}{\beta} \quad (\text{A11})$$

We quantify the scalar evolution parameter s by measuring the space-time trajectory of particle 2 and use the resulting value of s in Equation (A3) associated with particle 1.

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