C*-ALGEBRAS OF ORBIT-CLOSURES, THEN DECOMPOSITION OF NUCLEAR MAPS

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Preface

This dissertation is split into two parts. In the first part we expand upon work by Gábor Elek on C*-algebras of Uniformly Recurrent Subgroups. In the second we expand upon work by many hands on the decomposition of nuclear maps.

Part I generalizes Elek's construction. A finitely generated discrete group acts upon the set of its subgroups via conjugation. We apply a natural topology to this set, then take the closure of the orbit of a chosen subgroup. This gives a dynamical system which happens to have a nice correspondence with a Cayley-like graph of the subgroup's cosets. It is this graph upon which our C*-algebra is built, encoding information about the short-term evolution of the system, and using a process reminiscent of some common C*-algebra constructions. We make use of this similarity (as well as the differences) to apply techniques which have previously shown to be illuminating for these other constructions. This reveals properties of the C*-algebra and relates them to properties of the graph, the dynamical system, and the subgroup itself.

Part II adds to a chain of incremental papers. Nuclear maps between C*-algebras can be characterized by their ability to be approximately written as the composition of maps to and from matrices. Under certain conditions (such as quasidiagonality), we can guarantee that the maps in these decompositions behave nicely. In particular, we seek maps which preserve multiplication up to an arbitrary degree of accuracy. To that end, we find conditions both necessary and sufficient for such decompositions, and also relate them to a W*-analog.

Part I

C*-Algebras of Orbit-Closures of Subgroups

1 Introduction

This part is heavily based upon [Ele18], in which Gábor Elek defines the reduced C*-algebra of a Uniformly Recurrent Subgroup (URS) of a finitely generated discrete group. The concept of a URS was introduced by Glasner and Weiss in [GW15] as a topological analogue of an Invariant Random Subgroup; however, one of our goals is to explore beyond the scope of URSs.

Section 2First definitionssection.2 introduces the fundamental definitions and notation of the part. Sections 3Graphs and kernelssection.3 & 4The C*-algebrasection.4 retread the construction of Elek's C*-algebra; although we have dropped the minimal-under-group-action assumption of a URS, not much here is any different. Elek's construction bears resemblance to that of the uniform Roe algebra [BO08, Chap. 5] with an added layer of finiteness made possible by the group dynamics of the URS. These Roe algebras show up in the study of groups, groupoids, and coarse metric spaces. They are subalgebras of operators on functions on such spaces, built from "finite-width tubes" which limit interaction of distant coordinates (essentially, entries vanish far from the diagonal), thus providing insight into the space's small-scale structure. We embrace the similarities between these algebras later.

Elek remarks at the end of [Ele18, Sec. 6.4] that, by using a special coloring, one may define another C*-algebra from a URS, and that his preceding proof of simplicity then no longer requires what Elek calls genericity—a condition which describes a lack of self-similarity within the URS. We flesh out Elek's remark in Section 5Colorssection.5, along with a note that the new C*-algebra contains the original as a C*-subalgebra. We then use this containment in Section 6Coamenabilitysection.6 to strengthen [Ele18, Sec. 9.1] by dropping genericity.

Corollary 6.9theorem.6.9. *Suppose $\overline{\mathcal{O}_{\mathrm{K}}}$ is a URS. Then the following are equivalent:*

1. K *is coamenable.*
2. $C^*_\Gamma(\mathrm{K})$ *admits a faithful amenable trace.*
3. $C^*_\Gamma(\mathrm{K})$ *admits an amenable trace.*

We finally move beyond Elek in Section 7Covariant representationssection.7. Here we circle back to the Roe algebra by noting that, like it, our C*-algebra has a crossed-product-like structure. Crossed products are one of the elementary tools for constructing new C*-algebras. Akin to the semidirect product of groups, a crossed product integrates a C*-algebra and a group that acts upon it into a larger C*-algebra that contains the group as unitaries whose conjugation witnesses the action. However, our "crossed product" is very different from what we may expect, and its properties are explored in Section 8Crossed product consequencessection.8.

In Section 9Injective envelopessection.9 we see what more we can say about the C*-algebra by using Hamana's theory of injective envelopes. Injective envelopes are structures that show up in many related categories as unique super-objects that allow extensions of morphisms—of particular use is the extension of isomorphisms into what we call *pseudo-expectations*. These are more handy than conditional expectations, as we can always guarantee their existence. They are especially useful in analyzing crossed products, as we have a natural choice of subalgebra for the range (that is, the original C*-algebra). Using these tools, we are able to describe the C*-algebra's ideals in terms of the orbit-closure's topology, and expand upon Elek's proof that genericity implies simplicity.

Corollary 9.2theorem.9.2. *If Λ is topologically cheap (see Definition 2.4theorem.2.4), then $\mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda})$ is nonzero for every nontrivial (closed, two-sided) ideal $\mathcal{I} \trianglelefteq C^*_\Gamma(\Lambda)$. In particular, if $\overline{\mathcal{O}_\Lambda}$ is a URS, then $C^*_\Gamma(\Lambda)$ is simple.*

We end in Section 10Subalgebrassection.10 with a discussion of properties of other different kinds of subalgebras. First are Cartan C*-subalgebras, which are analagous to Cartan von Neumann algebras, and a special kind of maximal abelian subalgebra (MASA). MASAs are particularly informative: they are abelian, so they are well understood as being continuous functions on a topological space, and they are maximal, so much of their shape carries over into the original C*-algebra. Unlike general MASAs, whose classification remains a woolly task, the structure of the Cartan relation has been completely characterized in terms of groupoids. We are able to frame this global embedability property of a certain C*-subalgebra in terms of the self-similarity of an orbit-closure.

Theorem 10.3theorem.10.3. *Λ is topologically cheap iff $C(\overline{\mathcal{O}_\Lambda})$ is a Cartan subalgebra of $C^*_\Gamma(\Lambda)$.*

This is followed-up with a special case of both properties.

Theorem 10.6theorem.10.6. *Λ is cheap iff $C(\overline{\mathcal{O}_\Lambda})$ is a C^*-diagonal subalgebra of $C^*_\Gamma(\Lambda)$.*

Finally, we discuss when the orbit-closure of a subgroup of an intermediate group generates a C*-subalgebra of the C*-algebra of the orbit-closure within the full group.

Theorem 10.8theorem.10.8. *Suppose $\Lambda \le \mathrm{H} \le \Gamma$ and let $\mathcal{U}$ be Λ's orbit-closure within $\mathrm{Sub}\,\mathrm{H}$. Also suppose that $\mathcal{U}$ is open in $\overline{\mathcal{O}_\Lambda}$ and that the normalizer $N_\Gamma(\Lambda) \le \mathrm{H}$. Then $C^*_{\mathrm{H}}(\Lambda) \subseteq C^*_\Gamma(\Lambda)$.*

2 First definitions

Throughout this part, Γ will be a discrete group with finite generating set Q (for notational convenience, we assume $\gamma \in Q$ implies $\gamma^{-1} \in Q$) and unit e. We inductively define sets $Q^n := \{\gamma, \gamma\lambda, \lambda\gamma \mid \gamma \in Q^{n-1} \text{ and } \lambda \in Q\}$, starting with $Q^0 := \{e\}$. The *length* of an element $\gamma \in \Gamma$ is the smallest integer $l_Q(\gamma)$ such that $\gamma \in Q^{l_Q(\gamma)}$.

$\mathrm{Sub}\,\Gamma$ is the set of all subgroups of Γ, which we equip with the compact Hausdorff Chabauty (or Fell) topology [Bee93, Fel62], in which a net converges if its terms are eventually consistent in what they do and don't contain.

Definition 2.1. The **Chabauty topology** is a topology defined on a set S of subsets of another set G. It is induced by a clopen subbasis that comprises sets of the form $S_g = \{K \in S \mid g \in K\}$, and their complements.

Proposition 2.2. *A net $(\mathrm{K}_n) \subseteq \mathrm{Sub}\,\Gamma$ converges to a subgroup $\mathrm{K} \in \mathrm{Sub}\,\Gamma$ iff, for every $\gamma \in \Gamma$, there is an index m such that $n \ge m$ implies $\gamma \in \mathrm{K}_n$ iff $\gamma \in \mathrm{K}$.*

Proof. Any basic open subset U of $\mathrm{Sub}\,\Gamma$ takes the form

$$\left(\bigcap_{i=0}^{j-1} S_{\gamma_i}\right) \cap \left(\bigcap_{i=j}^{k-1} S \backslash S_{\gamma_i}\right) = \{\mathrm{K} \in \mathrm{Sub}\,\Gamma \mid \forall i \in [0,j)(\gamma_i \in \mathrm{K})\} \cap \{\mathrm{K} \in \mathrm{Sub}\,\Gamma \mid \forall i \in [j,k)(\gamma_i \notin \mathrm{K})\}$$
$$= \{\mathrm{K} \le \Gamma \mid \forall i \in [0,k)(\gamma_i \in \mathrm{K} \Leftrightarrow i < j)\}$$

for some $\gamma_i \in \Gamma$ and integers $0 \le j \le k$. Therefore, for any pair of subgroups $\mathrm{K}_n, \mathrm{K} \in U$ within this basic set, we have $\mathrm{K}_n \cap \{\gamma_i \mid i \in [0,k)\} = \mathrm{K} \cap \{\gamma_i \mid i \in [0,k)\} = \{\gamma_i \mid i \in [0,j)\}$, ergo $\gamma_i \in \mathrm{K}_n$ iff $\gamma_i \in \mathrm{K}$ for every $i \in [0,k)$. ∎

Λ shall be our designated subgroup of Γ, and $\mathcal{O}_\Lambda = \{\gamma\Lambda\gamma^{-1} \mid \gamma \in \Gamma\}$ shall be its orbit under conjugation. Naturally, its orbit-closure is denoted $\overline{\mathcal{O}_\Lambda}$.

Definition 2.3. [GW15] A **uniformly recurrent subgroup** is a closed subset $\mathcal{U} \subseteq \operatorname{Sub}\Gamma$ that is *minimal* under conjugation; that is, for every $\mathrm{K} \in \mathcal{U}$, its orbit-closure $\overline{\mathcal{O}_\mathrm{K}} = \mathcal{U}$.

The conjugation action of Γ on $\overline{\mathcal{O}_\Lambda}$ (as well as the induced action on $\ell^\infty(\overline{\mathcal{O}_\Lambda})$) is denoted α when needed. That is, for all $\gamma \in \Gamma$ and $\mathrm{K} \in \operatorname{Sub}\Gamma$,

$$\alpha_\gamma(\mathrm{K}) = \gamma\mathrm{K}\gamma^{-1}.$$

Evidently, the stabilizer of K is its normalizer $N_\Gamma(\mathrm{K})$. The *topological stabilizer* of K is the set Γ_K° of group-elements that fix some neighborhood of $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$; while we know of no group-theoretic description of this set, we can at least see $\mathrm{K} \leq \Gamma_\mathrm{K}^\circ \leq N_\Gamma(\mathrm{K})$. Thus, not even this topological stabilizer can be trivial for nontrivial K, so the action on $\overline{\mathcal{O}_\Lambda}$ cannot even be *topologically* free. However, it is nonetheless useful for our (topological) stabilizers to be as small as possible.

Definition 2.4. We shall say the subgroup $\Lambda \leq \Gamma$ is **cheap** if every element of its orbit-closure is equal to its own normalizer. That is, $\mathrm{K} = N_\Gamma(\mathrm{K})$ for every $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$. A nominally weaker condition is **topologically cheap**, which means $\mathrm{K} = \Gamma_\mathrm{K}^\circ$ for every $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$.

Remark 2.5. "Cheap Λ" generalizes the notion of a *generic URS* (in the sense of Elek) to apply when $\overline{\mathcal{O}_\Lambda}$ is not minimal.

The coset space Γ/Λ is equipped with the left-multiplication action by Γ. We define $\omega : \Gamma/\Lambda \to \mathcal{O}_\Lambda$ as the Γ-equivariant surjection that preserves Λ:

$$\omega(\gamma\Lambda) = \gamma\Lambda\gamma^{-1} = \alpha_\gamma(\Lambda)$$

for every $\gamma \in \Gamma$. If Λ is cheap, then ω is bijective; indeed, $\omega(\gamma\Lambda) = \omega(\Lambda)$ iff $\gamma \in N_\Gamma(\Lambda)$, while $\gamma\Lambda = \Lambda$ iff $\gamma \in \Lambda$.

3 Graphs and kernels

A *rooted-Q-labeled graph* (G, x, q) is a directed graph G with a distinguished vertex $x \in V(G)$ and a labeling $q : E(G) \to Q$ of its edges. A *rooted-Q-labeled graph isomorphism* between

$(G,x,q),(G',x',q')$ is a graph isomorphism that preserves the root and all labels; we write $(G,x,q) \cong (G',x',q')$ if such an isomorphism exists. The ball $B_r(y)$ of radius $r \in \mathbb{N}$ around a vertex $y \in V(G)$ in the rooted-Q-labeled graph (G,x,q) is a rooted-Q-labeled subgraph that comprises all vertices of G that are within a length-r (undirected) walk of y, and all edges connecting these vertices, equipped with y as a root and the restriction of q as a labeling. We shall occasionally use the same notation to refer only to the vertices within the ball, but context will make these instances clear.

Definition 3.1. The **Schreier graph** of a subgroup $\mathrm{K} \leq \Gamma$ is the rooted-Q-labeled graph $S_\Gamma^Q(\mathrm{K})$ that has the cosets Γ/K as it vertices, K as its root, and an edge labeled γ going from x to γx for every $x \in \Gamma/\mathrm{K}$ and $\gamma \in Q$. While the exact graph depends on the choice of Q, this turns out to be irrelevant for our purposes, so we shall usually suppress it.

We note that $\lambda\gamma\mathrm{K} = \gamma\mathrm{K}$ iff $\lambda\gamma\mathrm{K}\gamma^{-1} = \gamma\mathrm{K}\gamma^{-1}$, so we have a label-preserving graph isomorphism from $S_\Gamma(\mathrm{K})$ to $S_\Gamma(\alpha_\gamma(\mathrm{K}))$ that sends K to $\gamma^{-1}\alpha_\gamma(\mathrm{K})$; thus we shall treat them as being the same labeled graph with different choices of root.

For each integer $r \in \mathbb{N}$, we partition the vertices/cosets $V(S_\Gamma(\Lambda)) = \Gamma/\Lambda$ into equivalence classes $[x]_r = \{y \in \Gamma/\Lambda \mid B_r(x) \cong B_r(y)\}$. Since vertices all have degree $|Q|$, there are finitely many distinct classes for each r. We also define sets $B_{\mathrm{K},r} = \{\mathrm{K}' \in \overline{\mathcal{O}_\Lambda} \mid B_r(\mathrm{K}') \cong B_r(\mathrm{K})\}$ for every $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$ and $r \in \mathbb{N}$. The following Lemma shows that these form a clopen basis of $\overline{\mathcal{O}_\Lambda}$'s topology.

Lemma 3.2. *A net $(\mathrm{K}_n) \subseteq \operatorname{Sub}\Gamma$ converges to a subgroup $\mathrm{K} \in \operatorname{Sub}\Gamma$ iff, for every $r \in \mathbb{N}$, there is an index m such that $n > m$ implies $B_r(\mathrm{K}_n) \cong B_r(\mathrm{K})$.*

Proof. For every $\gamma \in \Gamma$, there are by definition $\gamma_i \in Q$ such that

$$\gamma = \gamma_{l_Q(\gamma)} \cdots \gamma_2\gamma_1.$$

For any $\mathrm{K} \in \operatorname{Sub}(\Gamma)$, there is a walk

$$(\mathrm{K}, \gamma_1\mathrm{K}, \gamma_2\gamma_1\mathrm{K}, \ldots, \gamma_{l_Q(\gamma)}^{-1}\gamma\mathrm{K}, \gamma\mathrm{K})$$

in $B_{l_Q(\gamma)}(\mathrm{K}) \subseteq S_\Gamma(\mathrm{K})$, and it is a cycle iff $\gamma \in \mathrm{K}$. If $B_r(\mathrm{K}) \cong B_r(\mathrm{K}_n)$, then a string of r-many labels corresponds to a cycle in one ball iff it corresponds to a cycle in the other, therefore $\mathrm{K} \cap Q^r = \mathrm{K}_n \cap Q^r$.

Conversely, suppose for sake of contraposition that $B_r(\mathrm{K}) \ncong B_r(\mathrm{K}_n)$. Then there are $\gamma, \lambda \in Q^r$

such that the vertices $\gamma \mathrm{K} = \lambda \mathrm{K}$, while $\gamma \mathrm{K}_n \neq \lambda \mathrm{K}_n$. Ergo $\lambda^{-1}\gamma \in \mathrm{K}$ while $\lambda^{-1}\gamma \notin \mathrm{K}_n$, therefore $\mathrm{K}_n \cap Q^{2r} \neq \mathrm{K} \cap Q^{2r}$. ∎

We denote by $\delta_x \in \ell^2(\Gamma/\Lambda)$ the indicator function of $x \in \Gamma/\Lambda$:

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{else.} \end{cases}$$

Definition 3.3. A **local kernel** is an operator $K \in \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ such that there is a $w \in \mathbb{N}$ satisfying the following for all vertices $x, y \in \Gamma/\Lambda$:

- If $y \notin B_w(x)$, then $K(\delta_x)(y) = 0$.
- If $[x]_w = [y]_w$ and $\gamma \in Q^w$, then $K(\delta_x)(\gamma x) = K(\delta_y)(\gamma y)$.

The smallest w satisfying these conditions is called the *width* of K (note that the conditions are still satisfied for any larger w).

While proving it directly is not too difficult, we shall see in Section 7Covariant representationssection.7 that the definition of a local kernel is independent of the choice of Q.

Proposition 3.4. *The set of local kernels forms a $*$-subalgebra of* $\mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$.

Proof. Let $c \in \mathbb{C}$, and K, K' be local kernels with respective widths w, w'. Then for all $x \in \Gamma/\Lambda$ and $y \notin B_{w+w'}(x)$, we have that $x \notin B_w(y)$ and $B_w(y) \cap B_{w'}(x) = \varnothing$; thus

$$\begin{aligned} (cK + K')(\delta_x)(y) &= cK(\delta_x)(y) + K'(\delta_x)(y) = c(0) + 0 = 0, \\ K^*(\delta_x)(y) &= \overline{K(\delta_y)(x)} = \overline{0} = 0, \\ KK'(\delta_x)(y) &= \sum_{z \in \Gamma/\Lambda} K(\delta_z)(y) K'(\delta_x)(z) \\ &= \sum_{z \in B_w(y) \cap B_{w'}(x)} K(\delta_z)(y) K'(\delta_x)(z) = 0. \end{aligned}$$

We check that linear combination preserves the second condition by supposing $x, y \in \Gamma/\Lambda$ and $\gamma \in \Gamma$ satisfy $[x]_{w+w'} = [y]_{w+w'}$ and $\gamma \in Q^{w+w'}$. We recall the parenthetical note from

Definition 3.3theorem.3.3, so

$$\begin{aligned}(cK+K')(\delta_x)(\gamma x) &= cK(\delta_x)(\gamma x)+K'(\delta_x)(\gamma x)\\ &= cK(\delta_y)(\gamma y)+K'(\delta_y)(\gamma y)=(cK+K')(\delta_y)(\gamma y).\end{aligned}$$

We check involution by supposing $x, y \in \Gamma/\Lambda$ and $\gamma \in \Gamma$ satisfy $[x]_{2w} = [y]_{2w}$ and $\gamma \in Q^w$. Then $[\gamma x]_w = [\gamma y]_w$ and $l_Q(\gamma^{-1}) = l_Q(\gamma) \leq w$, so

$$K^*(\delta_x)(\gamma x) = \overline{K(\delta_{\gamma x})(x)} = \overline{K(\delta_{\gamma x})(\gamma^{-1}\gamma x)} = \overline{K(\delta_{\gamma y})(\gamma^{-1}\gamma y)} = K^*(\delta_y)(\gamma y).$$

We must still address $\gamma \in Q^{2w}\backslash Q^w$, for which there are two cases. In the case $\gamma x \in B_w(x)$, we may use the above argument by replacing γ^{-1} with some $\lambda \in Q^w$ such that $\lambda\gamma x = x$. The other case is $\gamma x \notin B_w(x)$, which is equivalent to $\gamma y \notin B_w(y)$, and implies $K^*(\delta_x)(\gamma x) = K^*(\delta_y)(\gamma y) = 0$.

We finally check multiplication by supposing $x, y \in \Gamma/\Lambda$ and $\gamma \in \Gamma$ satisfy $[x]_{w+w'} = [y]_{w+w'}$ and $\gamma \in Q^{w+w'}$. Then for each $z \in B_{w'}(x)$ there is a $\lambda \in Q^{w'}$ (so $[\lambda x]_w = [\lambda y]_w$) such that $z = \lambda x$, and if $z \in B_w(\gamma x)$ then there is also $\lambda' \in Q^w$ such that $\gamma x = \lambda' z = \lambda'\lambda x$. Thus

$$\begin{aligned}KK'(\delta_x)(\gamma x) &= \sum_{z\in B_w(\gamma x)\cap B_{w'}(x)} K(\delta_z)(\gamma x)K'(\delta_x)(z)\\ &= \sum_{\lambda x\in B_w(\gamma x)\cap B_{w'}(x)} K(\delta_{\lambda x})(\lambda'\lambda x)K'(\delta_x)(\lambda x)\\ &= \sum_{\lambda y\in B_w(\gamma y)\cap B_{w'}(y)} K(\delta_{\lambda y})(\lambda'\lambda y)K'(\delta_y)(\lambda y)\\ &= KK'(\delta_y)(\gamma y).\end{aligned}$$

Therefore the set of local kernels is closed under $*$-algebraic operations. ∎

4 The C*-algebra

Definition 4.1. The C*-algebra $C^*_\Gamma(\Lambda) \subseteq \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ is the closure of the $*$-subalgebra of all local kernels. This generalizes the notion of the reduced C*-algebra of a URS (as defined by Elek) to apply when $\overline{\mathcal{O}_\Lambda}$ is not minimal.

Theorem 4.2. *Suppose $\Lambda' \in \overline{\mathcal{O}_\Lambda}$. Then $C^*_\Gamma(\Lambda')$ is a quotient of $C^*_\Gamma(\Lambda)$, with equality holding if $\overline{\mathcal{O}_{\Lambda'}} = \overline{\mathcal{O}_\Lambda}$.*

Proof. Since $\Lambda' \in \overline{\mathcal{O}_\Lambda}$, Lemma 3.2theorem.3.2 ensures that, for every radius $r \in \mathbb{N}$, we may find $\gamma_r \in \Gamma$ such that the balls $B_r(\Lambda') \cong B_r(\alpha_{\gamma_r}(\Lambda)) \cong B_r(\gamma_r\Lambda)$. Given a local kernel $K \in C^*_\Gamma(\Lambda)$ of width w, we shall use this fact to define a local kernel $K' \in C^*_\Gamma(\Lambda')$ (also of width w). Every vertex $x' \in \Gamma/\Lambda'$ of distance r from the root has corresponding $x \in B_r(\gamma_{r+w}\Lambda)$ with $B_w(x) \cong B_w(x')$. We set $K'(\delta_{x'})(\gamma x') = K(\delta_x)(\gamma x)$ for every $\gamma \in Q^w$.

By reversing the construction—with the added stipulation that $K(\delta_x)$ is identically 0 when no suitable x' exists—we may construct an appropriate K for any given local kernel $K' \in C^*_\Gamma(\Lambda')$. Ergo our map $K \mapsto K'$ is a surjection between the $*$-algebras of local kernels. Moreover, if $\overline{\mathcal{O}_{\Lambda'}} = \overline{\mathcal{O}_\Lambda}$, then $\Lambda \in \overline{\mathcal{O}_{\Lambda'}}$, therefore our map is bijective.

The map is also evidently linear and preserves adjoints. To show multiplicativity we first note the construction would not be changed by using w larger than K's width. Let $K, L \in C^*_\Gamma(\Lambda)$ be local kernels with respective widths w, v, so the product KL has width at most $w + v$. Choose $\gamma \in Q^{w+v}$ and $x' \in \Gamma/\Lambda'$. We can find $x \in \Gamma/\Lambda$ such that $B_{w+v}(x) \cong B_{w+v}(x')$, so $B_w(\lambda x') \cong B_w(\lambda x)$ for every $\lambda \in Q^v$. Thus

$$
\begin{aligned}
K'L'(\delta_{x'})(\gamma x') &= \sum_{y' \in \Gamma/\Lambda'} K'(\delta_{y'})(\gamma x')L'(\delta_{x'})(y') \\
&= \sum_{y' \in B_w(\gamma x') \cap B_v(x')} K'(\delta_{y'})(\gamma x')L'(\delta_{x'})(y') \\
&= \sum_{y \in B_w(\gamma x) \cap B_v(x)} K(\delta_y)(\gamma x)L(\delta_x)(y) = KL(\delta_x)(\gamma x).
\end{aligned}
$$

Therefore $K'L' = (KL)'$, so $K \mapsto K'$ is a $*$-epimorphism.

Finally we show it is continuous. For given local kernel K' of width w and $\epsilon > 0$, there is a finitely supported unit vector $f' \in \ell^2(\Gamma/\Lambda')$ such that $\|K'\| - \epsilon \leq \|K'(f')\|$. Since $\operatorname{supp}(f')$ is finite, it is contained in the ball $B_r(\Lambda')$ for some finite radius r. By our identification of the labeled graph $S_\Gamma(\Lambda)$ with $S_\Gamma(\alpha_{\gamma_{r+2w}}(\Lambda))$ and the fact that the choice of root has no bearing on local kernels, we may assume without loss of generality $\Lambda = \gamma_{r+2w}\Lambda$, so that there is a rooted-labeled graph isomorphism $\varphi : B_{r+2w}(\Lambda) \to B_{r+2w}(\Lambda')$. Then $f := f' \circ \varphi \in \ell^2(\Gamma/\Lambda)$ is supported on $B_r(\Lambda)$. Thus

$$\begin{aligned}
\|K(f)\|^2 = \sum_{x\in\Gamma/\Lambda} |K(f)(x)|^2 &= \sum_{x\in\Gamma/\Lambda} \left| \sum_{y\in\Gamma/\Lambda} f(y)K(\delta_y)(x) \right|^2 \\
&= \sum_{x\in B_{r+w}(\Lambda)} \left| \sum_{y\in B_w(x)} f(y)K(\delta_y)(x) \right|^2 \\
&= \sum_{x\in B_{r+w}(\Lambda)} \left| \sum_{y\in B_w(x)} f' \circ \varphi(y)K'(\delta_{\varphi(y)})(\varphi(x)) \right|^2 \\
&= \sum_{x'\in B_{r+w}(\Lambda')} \left| \sum_{y'\in B_w(x')} f'(y')K'(\delta_{y'})(x') \right|^2 \\
&= \|K'(f')\|^2.
\end{aligned}$$

This confirms our map $K \mapsto K'$ is bounded, therefore it extends to $C^*_\Gamma(\Lambda)$. Finally, as a $*$-homomorphism from a C*-algebra, the extension's codomain must be closed, therefore is $C^*_\Gamma(\Lambda')$.

Remark 4.3. There is a faithful conditional expectation from $\mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ to $\ell^\infty(\Gamma/\Lambda)$ that simply erases all entries not along the diagonal. When restricted to $C^*_\Gamma(\Lambda)$, we shall call this the *canonical conditional expectation* and denote it by E. To be clear,

$$E(T)(x) = T(\delta_x)(x).$$

5 Colors

A *colored-rooted-labeled graph* (G, x, q, c) is a rooted-labeled graph (G, x, q) with a coloring $c : V(G) \to \mathbb{Z}$ of its vertices; the coloring is called finite if the image of c is finite. Similarly, a *colored-rooted-labeled graph isomorphism* between $(G, x, q, c), (G', x', q', c')$ is a rooted-labeled graph isomorphism that also preserves the coloring, and we denote its existence by $(G, x, q, c) \cong^C (G', x', q', c')$. From here we define $B^c_r(y)$, $[x]^c_r$, *colored local kernels*, $C^{c*}_\Gamma(\Lambda) \subseteq \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ (which contains $C^*_\Gamma(\Lambda)$, since $[x]^c_w = [y]^c_w$ is stronger than $[x]_w = [y]_w$), and E^c analogously to the colorless versions.

Remark 5.1. For this section, we shall assume $\overline{\mathcal{O}_\Lambda}$ is a URS—that is, it is minimal. By [Ele18, Thm. 1], there is a minimal $\overline{\mathcal{O}_\Lambda}$-proper Bernoulli subshift $M \subseteq B^K(\overline{\mathcal{O}_\Lambda})$. Each element of

M is an $\overline{\mathcal{O}_\Lambda}$-regular (hence nonrepeating) finitely-colored Schreier graph, meaning it admits no non-trivial colored-labeled graph automorphism. The Γ-action on M is given by $\gamma.\big(S_\Gamma(\mathrm{K}), d\big) = \big(S_\Gamma(\alpha_\gamma(\mathrm{K})), \gamma.d\big)$, where each color $(\gamma.d)\big(\lambda\alpha_\gamma(\mathrm{K})\big) = d(\lambda\gamma\mathrm{K})$. Thus we have an injective analog to the Γ-equivariant map ω:

$$\omega_c : \Gamma/\Lambda \to M, \qquad \omega_c(\gamma\Lambda) = \gamma.(S_\Gamma(\Lambda), c).$$

The details of M's existence may be found in [Ele18, Sec. 3], but we only need to understand its description. The topology on M is in analogy to Lemma 3.2theorem.3.2: a net $\big((S_\Gamma(\mathrm{K}_n), d_n)\big) \subseteq M$ of colored Schreier graphs converges to $(S_\Gamma(\mathrm{K}), d)$ iff for every $r \in \mathbb{N}$ there is an index m such that $n > m$ implies $B_r^{d_n}(\mathrm{K}_n) \cong^C B_r^d(\mathrm{K})$. In fact, we can just think of M as the inverse inductive limit of colored-rooted-labeled balls: for each $r \in \mathbb{N}$, let $M_r = \{[x]_r^c \mid x \in \Gamma/\Lambda\}$ be the finite set of equivalence classes of vertices in our distinguished Schreier graph $(S_\Gamma(\Lambda), c) \in M$. (Since M is minimal, this is still independent of our choice of Λ.) Using discrete topologies and surjections $[x]_{r+1}^c \mapsto [x]_r^c$, we characterize $M = \varprojlim M_r$ by identifying the graph $(S_\Gamma(\mathrm{K}), d) \in M$ with the sequence $([x_n]_n^c)$ where each $B_n^c(x_n) \cong^C B_n^d(\mathrm{K})$.

In this section, the goal is the following:

Proposition 5.2. *$C_\Gamma^{c*}(\Lambda)$ is simple.*

This is exactly as Elek describes in [Ele18, Sec. 6.3]: we will show every nontrivial (closed, two sided) ideal of $C_\Gamma^{c*}(\Lambda)$ has nontrivial intersection with an abelian C*-subalgebra $\overline{\mathcal{A}} \cong C(M)$, which—since M is minimal—admits no Γ-invariant ideals.

Lemma 5.3. *If $w \in \mathbb{N}$, then there is a radius $r_w \in \mathbb{N}$ such that $[x]_{r_w}^c = [y]_{r_w}^c$ and $x \neq y$ together imply $y \notin B_w^c(x)$.*

Proof. For sake of contradiction, suppose there is a width $w \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, there are distinct $x_n, y_n \in \Gamma/\Lambda$ satisfying $y_n \in B_w^c(x_n)$ and $[x_n]_n^c = [y_n]_n^c$. Since a radius-w ball has at most $|Q^w| < \infty$ vertices, there must be some $\gamma \in Q^w$ such that $y_n = \gamma x_n$ holds for infinitely many n. Passing to a subsequence, we may assume it holds for all n. For each n, $([x_n]_r^c) \in \varprojlim M_r$ corresponds to some colored Schreier graph $G_n \in M$. By compactness of M, the sequence (G_n) has a subsequence (we again assume it is the whole sequence) converging to some $(S_\Gamma(\mathrm{K}), d) \in M$.

This contradicts $\overline{\mathcal{O}_\Lambda}$-regularity: For every radius r, by convergence there is an $n > r$ such that $B^d_{r+w}(\mathrm{K}) \cong^C B^c_{r+w}(x_n)$, hence

$$B^d_r(\mathrm{K}) \cong^C B^c_r(x_n) \cong^C B^c_r(y_n) = B^c_r(\gamma x_n) \cong^C B^d_r(\gamma \mathrm{K}).$$

Therefore $(S_\Gamma(\mathrm{K}), d)$ admits a non-trivial colored-labeled graph automorphism. ∎

Lemma 5.4. *$C(M)$ is $*$-isomorphic to a unital C^*-subalgebra of $C^{c*}_\Gamma(\Lambda)$.*

Proof. The family of characteristic functions $\chi_{[x]^c_w} \in \ell^\infty(\Gamma/\Lambda) \subseteq \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ on the equivalence classes for different $w \in \mathbb{N}$ and $x \in \Gamma/\Lambda$ generate a $*$-subalgebra $\mathcal{A}$ which contains the identity operator $\mathrm{id}_{\ell^2(\Gamma/\Lambda)} = \sum_{\xi \in M_0} \chi_\xi$.

By the inverse inductive limit characterization, a clopen basis of M's topology is given by the family of sets $C^x_w := \{(S_\Gamma(\mathrm{K}), d) \in M \mid B^d_w(\mathrm{K}) \cong^C B^c_w(x)\} = \{(\xi_r) \in \varprojlim M_r \mid \xi_w = [x]^c_w\}$ for different $w \in \mathbb{N}$ and $x \in \Gamma/\Lambda$. Thus $C(M)$ is generated by the characteristic functions $\chi_{C^x_w}$, so $\chi_{C^x_w} \mapsto \chi_{[x]^c_w}$ defines a Γ-equivariant $*$-isomorphism to the closure $\overline{\mathcal{A}}$.

As a multiplication operator, $\chi_{[x]^c_w} \in \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ is a projection given by

$$\chi_{[x]^c_w}(\delta_y)(z) = \begin{cases} 1 & \text{if } y = z \in [x]^c_w, \\ 0 & \text{else.} \end{cases}$$

Thus it is a colored local kernel, therefore $\overline{\mathcal{A}} \subset C^{c*}_\Gamma(\Lambda)$. ∎

For each $\gamma \in \Gamma$, let $U_\gamma \in \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ be the unitary enacting the induced Γ-action; that is, for all $f \in \ell^2(\Gamma/\Lambda)$ and $x \in \Gamma/\Lambda$,

$$U_\gamma(f)(x) = f(\gamma^{-1}x).$$

This is a local kernel of width (at most) $l_Q(\gamma)$, and $\mathrm{Ad}(U_\gamma)|_{\overline{\mathcal{A}}}$ corresponds to the induced action of

γ on $C(M)$:

$$\mathrm{Ad}(U_\gamma)(\chi_{[x]_r^c})(\delta_y)(z) = \chi_{[x]_r^c}(\delta_{\gamma^{-1}y})(\gamma^{-1}z) = \begin{cases} 1 & \text{if } \gamma^{-1}y = \gamma^{-1}z \in [x]_r^c. \\ 0 & \text{else,} \end{cases}$$
$$= \begin{cases} 1 & \text{if } y = z \in \gamma[x]_r^c, \\ 0 & \text{else,} \end{cases}$$
$$= \chi_{\gamma[x]_r^c}(\delta_y)(z)$$

for any $x, y, z \in \Gamma/\Lambda$.

We must now refer to *completely positive* maps, which describe those maps $\phi : \mathcal{A} \to \mathcal{B}$ between C*-algebras such that the matrix $[\phi(a_{i,j})] \in \mathbb{M}_n(\mathcal{B})$ is positive for every positive matrix $[a_{i,j}] \in \mathbb{M}_n(\mathcal{A})$ (see e.g. [BO08, Def. 1.5.1]). Notably, this includes $a \mapsto \sum_{i=1}^n b_i \pi_i(a) b_i^*$ for any $b_i \in \mathcal{B}$ and $*$-homomorphisms $\pi_i : \mathcal{A} \to \mathcal{B}$.

For each $r \in \mathbb{N}$, define the unital completely positive (u.c.p.) map $E_r^c : C_\Gamma^{c*}(\Lambda) \to C_\Gamma^{c*}(\Lambda)$ by

$$E_r^c(K) = \sum_{\xi \in M_r} \chi_\xi K \chi_\xi.$$

Note that, if K has finite width w, then

$$E^c(K) = \sum_{[x]_w^c \in M_w} K(\delta_x)(x) \chi_{[x]_w^c},$$

so $\mathrm{Im}(E^c) = \overline{\mathcal{A}}$.

Lemma 5.5. *$\lim_r E_r^c(K) = E^c(K)$ for every $K \in C_\Gamma^{c*}(\Lambda)$.*

Proof. Let $K \in C_\Gamma^{c*}(\Lambda)$ have finite width w. Find r_w as in Lemma 5.3theorem.5.3 so that $[x]_{r_w}^c = [y]_{r_w}^c$ and $x \neq y$ together imply $K(\delta_x)(y) = 0$. Thus, for every $r \geq r_w$,

$$E_r^c(K)(\delta_y) = \chi_{[y]_r^c} K(\delta_y) = \sum_{x \in [y]_r^c} K(\delta_y)(x)\delta_x = K(\delta_y)(y)\delta_y = E^c(K)(\delta_y).$$

Therefore the result is proven for local kernels, and the rest follows from density and continuity. ∎

Proof of Proposition 5.2theorem.5.2. Suppose $\mathcal{I} \trianglelefteq C_\Gamma^{c*}(\Lambda)$ is an ideal containing nonzero operator

T. Then $T^*T \in \mathcal{I}$, hence each $E^c_r(T^*T) \in \mathcal{I}$, hence the limit $E^c(T^*T) \in \mathcal{I}$. Since E^c is faithful, $E^c(T^*T)$ is also nonzero, so $\mathcal{I} \cap \overline{\mathcal{A}}$ is a nonzero Γ-invariant (due to the unitaries U_γ) ideal of $\overline{\mathcal{A}} \cong C(M)$. Thus $\mathrm{id}_{\ell^2(\Gamma/\Lambda)} \in \mathcal{I}$, therefore $\mathcal{I} = C^{c*}_\Gamma(\Lambda)$. ∎

6 Coamenability

Remark 6.1. We continue to assume $\overline{\mathcal{O}_\Lambda}$ is a URS for this section. We immediately use this assumption in the following lemma—a necessary component for our characterization of an amenable trace. We shall also assume Λ has infinte index in Γ (ergo $S_\Gamma(\Lambda)$ is an infinite graph), since the alternative is trivial.

Lemma 6.2. $C^{c*}_\Gamma(\Lambda)$ *contains no nonzero compact operators. (Hence neither does* $C^*_\Gamma(\Lambda)$*.)*

Proof. By definition, for any $L \in C^{c*}_\Gamma(\Lambda)$ with $L(\delta_x)(x') \neq 0$, there is some $K \in C^{c*}_\Gamma(\Lambda)$ of finite width w such that $\|L - K\| < L(\delta_x)(x')$. But for every $y \in [x]^c_w$ there is a coset $y' \in \Gamma/\Lambda$ such that $K(\delta_y)(y') = K(\delta_x)(x')$, hence $L(\delta_y)(y') \neq 0$. Thus, in order for L to be compact, $[x]^c_w$ must be finite, which contradicts the minimality of M.

Indeed, for sake of contradiction, suppose $F \subset S_\Gamma(\Lambda)$ is a finite subgraph such that the set $[F]$ of colored-labeled isomorphic subgraphs is finite. Then since $S_\Gamma(\Lambda)$ is an infinite graph, it has infinitely many vertices x such that the subgraph $B_1(x) \subset S_\Gamma(\Lambda)$ is disjoint from every subgraph in $[F]$. This allows us to choose one particular such vertex x_1 that has an infinite equivalence class $[x_1]^c_1$. Similarly, we inductively choose $x_n \in [x_{n-1}]^c_{n-1}$ so that $[x_n]^c_n$ is infinite and $B_n(x_n)$ is disjoint from every subgraph in $[F]$. The sequence $([x_n]^c_n) \subseteq M$ corresponds to a colored Schreier graph G which contains no subgraph colored-labeled isomorphic to F, therefore $(S_\Gamma(\Lambda), c)$ cannot be in G's orbit-closure. ∎

Remark 6.3. We shall also assume that Λ is *coamenable*; that is, the action of Γ on Γ/Λ is amenable (see [BO08, Def. 12.2.12]). In short, this means there exists a *Følner sequence* $(F_k) \subset \Gamma/\Lambda$ characterized by the property

$$\frac{|\gamma F_k \triangle F_k|}{|F_k|} \to 0$$

for every $\gamma \in \Gamma$ (where $\triangle$ denotes the symmetric difference and $|S|$ denotes the cardinality of S). This in fact implies that every $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$ is coamenable; more generally, if a subgroup in $\overline{\mathcal{O}_\mathrm{K}}$ is coamenable, then K is coamenable [Ele18, Prop. 5.2].

Because of the continuous Γ-equivariant ω, Λ is coamenable if Γ acts amenably on $\overline{\mathcal{O}_\Lambda}$. However, even in the cheap, minimal case, the converse is not true: If the Γ-action on $\overline{\mathcal{O}_\Lambda}$ is amenable, then $C^*_\Gamma(\Lambda)$ is a quotient of the reduced crossed product $C(\overline{\mathcal{O}_\Lambda}) \rtimes_{\alpha,r} \Gamma$ (see Section 7Covariant representationssection.7) that admits an amenable trace. This trace can be composed with the quotient map and restricted to $C^*_r(\Gamma) \subseteq C(\overline{\mathcal{O}_\Lambda}) \rtimes_{\alpha,r} \Gamma$ to imply Γ itself is amenable. Finally, since now we see $C(\overline{\mathcal{O}_\Lambda}) \rtimes_{\alpha,r} \Gamma$ is nuclear [BO08, Thm. 4.2.6.b], we conclude $C^*_\Gamma(\Lambda)$ must be nuclear. The catch is that in [Ele18, Sec. 10.1], Elek constructed a generic, coamenable URS with non-exact reduced C*-algebra.

Lemma 6.4. *There exists a Γ-invariant probability measure μ on M.*

Proof. We use the Følner sequence (F_k) and the Γ-equivariant injection ω_c to define probability measures μ_k on M by

$$\mu_k(U) = \mu_k\big(U \cap \operatorname{Im}(\omega_c)\big) = \frac{|\omega_c^{-1}(U) \cap F_k|}{|F_k|}$$

for any $U \subseteq M$. For any $\gamma \in \Gamma$,

$$\big|\omega_c^{-1}(\gamma U) \cap F_k\big| = \big|\big(\gamma\omega_c^{-1}(U)\big) \cap F_k\big| = \big|\gamma\big(\omega_c^{-1}(U) \cap (\gamma^{-1}F_k)\big)\big| = \big|\omega_c^{-1}(U) \cap (\gamma^{-1}F_k)\big|,$$

$$\big|\mu_k(U) - \mu_k(\gamma U)\big| \leq \frac{|\omega_c^{-1}(U) \cap (F_k \triangle \gamma^{-1}F_k)|}{|F_k|} \leq \frac{|F_k \triangle \gamma^{-1}F_k|}{|F_k|} \overset{k\to\infty}{\longrightarrow} 0.$$

Therefore, since M is compact, (μ_k) must have a subsequence (we shall assume itself) that weakly converges to our μ. ◻

For each $k \in \mathbb{N}$, we define $P_k \in \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ to be the finite-dimensional projection onto $\ell^2(F_k)$. We then define $\tau(T)$ as the limit of

$$\frac{\langle TP_k, P_k\rangle_{\mathrm{HS}}}{\|P_k\|_{\mathrm{HS}}^2} = \frac{\sum_{y\in\Gamma/\Lambda}\langle TP_k(\delta_y), P_k(\delta_y)\rangle}{\sum_{y\in\Gamma/\Lambda}\|P_k(\delta_y)\|^2} = \frac{\sum_{y\in F_k}\langle T(\delta_y), \delta_y\rangle}{\sum_{y\in F_k}\|\delta_y\|^2} = \sum_{x\in F_k}\frac{E^c(T)(x)}{|F_k|}.$$

It is no coincidence if this formula appears familiar.

Definition 6.5. [Bro06, Thm. 3.1.7.3] A trace τ' on a C*-algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ that contains no nonzero compact operators is called **amenable** if there exists a sequence $(P'_k) \subset \mathbb{B}(\mathcal{H})$ of finite rank projections such that, for every $a \in \mathcal{A}$,

$$\tau'(a) = \lim_{k\to\infty} \frac{\langle aP'_k, P'_k\rangle_{\mathrm{HS}}}{\|P'_k\|^2_{\mathrm{HS}}}, \qquad\qquad \lim_{k\to\infty} \frac{\|aP'_k - P'_k a\|^2_{\mathrm{HS}}}{\|P'_k\|^2_{\mathrm{HS}}} = 0.$$

The focus of this section is the following:

Theorem 6.6. *τ is a faithful amenable trace on $C^{c*}_\Gamma(\Lambda)$.*

As in the previous section, the proof is just a translation of what Elek did in the generic case [Ele18, Sec. 9.1]. Of course, first we need the following:

Lemma 6.7. *τ is a well-defined trace on colored local kernels.*

Proof. For any $(S_\Gamma(\mathrm{K}), d) \in M$ and $r \in \mathbb{N}$, by minimality there is a coset $x_{\mathrm{K},r} \in \Gamma/\Lambda$ such that $B^d_r(\mathrm{K}) \cong^C B^c_r(x_{\mathrm{K},r})$. Thus $\mu_k\big(\{(S_\Gamma(\mathrm{K}'), d') \in M \mid B^{d'}_r(\mathrm{K}') \cong^C B^d_r(\mathrm{K})\}\big)$ is equal to

$$\mu_k\big(\omega_c([x_{\mathrm{K},r}]^c_r)\big) = \int_{\operatorname{Im}\omega_c} \chi_{\omega_c([x_{\mathrm{K},r}]^c_r)}(G)\,\mathrm{d}\mu_k(G) = \sum_{y\in F_k} \frac{\chi_{[x_{\mathrm{K},r}]^c_r}(y)}{|F_k|} = \frac{|[x_{\mathrm{K},r}]^c_r \cap F_k|}{|F_k|}.$$

For any $L \in C^{c*}_\Gamma(\Lambda)$ with finite width w, this therefore shows

$$\begin{aligned}\frac{\langle LP_k, P_k\rangle_{\mathrm{HS}}}{\|P_k\|^2_{\mathrm{HS}}} &= \sum_{[y]^c_w\in M_w} \sum_{x\in[y]^c_w} \chi_{F_k}(x)\frac{E^c(L)(x)}{|F_k|} = \sum_{[y]^c_w\in M_w} E^c(L)(y)\frac{\big|[y]^c_w \cap F_k\big|}{|F_k|}\\ &= \sum_{[y]^c_w\in M_w} E^c(L)(y)\mu_k\big(\{(S_\Gamma(\mathrm{K}), d) \in M \mid B^d_w(\mathrm{K}) \cong^C B^c_w(y)\}\big)\\ &= \int_M E^c(L)(x_{\mathrm{K},w})\,\mathrm{d}\mu_k\big((S_\Gamma(\mathrm{K}), d)\big) \xrightarrow{k\to\infty} \int_M E^c(L)(x_{\mathrm{K},w})\,\mathrm{d}\mu\big((S_\Gamma(\mathrm{K}), d)\big).\end{aligned}$$

∎

Lemma 6.8. *For any $L \in C^{c*}_\Gamma(\Lambda)$ with finite width w,*

$$\lim_{k\to\infty} \frac{\|LP_k - P_kL\|^2_{\mathrm{HS}}}{\|P_k\|^2_{\mathrm{HS}}} = 0.$$

Proof. Note that

$$\begin{aligned}\|LP_k - P_kL\|^2_{\mathrm{HS}} &= \operatorname{Tr}\big(|LP_k - P_kL|^2\big) = \operatorname{Tr}\big(P_kL^*LP_k - P_kL^*P_kL - L^*P_kLP_k + L^*P_kP_kL\big)\\ &= \operatorname{Tr}\big(|LP_k|^2 - |P_kLP_k|^2 - |P_kL^*P_k|^2 + |L^*P_k|^2\big) = \operatorname{Tr}(|P_k^\perp LP_k|^2) + \operatorname{Tr}(|P_k^\perp L^*P_k|^2).\end{aligned}$$

Let ∂F_k denote the set of vertices of F_k adjacent to vertices not in F_k (thus the Følner condition yields $\frac{|\partial F_k|}{|F_k|} \leq \sum_{\gamma\in Q} \frac{|F_k \setminus \gamma F_k|}{|F_k|} \to 0$) and $Q_n = \{(x,y) \in \Gamma/\Lambda \times \Gamma/\Lambda \mid |P_k^\perp L P_k|^2(\delta_x)(y) \neq 0\}$. Recalling that Q is our finite symmetric set that generates Γ (thus every vertex in G has at most $|Q|$ neighbors), we find an upper bound on $|Q_n|$:

Suppose $(x,y) \in Q_n$, ergo $0 \neq |P_k^\perp L P_k|^2(\delta_x)(y) = \sum_{z\in\Gamma/\Lambda} \big(P_k^\perp L^* P_k(\delta_z)(y)\big)\big(P_k^\perp L P_k(\delta_x)(z)\big)$. We deduce $P_k^\perp L P_k(\delta_x)$ is not identically zero, so $x \in F_k$ and there exists at least one vertex $z \in B_w(x)\setminus F_k$. Thus there are at most $|Q|^{w-1}|\partial F_k|$ possibilities for x. Moreover, since $P_k L^* P_k^\perp(\delta_z)(y) \neq 0$, the vertex $z \in B_w(y)$, implying $y \in B_{2w}(x)$. Given x, this leaves at most $|Q|^{2w}$ possibilities for y.

Using these bounds,

$$\frac{\big|\mathrm{Tr}\big(|P_k^\perp L P_k|^2\big)\big|}{\|P_k\|_{\mathrm{HS}}^2} \leq \sum_{(x,y)\in Q_n} \frac{\big||P_k^\perp L P_k|^2(\delta_x)(y)\big|}{\|P_k\|_{\mathrm{HS}}^2} \leq \|L\|^2 |Q|^{3w-1} \frac{|\partial F_k|}{|F_k|} \xrightarrow{k\to\infty} 0.$$

Similarly, $\big|\mathrm{Tr}\big(|P_k^\perp L^* P_k|^2\big)\big| / \|P_k\|_{\mathrm{HS}}^2 \to 0$, therefore the result follows. ◻

Proof of Theorem 6.6theorem.6.6. Density and continuity generalize the prior lemmas to all of $C^{c*}_\Gamma(\Lambda)$, which we know to be simple and contain no nonzero compact operators. Therefore τ is a faithful amenable trace. ◻

Corollary 6.9. *Suppose $\overline{\mathcal{O}_{\mathrm{K}}}$ is a URS. Then the following are equivalent:*

1. K *is coamenable.*
2. $C^*_\Gamma(\mathrm{K})$ *admits a faithful amenable trace.*
3. $C^*_\Gamma(\mathrm{K})$ *admits an amenable trace.*

Proof. (2)⇒(3) is vacuous, while (1)⇒(2) combines the previous theorem with the fact that $C^*_\Gamma(\mathrm{K}) \subseteq C^{d*}_\Gamma(\mathrm{K})$. (3)⇒(1) is a minor alteration of [BO08, Prop. 6.3.2]:

Suppose $C^*_\Gamma(\mathrm{K}) \subset \mathbb{B}\big(\ell^2(\Gamma/\mathrm{K})\big)$ has an amenable trace τ. Then τ can be extended to a state on $\mathbb{B}\big(\ell^2(\Gamma/\mathrm{K})\big)$ such that $\tau(uTu^*) = \tau(T)$ for every operator $T \in \mathbb{B}\big(\ell^2(\Gamma/\mathrm{K})\big)$ and unitary $u \in C^*_\Gamma(\mathrm{K})$. For any multiplication operator $g \in \ell^\infty(\Gamma/\mathrm{K})$ and vector $f \in \ell^2(\Gamma/\mathrm{K})$,

$$\begin{aligned} U_\gamma g U_\gamma^*(f)(x) &= g(\gamma^{-1}.f)(\gamma^{-1}x) = g(\gamma^{-1}x)(\gamma^{-1}.f)(\gamma^{-1}x) = (\gamma.g)(x)f(x) \\ &= (\gamma.g)(f)(x). \end{aligned}$$

Thus $\tau(\gamma.g) = \tau(U_\gamma g U_\gamma^*) = \tau(g)$, so $g \mapsto \tau(g)$ is a Γ-invariant mean on $\ell^\infty(\Gamma/\mathrm{K})$, therefore K is coamenable. ◻

Remark 6.10. Note that, even for nonminimal orbit-closures, the proof above of (3)⇒(1) still applies. We also have from Theorem 4.2theorem.4.2 that any $C^*_\Gamma(\mathrm{K})$ with $\Lambda \in \overline{\mathcal{O}_\mathrm{K}}$ will also admit an amenable trace.

7 Covariant representations

Let $\beta : \Gamma \to \mathrm{Aut}(\mathcal{A})$ denote a Γ-action on a C*-algebra $\mathcal{A}$. A *covariant representation* $(\pi, u, \mathcal{H})$ of $\mathcal{A}$ consists of a $*$-homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ and a unitary representation $u : \Gamma \to \mathbb{B}(\mathcal{H})$ such that $u_\gamma \pi(a) u_\gamma^* = \pi \circ \beta_\gamma(a)$ for every $\gamma \in \Gamma$ and $a \in \mathcal{A}$. We refer the reader to [BO08, Ch. 4.1] for further discussion. Suffice it to say that in this section we show $C^*_\Gamma(\Lambda)$ is a quotient of the full crossed product $C(\overline{\mathcal{O}_\Lambda}) \rtimes_\alpha \Gamma$.

For every $r \in \mathbb{N}$, $\gamma \in Q^r$, and $x, y \in \Gamma/\Lambda$, define

$$L^r_{x,\gamma}(\delta_y) = \begin{cases} \delta_{\gamma y} & \text{if } y \in [x]_r, \\ 0 & \text{else;} \end{cases}$$

thus $L^r_{x,\gamma}$ is a local kernel of width r. Moreover, every local kernel K of width r is a linear combination of operators of this form:

$$K = \sum_{[x]_r \subseteq \Gamma/\Lambda} \sum_{\gamma \in Q^r} \frac{\big(K(\delta_x)(\gamma x)\big) L^r_{x,\gamma}}{|\{\lambda \in Q^r \mid \lambda x = \gamma x\}|}.$$

Define $\mu : C(\overline{\mathcal{O}_\Lambda}) \to \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$ as taking a function, restricting it to the orbit of Λ, passing to the coset-space, then using it as a multiplication operator:

$$\mu(g)(f)(\gamma\Lambda) = (g \circ \omega)(f)(\gamma\Lambda) = g(\gamma\Lambda\gamma^{-1}) f(\gamma\Lambda)$$

for any $g \in C(\overline{\mathcal{O}_\Lambda})$, $f \in \ell^2(\Gamma/\Lambda)$, and $\gamma \in \Gamma$.

Lemma 7.1. *μ is a $*$-monomorphism into $C^*_\Gamma(\Lambda)$.*

Proof. μ is a $*$-homomorphism by composition. μ is injective since $g \circ \omega = h \circ \omega$ for some $g, h \in C(\overline{\mathcal{O}_\Lambda})$ implies $g|_{\mathcal{O}_\Lambda} = h|_{\mathcal{O}_\Lambda}$, and $\mathcal{O}_\Lambda$ is dense in $\overline{\mathcal{O}_\Lambda}$.

By Lemma 3.2theorem.3.2 and compactness of $\overline{\mathcal{O}_\Lambda}$, $C(\overline{\mathcal{O}_\Lambda})$ is the closed linear span of the characteristic functions $\chi_{B_{\mathrm{K},r}}$. These have images given by

$$\begin{aligned}\mu(\chi_{B_{\omega(x),r}})(\delta_y) = \chi_{B_{\omega(x),r}}(\omega(y))\delta_y &= \begin{cases} \delta_y & \text{if } \omega(y) \in B_{\omega(x),r}, \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} \delta_{ey} & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases} \\ &= L^r_{x,e}(\delta_y). \end{aligned}$$

Therefore $C(\overline{\mathcal{O}_\Lambda}) \cong \mu(C(\overline{\mathcal{O}_\Lambda})) \subset C^*_\Gamma(\Lambda)$. ∎

Recall from Section 5Colorssection.5 that we have unitaries $U_\gamma \in C^*_\Gamma(\Lambda)$ enacting the induced Γ-action on $\ell^2(\Gamma/\Lambda)$. These shall form our unitary representation $U : \Gamma \to C^*_\Gamma(\Lambda)$.

Theorem 7.2. *$(U, \mu, \ell^2(\Gamma/\Lambda))$ is a covariant representation of the Γ-C^*-algebra $C(\overline{\mathcal{O}_\Lambda})$, and its image is $C^*_\Gamma(\Lambda)$.*

Proof. We first show covariance:

$$\begin{aligned}U_\gamma \mu(g) U^*_\gamma(f)(x) = \mu(g) U^*_\gamma(f)(\gamma^{-1}x) &= g(\omega(\gamma^{-1}x)) U^*_\gamma(f)(\gamma^{-1}x) \\ &= g\big(\alpha_{\gamma^{-1}}(\omega(x))\big) f(x) = \alpha_\gamma(g)(\omega(x)) f(x) = \mu(\alpha_\gamma(g))(f)(x).\end{aligned}$$

for all $\gamma \in \Gamma$, $g \in C(\overline{\mathcal{O}_\Lambda})$, $f \in \ell^2(\Gamma/\Lambda)$, and $x \in \Gamma/\Lambda$.

We next show that the spanning elements $L^r_{x,\gamma}$ are in $U(\Gamma)\mu(C(\overline{\mathcal{O}_\Lambda}))$; we already know from Lemma 7.1theorem.7.1 that this is true for $\gamma = e$. But

$$\begin{aligned}L^r_{x,\gamma}(\delta_y) &= \begin{cases} U_\gamma \delta_y & \text{if } y \in [x]_r, \\ , & \text{else,} \end{cases} \\ &= U_\gamma L^r_{x,e}(\delta_y).\end{aligned}$$

Therefore $C^*\big(\mu(C(\overline{\mathcal{O}_\Lambda}))U(\Gamma)\big) = C^*_\Gamma(\Lambda)$. ◻

Remark 7.3. In the future, we shall suppress μ, identifying $C(\overline{\mathcal{O}_\Lambda})$ itself as being a C*-subalgebra of $C^*_\Gamma(\Lambda)$. Also, we now have a way to define $C^*_\Gamma(\Lambda)$ even if Γ is not finitely generated, and most of the rest of this part still applies. Moreover, Theorem 4.2theorem.4.2 and the representations described here imply a natural generalization to *any* Γ-invariant subset $\mathcal{O} \subseteq \operatorname{Sub}\Gamma$ by representing $C_0(\mathcal{O}) \rtimes \Gamma$ on $\bigoplus_{\Lambda\in\mathcal{O}} \ell^2(\Gamma/\Lambda)$.

Unfortunately, U is not necessarily faithful, which limits the usefulness of such a crossed-product-like form. Fortunately, we have a workaround.

Lemma 7.4. *Suppose* N *is a subgroup of* Λ *and a normal subgroup of* Γ*, and* $\pi_\mathrm{N} : \Gamma \to \Gamma/\mathrm{N}$ *is the corresponding quotient map. Then* $S^{\pi_\mathrm{N}(Q)}_{\Gamma/\mathrm{N}}(\Lambda/\mathrm{N})$ *is a rooted graph isomorphic to* $S^Q_\Gamma(\Lambda)$*,* $C^*_{\Gamma/\mathrm{N}}(\Lambda/\mathrm{N})$ *is* $*$*-isomorphic to* $C^*_\Gamma(\Lambda)$*, and* $\overline{\mathcal{O}_{\Lambda/\mathrm{N}}}$ *is homeomorphic to* $\overline{\mathcal{O}_\Lambda}$*.*

Proof. $\mathrm{N} = \gamma\mathrm{N}\gamma^{-1} \subseteq \gamma\Lambda\gamma^{-1}$ for every $\gamma \in \Gamma$, so $\mathrm{N} \subseteq \mathrm{K}$ for every $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$. By surjectivity, Γ/N is generated by $\pi_\mathrm{N}(Q) := \bigsqcup_{\gamma\in Q}\{\pi_\mathrm{N}(Q)\}$. (The disjoint union is a technicality to ensure vertices' degrees remain the same.)

Suppose $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$ and $\pi_\mathrm{N}(\gamma) \in \mathrm{K}/\mathrm{N}$, so there is an element $\lambda \in \mathrm{K}$ such that $\pi_\mathrm{N}(\lambda) = \pi_\mathrm{N}(\gamma)$. Then $\lambda^{-1}\gamma \in \mathrm{N} \subseteq \mathrm{K}$, hence $\gamma = \lambda\lambda^{-1}\gamma \in \mathrm{K}$. This shows

$$\mathrm{K} = \pi_\mathrm{N}^{-1}(\mathrm{K}/\mathrm{N}), \tag{7.1}$$

which completes the chain of equivalent statements establishing a Γ-invariant bijection $\gamma\Lambda \mapsto \pi_\mathrm{N}(\gamma)(\Lambda/\mathrm{N})$ between Λ's and Λ/N's left cosets:

$$\begin{aligned}\gamma\Lambda = \lambda\Lambda \Leftrightarrow \Lambda = \gamma^{-1}\lambda\Lambda \Leftrightarrow \gamma^{-1}\lambda \in \Lambda \Leftrightarrow \pi_\mathrm{N}(\gamma^{-1}\lambda) \in \Lambda/\mathrm{N}\\ \Leftrightarrow \Lambda/\mathrm{N} = \pi_\mathrm{N}(\gamma^{-1}\lambda)(\Lambda/\mathrm{N}) \Leftrightarrow \pi_\mathrm{N}(\gamma)(\Lambda/\mathrm{N}) = \pi_\mathrm{N}(\lambda)(\Lambda/\mathrm{N}).\end{aligned}$$

Therefore we have our rooted graph isomorphism, which moreover sends edges labeled γ to edges labeled $\pi_\mathrm{N}(\gamma)$. The definition of a local kernel is tied to the rooted-labeled structure of the Schreier graph, therefore this also provides our $*$-isomorphism.

We showed with equation (7.1Covariant representationsequation.7.1) that the map $\mathrm{K} \mapsto \mathrm{K}/\mathrm{N}$ is bijective, and it is even evidently Γ-equivariant for the natural Γ-action on $\overline{\mathcal{O}_{\Lambda/\mathrm{N}}}$. Next we show it is bicontinuous in the Chabauty topology.

Let $(\mathrm{K}_n) \subseteq \overline{\mathcal{O}_\Lambda}$ be a net converging to some $\mathrm{K} \leq \Gamma$. Suppose $s \notin \pi_{\mathrm{N}}(\mathrm{K})$. Then $\mathrm{K} \cap \pi_{\mathrm{N}}^{-1}(s) = \varnothing$, so for every $\gamma \in \pi_{\mathrm{N}}^{-1}(s)$ there is an index m such that $n > m$ implies $\gamma \notin \mathrm{K}_n$. Since $\pi_{\mathrm{N}}^{-1}(s)$ is not a subset of K_n, equation (7.1Covariant representationsequation.7.1) tells us they are disjoint. Thus $n > m$ also implies $s = \pi_{\mathrm{N}}(\gamma) \notin \pi_{\mathrm{N}}(\mathrm{K}_n)$. Similarly, $t \in \pi_{\mathrm{N}}(\mathrm{K})$ implies $t \in \pi_{\mathrm{N}}(\mathrm{K}_n)$ for big enough n. Therefore $(\pi_{\mathrm{N}}(\mathrm{K}_n))$ converges to $\pi_{\mathrm{N}}(\mathrm{K})$, establishing continuity.

Conversely, let $(\mathrm{I}_n) \subseteq \overline{\mathcal{O}_{\Lambda/\mathrm{N}}}$ be a net converging to $\mathrm{I} \leq \Gamma/\mathrm{N}$. Suppose $\gamma \in \pi_{\mathrm{N}}^{-1}(\mathrm{I})$, ergo $\pi_{\mathrm{N}}(\gamma) \in \mathrm{I}$. Then there is an index m such that $n > m$ implies $\pi_{\mathrm{N}}(\gamma) \in \mathrm{I}_n$, which in turn implies $\gamma \in \pi_{\mathrm{N}}^{-1}(\mathrm{I}_n)$. Similarly, $\lambda \notin \pi_{\mathrm{N}}^{-1}(\mathrm{I})$ implies $\lambda \notin \pi_{\mathrm{N}}^{-1}(\mathrm{I}_n)$ for big enough n. Thus $(\pi_{\mathrm{N}}^{-1}(\mathrm{I}_n))$ converges to $\pi_{\mathrm{N}}^{-1}(\mathrm{I})$, therefore the map is open by equation (7.1Covariant representationsequation.7.1). ∎

Lemma 7.5. Λ *contains no nontrivial normal subgroups of* Γ *iff the representation* U *is faithful.*

Proof. U is faithful iff the identity e is the only group element that fixes every coset. Left-multiplication by group element γ fixes coset $\lambda\Lambda$ iff it fixes $\lambda\Lambda\lambda^{-1}$, ergo $\gamma \in \lambda\Lambda\lambda^{-1}$. Thus the representation is faithful iff the intersection $\bigcap_{\lambda\in\Gamma} \lambda\Lambda\lambda^{-1}$ is trivial. Therefore the forward direction is proven by observing $\bigcap_{\lambda\in\Gamma} \lambda\Lambda\lambda^{-1} \trianglelefteq \Gamma$. For the reverse, we note that $\mathrm{N} = \bigcap_{\lambda\in\Gamma} \lambda\mathrm{N}\lambda^{-1} \leq \bigcap_{\lambda\in\Gamma} \lambda\Lambda\lambda^{-1}$ for any subgroup $\mathrm{N} \trianglelefteq \Gamma$ of Λ. ∎

Remark 7.6. Lemma 7.4theorem.7.4 shows that nothing is lost by passing to $\Gamma/\cap\mathcal{O}_\Lambda$, and Lemma 7.5theorem.7.5 gives us incentive to do so. Hence, from here on, we shall assume $\cap\mathcal{O}_\Lambda = \{e\}$.

8 Crossed product consequences

The faithfulness of γ means $C^*_\Gamma(\Lambda)$ contains a copy of the group $*$-algebra $\mathbb{C}[\Gamma]$. However, the covariant representation itself is never faithful (for nontrivial Λ).

Proposition 8.1. *Let* $S_\gamma = \{\mathrm{K} \in \operatorname{Sub}\Gamma \mid \gamma \in \mathrm{K}\}$ *be a subbasic set of the Chabauty topology as in Definition 2.1theorem.2.1. Then, as a multiplication operator,*

$$\chi_{S_\gamma\cap\overline{\mathcal{O}_\Lambda}} = \chi_{S_\gamma\cap\overline{\mathcal{O}_\Lambda}} U_\gamma.$$

Proof. A simple calculation:

$$\begin{aligned}
\chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}}(\delta_x) &= \begin{cases} \delta_x & \text{if } \gamma \in \omega(x), \\ 0 & \text{else,} \end{cases} \\
&= \begin{cases} \delta_{\gamma x} & \text{if } \gamma \in \omega(\gamma x), \\ 0 & \text{else,} \end{cases} \\
&= \chi_{S_\gamma}\left(\omega(\gamma x)\right)(\delta_{\gamma x}) = \chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}}(\delta_{\gamma x}) \\
&= \chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}} U_\gamma(\delta_x).
\end{aligned}$$

∎

Corollary 8.2. *If $\Lambda \neq \{e\}$, then $C^*_\Gamma(\Lambda)$ is neither the full nor reduced crossed product of $C(\overline{\mathcal{O}_\Lambda})$ with Γ. It does not even admit the reduced crossed product as a quotient.*

Information is sparse on such exotic, crossed-product-like C*-algebras. We are at least able to claim $U : \Gamma \to C^*_\Gamma(\Lambda)$ is a D_Λ-representation in the sense of [BG13, Def. 2.1], where $D_\Lambda \trianglelefteq \ell^\infty(\Gamma)$ is the ideal generated by functions supported on subsets of the form $\bigcup_{i=1}^n \lambda'_n \Lambda \lambda_n$. Indeed, $\ell^2(\Gamma/\Lambda)$ is the closed linear span of the indicator functions $\delta_{\lambda\Lambda}$, and

$$\pi_{\delta_{\lambda\Lambda}, \delta_{\lambda'\Lambda}}(\gamma) = \delta_{\lambda\Lambda}(\gamma^{-1}\lambda'\Lambda) = \chi_{\lambda'\Lambda\lambda^{-1}}(\gamma),$$

where $\pi_{f,g} \in \ell^\infty(\Gamma)$ is defined for every $f, g \in \ell^2(\Gamma/\Lambda)$ by

$$\pi_{f,g}(\gamma) = \langle U_\gamma f, g \rangle = \sum_{x \in \Gamma/\Lambda} f(\gamma^{-1}x)\overline{g(x)}.$$

Theorem 8.3. *The canonical conditional expectation E is a faithful conditional expectation onto $C(\overline{\mathcal{O}_\Lambda})$.*

Proof. That E is the identity on $C(\overline{\mathcal{O}_\Lambda})$ comes straight from the definitions. Referring to the proof of Lemma 7.1theorem.7.1,

$$E(L^r_{x,\gamma})(y) = \begin{cases} 1 & \text{if } [x]_r = [y]_r \text{ and } y = \gamma y, \\ 0 & \text{else,} \end{cases}$$
$$= \begin{cases} 1 & \text{if } B_r(\omega(x)) \cong B_r(\omega(y)) \text{ and } \gamma \in \omega(y), \\ 0 & \text{else,} \end{cases}$$
$$= \chi_{B_{\omega(x),r} \cap S_\gamma} \circ \omega(y).$$

Therefore, by continuity, for every operator $K \in C^*_\Gamma(\Lambda)$ there exists $f_K \in C(\overline{\mathcal{O}_\Lambda})$ such that $E(K) = f_K \circ \omega$. ∎

Proposition 8.4. *Suppose Γ acts amenably on $\overline{\mathcal{O}_\Lambda}$. Then every trace on $C^*_\Gamma(\Lambda)$ is amenable.*

Proof. We have that the usual crossed product $C(\overline{\mathcal{O}_\Lambda}) \rtimes_\alpha \Gamma = C(\overline{\mathcal{O}_\Lambda}) \rtimes_{\alpha,r} \Gamma$ is nuclear by [BO08, Thm. 4.3.4], hence every trace on it is amenable by [BO08, Prop. 6.3.4]. Therefore, since $C^*_\Gamma(\Lambda)$ is a quotient thereof, the result follows from [BO08, Prop. 6.3.6]. ∎

Theorem 8.5. *If Λ is cheap (Definition 2.4theorem.2.4), then its C^*-algebra has trivial center:*

$$Z(C^*_\Gamma(\Lambda)) = \mathbb{C}\mathrm{id}_{\ell^2(\Gamma/\Lambda)}.$$

Proof. Let $K \in Z(C^*_\Gamma(\Lambda))$ be a central operator. Then for every $\gamma \in \Gamma$ and $y, z \in \Gamma/\Lambda$,

$$K(\delta_y)(z) = U_\gamma K(\delta_y)(\gamma z) = K U_\gamma(\delta_y)(\gamma z) = K(\delta_{\gamma y})(\gamma z),$$

therefore K is constant along "diagonals."

Also,

$$L^r_{x,\gamma} K(\delta_y)(z) = \sum_{w \in \Gamma/\Lambda} L^r_{x,\gamma}(\delta_w)(z) K(\delta_y)(w) = \begin{cases} K(\delta_y)(\gamma^{-1} z) & \text{if } \gamma^{-1} z \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

must be equal to

$$KL^r_{x,\gamma}(\delta_y)(z) = \sum_{w\in\Gamma/\Lambda} K(\delta_w)(z)L^r_{x,\gamma}(\delta_y)(w) = \begin{cases} K(\delta_{\gamma y})(z) & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

for every $x, y, z \in \Gamma/\Lambda$, $r \in \mathbb{N}$, and $\gamma \in Q^r$. In particular, if $\omega(x) \neq \omega(z)$, then $z \notin [x]_R$ for some $R \in \mathbb{N}$, so

$$K(\delta_x)(z) = KL^R_{x,e}(\delta_x)(z) = L^R_{x,e}K(\delta_x)(z) = 0.$$

If Λ is cheap, then $\omega(x) = \omega(z)$ only if $x = z$, therefore K is nonzero only along the main diagonal. ∎

9 Injective envelopes

The *injective envelope* of a C*-algebra $\mathcal{A}$ is the minimal C*-algebra $I(\mathcal{A})$ containing $\mathcal{A}$ as a C*-subalgebra such that every u.c.p. map to $I(\mathcal{A})$ from an operator subsystem $\mathcal{T}$ of operator system $\mathcal{S}$ can be extended to a u.c.p. map on all of $\mathcal{S}$. Automorphisms extend uniquely to injective envelopes, so a Γ-action on $\mathcal{A}$ uniquely extends to $I(\mathcal{A})$. This all originates in [Ham79], and more can be found in [HP11, Kaw17, PZ15, Zar19].

The injective envelope of a commutative C*-algebra is also commutative [HP11, Cor. 2.18]. We write $\widetilde{\mathcal{O}_\Lambda}$ for the character space of the injective envelope $I\big(C(\overline{\mathcal{O}_\Lambda})\big) \cong C(\widetilde{\mathcal{O}_\Lambda})$ of $C(\overline{\mathcal{O}_\Lambda})$.

By definition, there always exists at least one u.c.p. map $C^*_\Gamma(\Lambda) \to C(\widetilde{\mathcal{O}_\Lambda})$ which restricts to the identity on $C(\overline{\mathcal{O}_\Lambda})$; any such map is known as a *pseudo-expectation.* Since $C(\overline{\mathcal{O}_\Lambda})$ is in the multiplicative domain, any pseudo-expectation is a $C(\overline{\mathcal{O}_\Lambda})$-bimodule. See [**?**, Zar19] for further discussion.

Proposition 9.1. *Suppose the Γ-action on $\widetilde{\mathcal{O}_\Lambda}$ is free. Then $\Lambda = \{e\}$.*

Proof. By [HP11, Thm. 2.21], the injective envelope $C(\widetilde{\mathcal{O}_\Lambda})$ is a Γ-C*-subalgebra of the injective envelope $I(C^*_\Gamma(\Lambda))$. Moreover, by the uniqueness of the automorphisms [Ham79, Cor. 4.2], the action of Γ on $I(C^*_\Gamma(\Lambda))$ is still given by the inner automorphisms associated to the unitaries $U_\gamma \in C^*_\Gamma(\Lambda)$.

Let $\phi : C^*_\Gamma(\Lambda) \to C(\widetilde{\mathcal{O}_\Lambda})$ be a pseudo-expectation. Then we can extend ϕ to a u.c.p. map on $I(C^*_\Gamma(\Lambda))$. Note that ϕ is the identity on $C(\overline{\mathcal{O}_\Lambda})$, hence is a conditional expectation onto $C(\widetilde{\mathcal{O}_\Lambda})$ by *rigidity* of the injective envelope [Ham79, Def. 2.2].

Let $\gamma \in \Gamma \backslash \{e\}$. Since any point $x \in \widetilde{\mathcal{O}_\Lambda}$ is not equal to the point $\gamma^{-1}.x$, they can be separated by a continuous function f; ergo, $f(x) \neq (\gamma.f)(x)$. By $C(\widetilde{\mathcal{O}_\Lambda})$-bimodularity,

$$\phi(U_\gamma)f = \phi(U_\gamma)\phi(f) = \phi(U_\gamma f) = \phi((\gamma.f)U_\gamma) = (\gamma.f)\phi(U_\gamma),$$

so $\phi(U_\gamma)(x)$ must be 0. Since x was arbitrary, we in fact have that $\phi(U_\gamma)$ is identically 0, hence (using Proposition 8.1theorem.8.1) so is

$$\chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}} = \phi(\chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}}) = \phi(\chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}} U_\gamma) = \chi_{S_\gamma \cap \overline{\mathcal{O}_\Lambda}} \phi(U_\gamma).$$

Therefore $S_\gamma \cap \overline{\mathcal{O}_\Lambda} = \{\mathrm{K} \in \overline{\mathcal{O}_\Lambda} \mid \gamma \in \mathrm{K}\} = \varnothing$. ∎

We shall work our way toward the following corollary. Compare it with [Ele18, Thm. 7], and the *intersection property* discussed in [Bry17, Kaw17, KS19].

Corollary 9.2. *If Λ is topologically cheap, then $\mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda})$ is nonzero for every nontrivial (closed, two-sided) ideal $\mathcal{I} \trianglelefteq C^*_\Gamma(\Lambda)$. In particular, if $\overline{\mathcal{O}_\Lambda}$ is a URS, then $C^*_\Gamma(\Lambda)$ is simple.*

This shall be a consequence of Proposition 9.3theorem.9.3 and Corollary 9.5theorem.9.5. Compare Proposition 9.3theorem.9.3 with [KS19, Sec. 6]'s results for C*-dynamical injective envelopes, and note that we can achieve both directions in that category by mimicking Lemma 6.5 of that paper.

Proposition 9.3. *If every pseudo-expectation is faithful, then $\mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda})$ is nonzero for every nontrivial ideal $\mathcal{I} \trianglelefteq C^*_\Gamma(\Lambda)$.*

Proof. For sake of contraposition, suppose there is nontrivial ideal $\mathcal{I} \triangleleft C^*_\Gamma(\Lambda)$ with $\mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda}) = \{0\}$. Let $\pi_\mathcal{I} : C^*_\Gamma(\Lambda) \to C^*_\Gamma(\Lambda)/\mathcal{I}$ be the quotient map. Then $\pi_\mathcal{I}$ is injective on $C(\overline{\mathcal{O}_\Lambda})$, hence there is a u.c.p. $\psi : C^*_\Gamma(\Lambda)/\mathcal{I} \to C(\widetilde{\mathcal{O}_\Lambda})$ such that $\psi \circ \pi_\mathcal{I}$ is the identity on $C(\overline{\mathcal{O}_\Lambda})$. Therefore $\psi \circ \pi_\mathcal{I}$ is an unfaithful pseudo-expectation. ∎

Since the action on $\overline{\mathcal{O}_\Lambda}$ is never topologically free, [PZ15, Thm. 4.6] states $C(\overline{\mathcal{O}_\Lambda})$ is never a *maximal abelian subalgebra* (MASA, an abelian subalgebra not contained in any other abelian subalgebra) of the reduced crossed product $C(\overline{\mathcal{O}_\Lambda}) \rtimes_{\alpha,r} \Gamma$. Contrast that with the following:

Theorem 9.4. *Λ is topologically cheap iff $C(\overline{\mathcal{O}_\Lambda})$ is a MASA of $C^*_\Gamma(\Lambda)$.*

Proof. $\Rightarrow$: Proceeding by contraposition, suppose there is a $T \in C^*_\Gamma(\Lambda) \backslash C(\overline{\mathcal{O}_\Lambda})$ that commutes with all operators in $C(\overline{\mathcal{O}_\Lambda})$. Then there are $\gamma, \lambda \in \Gamma$ such that $c := T(\delta_{\lambda\Lambda})(\gamma\Lambda) \neq 0$ and $\lambda\Lambda \neq \gamma\Lambda$. Changing root if necessary, we may write $\lambda = e$ and $\gamma \notin \Lambda$. For every $f \in C(\overline{\mathcal{O}_\Lambda})$,

$$f\big(\alpha_\gamma(\Lambda)\big) = f\big(\alpha_\gamma(\Lambda)\big)T(\delta_\Lambda)(\gamma\Lambda)/c = \big(fT\big)(\delta_\Lambda)(\gamma\Lambda)/c = \big(Tf\big)(\delta_\Lambda)(\gamma\Lambda)/c =$$
$$T\big((f \circ \omega)\delta_\Lambda\big)(\gamma\Lambda)/c = T\big(f(\omega(\Lambda))\delta_\Lambda\big)(\gamma\Lambda)/c = f(\Lambda)T(\delta_\Lambda)(\gamma\Lambda)/c = f(\Lambda), \tag{9.1}$$

so $\Lambda = \alpha_\gamma(\Lambda)$, ergo $\gamma \in N_\Gamma(\Lambda)$.

There must be a local kernel L of width $w \geq l_Q(\gamma)$ with $\|T - L\| < c/2$. By Lemma 3.2theorem.3.2, $B_{\Lambda,w}$ is a neighborhood of Λ. Suppose $\mathrm{K} \in B_{\Lambda,w}$, so $\gamma \notin \mathrm{K}$. By Theorem 4.2theorem.4.2 there is a surjection $\varphi : C^*_\Gamma(\Lambda) \to C^*_\Gamma(\mathrm{K})$. Thus $\varphi(T)$ commutes with all operators in $\varphi(C(\overline{\mathcal{O}_\Lambda})) = C(\overline{\mathcal{O}_\mathrm{K}})$, and $\varphi(T)(\delta_\mathrm{K})(\gamma\mathrm{K}) \neq 0$ since

$$c > \|\varphi(T - L)\| + \|L - T\| \geq |\varphi(T - L)(\delta_\mathrm{K})(\gamma\mathrm{K})| + |(L - T)(\delta_\Lambda)(\gamma\Lambda)|$$
$$\geq |\varphi(T)(\delta_\mathrm{K})(\gamma\mathrm{K}) - T(\delta_\Lambda)(\gamma\Lambda)| = |\varphi(T)(\delta_\mathrm{K})(\gamma\mathrm{K}) - c|.$$

Combining these facts, we may repeat equation (9.1Injective envelopesequation.9.1) to show $\gamma \in N_\Gamma(\mathrm{K})$. Therefore γ is in the topological stabilizer Γ°_Λ.

$\Leftarrow$: Using contraposition again, suppose there is a subgroup $\Lambda' \in \overline{\mathcal{O}_\Lambda}$ and $\gamma \in \Gamma^\circ_{\Lambda'} \backslash \Lambda'$. Ergo there is neighborhood $V \subseteq \overline{\mathcal{O}_\Lambda}$ of Λ' such that $\alpha_\gamma|_V = \mathrm{id}_V$. Since $V \backslash S_\gamma = \{\mathrm{K} \in V \mid \gamma \notin \mathrm{K}\}$ is open, it intersects $\mathcal{O}_\Lambda$; ergo there is a subgroup in $\mathcal{O}_\Lambda$ (which our freedom of root allows us to assume is Λ) such that $\gamma \in \Gamma^\circ_\Lambda \backslash \Lambda$. Let $f \in C(\overline{\mathcal{O}_\Lambda})$ be equal to 1 at Λ and supported on $V \backslash S_\gamma$. Thus $\Lambda \neq \gamma\Lambda$, yet

$$fU_\gamma(\delta_\Lambda)(\gamma\Lambda) = f(\gamma\Lambda\gamma^{-1})U_\gamma(\delta_\Lambda)(\gamma\Lambda) = f(\Lambda)\delta_{\gamma\Lambda}(\gamma\Lambda) = 1 \neq 0,$$

so $fU_\gamma \notin C(\overline{\mathcal{O}_\Lambda})$. However, for any $g \in C(\overline{\mathcal{O}_\Lambda})$ and $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$,

$$\begin{aligned}\big(fU_\gamma g U_\gamma^*\big)(\mathrm{K}) = f(\mathrm{K})g(\alpha_\gamma^{-1}(\mathrm{K})) &= \begin{cases} f(\mathrm{K})g(\alpha_\gamma^{-1}(\mathrm{K})) & \text{if } \mathrm{K} \in V, \\ 0 & \text{else,} \end{cases} \\ &= \begin{cases} f(\mathrm{K})g(\mathrm{K}) & \text{if } \mathrm{K} \in V, \\ 0 & \text{else,} \end{cases} \\ &= \big(gf\big)(\mathrm{K}),\end{aligned}$$

therefore fU_γ commutes with g. ◻

Corollary 9.5. *If Λ is topologically cheap, then there is a unique pseudo-expectation $C^*_\Gamma(\Lambda) \to C(\widetilde{\mathcal{O}_\Lambda})$ (which must be the canonical conditional expectation). Moreover, for any $T \in \mathbb{M}_n(C^*_\Gamma(\Lambda))$,*

$$\|T\| = \sup\big\{\|LTC\| \;\big|\; L, R^t \in C(\overline{\mathcal{O}_\Lambda}, \mathbb{M}_{1\times n}) \text{ and } \|L\| = \|R\| = 1\big\}.$$

Proof. Combine the previous theorem with [PZ15, Thm. 1.4]. ◻

Question 9.6. *What conditions on Λ guarantee there is a bijection between ideals of $C^*_\Gamma(\Lambda)$ and Γ-invariant ideals of $C(\overline{\mathcal{O}_\Lambda})$? (cf.* ideal separation property *[Kaw17, KS19, Sie10])*

To prove a bijection, we simply need that for every ideal $\mathcal{I} \triangleleft C^*_\Gamma(\Lambda)$, subgroup $K \in \mathcal{I}$, and $\mathcal{J} := \mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda})$,

$$K \in C^*_\Gamma(\Lambda)\mathcal{J}C^*_\Gamma(\Lambda). \tag{9.2}$$

Corollary 9.2theorem.9.2 suggests the answer is related to cheapness, as does the following observation: If Λ is cheap, [Ele18, Lm. 6.6] tells us $E(\mathcal{I}) \subseteq \mathcal{J}$. For any $\gamma \in \Gamma$ and $x \in \Gamma/\Lambda$,

$$E(KU_\gamma)(\omega(x)) = KU_\gamma(\delta_x)(x) = K(\delta_{\gamma x})(x).$$

Thus we have "slices" of K in $\mathcal{J}$. These slices are not necessarily "parallel," but we can remove all double-counting.

Order $Q^r = \{\gamma_1, \gamma_2, \ldots, \gamma_{|Q^r|}\}$. For an integer $m \in [1, |Q^r|]$, define $K_m^r \in C(\overline{\mathcal{O}_\Lambda})$ by

$$K_m^r(\Lambda') = \begin{cases} E(KU_{\gamma_m})(\Lambda') & \text{if } \gamma_m\Lambda' \neq \gamma_i\Lambda' \text{ for every } i < m, \\ 0 & \text{else.} \end{cases}$$

Note that $K_m^r \in \mathcal{J}$ since it is simply the product of $E(KU_{\gamma_m})$ with the (continuous) characteristic function of $\overline{\mathcal{O}_\Lambda} \setminus \bigcup_{i=1}^m S_{\gamma_i^{-1}\gamma_m}$.

Now, for every $x, y \in \Gamma/\Lambda$,

$$K_m^r U_{\gamma_m}^*(\delta_x)(y) = K_m^r(\delta_{\gamma_m^{-1}x})(y) = \begin{cases} 0 & \text{if } x \neq \gamma_m y, \\ 0 & \text{if } \gamma_m y = \gamma_i y \text{ for some } i < m \\ K(\delta_x)(y) & \text{else.} \end{cases}$$

Define $K^r = \sum_{m=1}^{|Q^r|} K_m^r U_{\gamma_m}^*$, so

$$K^r(\delta_x)(y) = \begin{cases} K(\delta_x)(y) & \text{if } y \in B_r(x), \\ 0 & \text{else.} \end{cases}$$

Thus the sequence $(K^n) \subset \mathcal{I}$ strongly converges to K. If it converges in norm, then we have $K \in C_\Gamma^*(\Lambda)\mathcal{J}C_\Gamma^*(\Lambda)$. K is already the limit of local kernels, so it seems feasible.

However, the following throws a wrench into things:

Theorem 9.7. *Suppose Λ is topologically cheap and $\mathcal{I}$ is a nonzero ideal of $C_\Gamma^*(\Lambda)$. Then $\mathcal{I}$ is strongly dense in $C_\Gamma^*(\Lambda) \subset \mathbb{B}\big(\ell^2(\Gamma/\Lambda)\big)$.*

Proof. By Corollary 9.2theorem.9.2, $\mathcal{I} \cap C(\overline{\mathcal{O}_\Lambda})$ is a nonzero Γ-invariant ideal of $C(\overline{\mathcal{O}_\Lambda})$, hence it must be $C_0(V)$ for some nonempty Γ-invariant open subspace $V \subseteq \overline{\mathcal{O}_\Lambda}$. Since V is open, it must intersect $\mathcal{O}_\Lambda$, and since V is Γ-invariant, it must contain all of $\mathcal{O}_\Lambda$, ergo $\alpha_\gamma(\Lambda) \in V$ for every $\gamma \in \Gamma$.

Let $F \subset \Gamma$ be a finite subset. Since $\overline{\mathcal{O}_\Lambda}$ is compact and Hausdorff, there is a function $f_F \in C_0(V)$ such that $f_F(\alpha_\gamma(\Lambda)) = 1$ for each $\gamma \in F$, and so

$$(f_F - \mathrm{id}_{\Gamma/\Lambda})(\delta_{\gamma\Lambda}) = f_F(\alpha_\gamma(\Lambda))\delta_{\gamma\Lambda} - \delta_{\gamma\Lambda} = 0.$$

Since $\ell^2(\Gamma/\Lambda)$ is the closed linear span of $\{\delta_{\gamma\Lambda} \mid \gamma \in \Gamma\}$, the net $(f_F) \subset \mathcal{I}$ (ordered by inclusion of F) strongly converges to $\mathrm{id}_{\Gamma/\Lambda}$. Therefore the result follows from strong-continuity of multiplication.

10 Subalgebras

In this section we explore some assorted facts related to C*-subalgebras of $C^*_\Gamma(\Lambda)$, starting with Cartan subalgebras, as introduced in [Ren08, Def. 5.1]. These arise from the study of groupoid C*-algebras as a C*-analogue of the W* concept introduced in [FM77], and have recently come under study for their relation to the Universal Coefficient Theorem. (See e.g. [Li20].)

Definition 10.1. A C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ of a unital C*-algebra $\mathcal{A}$ is called a **Cartan subalgebra** if the following hold:

1. $\mathcal{B}$ is a MASA of $\mathcal{A}$.
2. There is a faithful conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.
3. $\mathcal{B}$ is *regular in* $\mathcal{A}$ [PZ15, Sec. 1.1]; that is, it is the closed linear span of the set

$$\big\{a \in \mathcal{A} \;\big|\; a\mathcal{B}a^*, a^*\mathcal{B}a \subseteq \mathcal{B}\big\}.$$

Proposition 10.2. *$C(\overline{\mathcal{O}_\Lambda})$ is regular in $C^*_\Gamma(\Lambda)$.*

Proof. This is evident from Section 7Covariant representationssection.7 and the fact $C(\overline{\mathcal{O}_\Lambda})$ is Γ-invariant.

Theorem 10.3. *Λ is topologically cheap iff $C(\overline{\mathcal{O}_\Lambda})$ is a Cartan subalgebra of $C^*_\Gamma(\Lambda)$.*

Proof. The requirements have been established previously: The first (and the backwards direction) in Theorem 9.4theorem.9.4, the second in Theorem 8.3theorem.8.3, and the third just now in Proposition 10.2theorem.10.2.

Definition 10.4. [Kum86, Def. 1.3] A **C*-diagonal subalgebra** $\mathcal{B}$ of unital $\mathcal{A}$ is a special case of a Cartan subalgebra, inspired by the subalgebra of $\mathbb{M}_n$ consisting of diagonal matrices, and is characterized by the following requirements:

1. $\mathcal{B}$ is abelian and contains $1_\mathcal{A}$.
2. There is a faithful conditional expectation $P : \mathcal{A} \to \mathcal{B}$.

3. The kernel of P is the closed linear span of

$$\mathcal{N}_P := \{a \in \mathcal{A} \mid a^*\mathcal{B}a, a\mathcal{B}a^* \subseteq \mathcal{B} \text{ and } a^2 = 0\}.$$

Lemma 10.5. *Suppose $T \in C^*_\Gamma(\Lambda)$ satisfies $TC(\overline{\mathcal{O}_\Lambda})T^* \subseteq C(\overline{\mathcal{O}_\Lambda})$, and $z \in \Gamma/\Lambda$. Then there exists at most one coset $x \in \Gamma/\Lambda$ such that $T(\delta_z)(x) \neq 0$.*

Proof. For each $x, y \in \Gamma/\Lambda$, define $\tau_{x,y} : C(\overline{\mathcal{O}_\Lambda}) \to \mathbb{C}$ by $\tau_{x,y}(f) = TfT^*(\delta_x)(y)$, so for any $f, g \in C(\overline{\mathcal{O}_\Lambda})$ and $c \in \mathbb{C}$,

$$\begin{aligned}
\tau_{x,y}(cf+g) &= T(cf+g)T^*(\delta_x)(y) = cTfT^*(\delta_x)(y) + TgT^*(\delta_x)(y) = c\tau_{x,y}(f) + \tau_{x,y}(g), \\
|\tau_{x,y}(f)| &\leq \|f\|_\infty \|T\|^2, \\
\tau_{x,y}(f) &= \sum_{z \in \Gamma/\Lambda} \big(Tf\big)(\delta_z)(y) T^*(\delta_x)(z) = \sum_{z \in \Gamma/\Lambda} \sum_{w \in \Gamma/\Lambda} T(\delta_w)(y) f(\delta_z)(w) \overline{T(\delta_z)(x)} \\
&= \sum_{z \in \Gamma/\Lambda} T(\delta_z)(y) f\big(\omega(z)\big) \overline{T(\delta_z)(x)}.
\end{aligned} \tag{10.1}$$

Thus $\tau_{x,y}$ is a bounded linear functional, hence represents integration against a unique regular Borel measure $\mu_{x,y}$ on $\overline{\mathcal{O}_\Lambda}$ by the Riesz-Markov-Kakutani Representation Theorem [Kad18, Thm. 8.4.2.4][Con90, Cor. 3.5].

If $y \neq x$, then every $\tau_{x,y}(f) = 0$ since $TfT^* \in C(\overline{\mathcal{O}_\Lambda})$ by assumption; in particular, for any $z \in \Gamma/\Lambda$,

$$0 = \lim_r 0 = \lim_r \tau_{x,y}\left(\chi_{B_{\omega(z),r}}\right) = \lim_r \int \chi_{B_{\omega(z),r}} \, \mathrm{d}\mu_{x,y} = \int \chi_{\{\omega(z)\}} \, \mathrm{d}\mu_{x,y} = T(\delta_z)(y)\overline{T(\delta_z)(x)}$$

using the basic clopen subsets $B_{\omega(z),r} = \{\mathrm{K} \in \overline{\mathcal{O}_\Lambda} \mid B_r(\mathrm{K}) \cong B_r(\omega(z))\}$, the Dominated Convergence Theorem [Kad18, Thm. 4.4.2.1], and equation (10.1Subalgebrasequation.10.1).

Therefore, for any pair of distinct $x, y \in \Gamma/\Lambda$, at least one of $T(\delta_z)(y), T(\delta_z)(x)$ is zero. ∎

Theorem 10.6. Λ *is cheap iff $C(\overline{\mathcal{O}_\Lambda})$ is a C^*-diagonal subalgebra of $C^*_\Gamma(\Lambda)$.*

Proof. $\Rightarrow$: Once again, we reference the canonical conditional expectation E from Remark 4.3theorem.4.3 and the spanning elements $L^r_{x,\gamma}$ from Section 7Covariant representationssection.7. Suppose $T \notin$

$\ker(E)$ and $T^2 = 0$. Then there is a coset $z \in \Gamma/\Lambda$ such that $T(\delta_z)(z) = E(T)(\omega(z)) \neq 0$, but

$$0 = T^2(\delta_z)(z) = \sum_{x \in \Gamma/\Lambda} T(\delta_x)(z)T(\delta_z)(x).$$

Thus there must be another coset $x \in \Gamma/\Lambda$ such that $T(\delta_z)(x) \neq 0$. Then $TC(\overline{\mathcal{O}_\Lambda})T^*$ is not a subset of $C(\overline{\mathcal{O}_\Lambda})$ by Lemma 10.5theorem.10.5, so $T \notin \mathcal{N}_E$. Therefore $\mathcal{N}_E \subseteq \ker(E)$.

By the definitions, if $\gamma x \neq x$, then $E(L^r_{x,\gamma})$ is identically 0. Moreover, every local kernel in $\ker(E)$ of width w is a linear combination of $L^w_{x,\gamma}$'s with $\gamma x \neq x$. Any $K \in \ker(E)$ is the limit of a net (K_n) of local kernels; thus $K = K - E(K)$ is the limit of the net $(K_n - E(K_n)) \subseteq \ker(E)$ of local kernels. Therefore we see $\ker(E) = \overline{\operatorname{span}}\{L^r_{x,\gamma} \mid \gamma x \neq x\}$.

For cheap Λ and every $\gamma \in \Gamma$, [Ele18, Prop. 2.3] provides an $r_\gamma > l_Q(\gamma)$ such that $\gamma y \notin [y]_{r_\gamma}$ for all $y \in \Gamma/\Lambda$. For any $r \geq r_\gamma \geq n \geq l_Q(\gamma)$,

$$\begin{aligned}
(L^r_{x,\gamma})^2(\delta_y) &= \begin{cases} L^r_{x,\gamma}(\delta_{\gamma y}) & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases} \\
&= \begin{cases} \delta_{\gamma^2 y} & \text{if } \gamma y, y \in [x]_r, \\ 0 & \text{else,} \end{cases} \\
&= 0, \\
L^n_{x,\gamma} &= \sum_{[z]_r \subseteq [x]_n} L^r_{z,\gamma}.
\end{aligned}$$

Therefore $\ker(E) = \overline{\operatorname{span}}\{L^r_{x,\gamma} \mid \gamma x \neq x \text{ and } (L^r_{x,\gamma})^2 = 0\}$.

For any $f \in C(\overline{\mathcal{O}_\Lambda})$,

$$(L^r_{x,\gamma})^* f L^r_{x,\gamma}(\delta_y) = \begin{cases} (L^r_{x,\gamma})^* f(\delta_{\gamma y}) & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

$$= \begin{cases} f \circ \omega(\gamma y)(L^r_{x,\gamma})^*(\delta_{\gamma y}) & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

$$= \begin{cases} f \circ \omega(\gamma y)\delta_y & \text{if } y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

$$L^r_{x,\gamma} f (L^r_{x,\gamma})^*(\delta_y) = \begin{cases} L^r_{x,\gamma} f(\delta_{\gamma^{-1}y}) & \text{if } \gamma^{-1}y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

$$= \begin{cases} f \circ \omega(\gamma^{-1}y) L^r_{x,\gamma}(\delta_{\gamma^{-1}y}) & \text{if } \gamma^{-1}y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

$$= \begin{cases} f \circ \omega(\gamma^{-1}y)\delta_y & \text{if } \gamma^{-1}y \in [x]_r, \\ 0 & \text{else,} \end{cases}$$

therefore $\{L^r_{x,\gamma} \mid \gamma x \neq x \text{ and } (L^r_{x,\gamma})^2 = 0\} \subseteq \mathcal{N}_E$.

$\Leftarrow$: Conversely, suppose Λ is not cheap. Then there is a subgroup $\mathrm{K} \in \overline{\mathcal{O}_\Lambda}$ and $\gamma \notin \mathrm{K}$ such that $\alpha_\gamma(\mathrm{K}) = \mathrm{K}$. Let $r \geq \max\{l_Q(\gamma), l_Q(\gamma^2)\}$, and $q : C^*_\Gamma(\Lambda) \to C^*_\Gamma(\mathrm{K})$ be the quotient map given in Theorem 4.2theorem.4.2. Let $r \geq \max\{l_Q(\gamma), l_Q(\gamma^2)\}$. Lemma 3.2theorem.3.2 allows us to find $y \in \Gamma/\Lambda$ such that $B_r(y) \cong B_r(\mathrm{K})$, hence $\gamma y \neq y$. We shall use separate cases depending on whether γ^2 is in K.

Suppose $\gamma^2 \in \mathrm{K}$, so also $\gamma^2 y = y$. Choose $T \in C^*_\Gamma(\Lambda)$ such that $TC(\overline{\mathcal{O}_\Lambda})T^* \subseteq C(\overline{\mathcal{O}_\Lambda})$ and $q(T)(\delta_\mathrm{K})(\gamma\mathrm{K}) \neq 0$. There are local kernels T_n of respective widths (at least) n such that $T_n \to T$, and there are cosets $x_n \in \Gamma/\Lambda$ with respective balls $B_n(x_n) \cong B_n(\mathrm{K})$. We use these to see that, for every $\lambda \in \Gamma$,

$$
\begin{aligned}
q(T)(\delta_{\mathrm{K}})(\lambda\mathrm{K}) &= \lim_n q(T_n)(\delta_{\mathrm{K}})(\lambda\mathrm{K}) = \lim_n T_{n+l_Q(\lambda)}(\delta_{x_{n+l_Q(\lambda)}})(\lambda x_{n+l_Q(\lambda)}) \\
&= \lim_n U_\lambda^* T_{n+l_Q(\lambda)}(\delta_{x_{n+l_Q(\lambda)}})(x_{n+l_Q(\lambda)}) = \lim_n E(U_\lambda^* T_{n+l_Q(\lambda)})(\omega(x_{n+l_Q(\lambda)})) \\
&= \lim_n E(U_\lambda^* T_{n+l_Q(\lambda)})(\mathrm{K}) = E(U_\lambda^* T)(\mathrm{K}) = \lim_n E(U_\lambda^* T)(\omega(x_{n+l_Q(\lambda)})) \\
&= \lim_n T(\delta_{x_n})(\lambda x_n).
\end{aligned}
$$

Then, using Lemma 10.5theorem.10.5,

$$
\begin{aligned}
q(T)^2(\delta_{\mathrm{K}})(\mathrm{K}) &= \sum_{s\in\Gamma/\mathrm{K}} q(T)(\delta_s)(\mathrm{K})q(T)(\delta_{\mathrm{K}})(s) = q(T)(\delta_{\gamma\mathrm{K}})(\mathrm{K})q(T)(\delta_{\mathrm{K}})(\gamma\mathrm{K}) \\
&= q(T)(\delta_{\gamma\mathrm{K}})(\gamma^2\mathrm{K})q(T)(\delta_{\mathrm{K}})(\gamma\mathrm{K}) = q(T)(\delta_{\mathrm{K}})(\gamma\mathrm{K})q(T)(\delta_{\mathrm{K}})(\gamma\mathrm{K}) \neq 0.
\end{aligned}
$$

Thus $T^2 \neq 0$, hence $T \notin \mathcal{N}_E$. Since $q(L^r_{y,\gamma})(\delta_{\mathrm{K}})(\gamma\mathrm{K}) = L^r_{y,\gamma}(\delta_y)(\gamma y) \neq 0$, this means $L^r_{y,\gamma}$ cannot be in the closed linear span of $\mathcal{N}_E$. In proving the other direction we showed $L^r_{y,\gamma} \in \ker(E)$, therefore we have proved this direction in this case.

Now suppose $\gamma^2 \notin \mathrm{K}$, so $L^r_{y,\gamma^2} \in \ker(E)$. The evaluation map $g \mapsto g(\mathrm{K})$ is a pure state ρ on $C(\overline{\mathcal{O}_\Lambda})$, which we may extend to ρ_0 on $C^*_\Gamma(\Lambda)$ by $\rho_0(T) = E(T)(\mathrm{K})$. Define $V = (U_\gamma + U_\gamma^*)/\sqrt{2}$ and state ρ_1 on $C^*_\Gamma(\Lambda)$ by $\rho_1(T) = E\big(VgV^*\big)(\mathrm{K})$. For any $x \in \Gamma/\Lambda$ such that $\gamma \notin \omega(x)$ (ergo $\gamma x \neq x$) and any $g \in C(\overline{\mathcal{O}_\Lambda})$,

$$
\begin{aligned}
E\big(VgV^*\big)(\omega(x)) &= VgV^*(\delta_x)(x) = \big(g(\delta_{\gamma x})(\gamma x) + g(\delta_{\gamma^{-1}x})(\gamma x) + g(\delta_{\gamma x})(\gamma^{-1}x) + g(\delta_{\gamma^{-1}x})(\gamma^{-1}x)\big)/2 \\
&= \big(g\circ\omega(\gamma x)(\delta_{\gamma x} + \delta_{\gamma^{-1}x})(\gamma x) + g\circ\omega(\gamma^{-1}x)(\delta_{\gamma x} + \delta_{\gamma^{-1}x})(\gamma^{-1}x)\big)/2 \\
&= \big(g\circ\alpha_\gamma\circ\omega(x) + g\circ\alpha_{\gamma^{-1}}\circ\omega(x)\big)/2.
\end{aligned}
$$

Thus $\rho_1(g) = \big(g(\mathrm{K}) + g(\mathrm{K})\big)/2 = g(\mathrm{K})$, so ρ_1 is also an extension of ρ. We have

$$
\begin{aligned}
.E\left(VL^r_{y,\gamma^2}V^*\right)(\omega(x)) &= \left(L^r_{y,\gamma^2}(\delta_{\gamma x} + \delta_{\gamma^{-1}x})(\gamma x) + L^r_{y,\gamma^2}(\delta_{\gamma x} + \delta_{\gamma^{-1}x})(\gamma^{-1}x)\right)/2 \\
&\geq L^r_{y,\gamma^2}(\delta_{\gamma^{-1}x})(\gamma x)/2 = \begin{cases} 1/2 & \text{if } B_r(\alpha_{\gamma^{-1}}\circ\omega(x)) \cong B_r(\mathrm{K}), \\ 0 & \text{else,} \end{cases}
\end{aligned}
$$

Thus $\rho_1(L^{r_\gamma}_{y,\gamma^2}) \neq 0 = \rho_0(L^{r_\gamma}_{y,\gamma^2})$, hence the two extensions are distinct. Therefore $C(\overline{\mathcal{O}_\Lambda})$ is not a diagonal subalgebra of $C^*_\Gamma(\Lambda)$ by [Kum86, Prop. 1.4]. ◻

We end this section with a discussion of the relationship between the C*-algebras when there is an intermediate group $\Lambda \leq \mathrm{H} \leq \Gamma$.

Proposition 10.7. *Suppose there is a subgroup* $\mathrm{H} \leq \Gamma$ *such that* $\Lambda \leq \mathrm{H}$. *Then there is a u.c.p. map* $E_{\mathrm{H}} : C^*_\Gamma(\Lambda) \to C^*_{\mathrm{H}}(\Lambda)$.

Proof. First let $\mathcal{U}$ be Λ's orbit-closure within the dynamical system of H's subgroups, and note that $\mathcal{U} \subseteq \overline{\mathcal{O}_\Lambda}$. For each $\gamma \in \Gamma$ and $g \in C(\overline{\mathcal{O}_\Lambda})$, define

$$E_{\mathrm{H}}(gU_\gamma) = \chi_{\mathrm{H}}(\gamma) g|_{\mathcal{U}} U_\gamma \in C^*_{\mathrm{H}}(\Lambda).$$

For cosets $x, \gamma x \in \mathrm{H}/\Lambda$, there must be $\eta, \eta' \in \mathrm{H}$ such that $\eta\Lambda = x$ and $\eta'\Lambda = \gamma x = \gamma\eta\Lambda$, hence $\gamma \in \eta'\Lambda\eta^{-1} \subseteq \mathrm{H}$. Thus, for any finite sum $\sum_{\gamma\in\Gamma} g_\gamma U_\gamma$ and $f \in \ell^2(\mathrm{H}/\Lambda) \subseteq \ell^2(\Gamma/\Lambda)$,

$$\begin{aligned}
\left\| \sum_{\gamma\in\Gamma} E_{\mathrm{H}}\left(g_\gamma U_\gamma\right)(f) \right\|^2 = \left\| \sum_{\eta\in\mathrm{H}} g_\eta \circ \omega(x) f(\eta^{-1}x) \right\|^2 &= \sum_{x\in\mathrm{H}/\Lambda} \left| \sum_{\eta\in\mathrm{H}} g_\eta \circ \omega(x) f(\eta^{-1}x) \right|^2 \\
&= \sum_{x\in\mathrm{H}/\Lambda} \left| \sum_{\gamma\in\Gamma} g_\gamma \circ \omega(x) f(\gamma^{-1}x) \right|^2 \\
&\leq \sum_{x\in\Gamma/\Lambda} \left| \sum_{\gamma\in\Gamma} g_\gamma \circ \omega(x) f(\gamma^{-1}x) \right|^2 = \left\| \sum_{\gamma\in\Gamma} g_\gamma U_\gamma(f) \right\|^2,
\end{aligned}$$

so E_{H} is well-defined and contractive, therefore may be continuously extended to a linear map on $C^*_\Gamma(\Lambda)$.

Similarly, for matrices $\left[f_i\right] \in \mathbb{M}_{n,1}\big(\ell^2(\mathrm{H}/\Lambda)\big) \subseteq \mathbb{M}_{n,1}\big(\ell^2(\mathrm{H}/\Lambda)\big)$ and positive $\left[T_{i,j}\right] \in \mathbb{M}_n(C^*_\Gamma(\Lambda))$ with $T_{i,j} = \sum_{\gamma\in\Gamma} g_{\gamma;i,j} U_\gamma$,

$$\begin{aligned}
\left\langle \Big[E_{\mathrm{H}}\left(T_{i,j}\right)\Big] \Big[f_i\Big], \Big[f_i\Big] \right\rangle &= \sum_{i,j=1}^n \left\langle E_{\mathrm{H}}\left(T_{i,j}\right)(f_j), f_i \right\rangle = \sum_{i,j=1}^n \sum_{x\in\mathrm{H}/\Lambda} \overline{f_i(x)} \sum_{\eta\in\mathrm{H}} f_j(\eta^{-1}x) g_{\eta;i,j} \circ \omega(x) \\
&= \sum_{i,j=1}^n \sum_{x\in\Gamma/\Lambda} \overline{f_i(x)} \sum_{\gamma\in\Gamma} f_j(\gamma^{-1}x) g_{\gamma;i,j} \circ \omega(x) = \left\langle \Big[T_{i,j}\Big] \left(\Big[f_i\Big]\right), \Big[f_i\Big] \right\rangle \geq 0,
\end{aligned}$$

so $\left[E_{\mathrm{H}}\left(T_{i,j}\right)\right]$ is also a positive matrix. Therefore E_{H} is completely positive.

Theorem 10.8. *Let* H *and* $\mathcal{U}$ *be as above, and suppose* $\mathcal{U}$ *is open in* $\overline{\mathcal{O}_\Lambda}$ *and that the normalizer* $N_\Gamma(\Lambda) \leq \mathrm{H}$*. Then* $C^*_{\mathrm{H}}(\Lambda) \subseteq C^*_\Gamma(\Lambda)$*.*

Proof. We have $C(\mathcal{U}) \subseteq C(\overline{\mathcal{O}_\Lambda})$, so any finite sum $\sum_{\eta\in\mathrm{H}} g_\eta U_\eta \in C^*_{\mathrm{H}}(\Lambda)$ is formally equivalent to an operator in $C^*_\Gamma(\Lambda)$. The result shall follow once we show these two interpretations have equal norm. We simply note that if $\alpha_\gamma(\Lambda) \in \mathcal{U}$, then it is equal to $\alpha_\eta(\Lambda)$ for some $\eta \in \mathrm{H}$, thus $\eta^{-1}\gamma \in N_\Gamma(\Lambda)$, so $\gamma \in \mathrm{H}$. Therefore, for any $f \in \ell^2(\Gamma/\Lambda)$,

$$\begin{aligned}\left\| \sum_{\eta\in\mathrm{H}} g_\eta U_\eta(f) \right\|^2_{\ell^2(\Gamma/\Lambda)} &= \sum_{x\in\Gamma/\Lambda} \left| \sum_{\eta\in\mathrm{H}} g_\eta \circ \omega(x) f(\eta^{-1}x) \right|^2 = \sum_{x\in\mathrm{H}/\Lambda} \left| \sum_{\eta\in\mathrm{H}} g_\eta \circ \omega(x) f(\eta^{-1}x) \right|^2 \\ &= \left\| E_{\mathrm{H}} \left(\sum_{\eta\in\mathrm{H}} g_\eta U_\eta \right) \left(f|_{\mathrm{H}/\Lambda} \right) \right\|^2_{\ell^2(\mathrm{H}/\Lambda)} .\end{aligned}$$

Part II

Approximately Multiplicative Decompositions of Nuclear Maps

1 Background

Nuclearity was originally defined for C*-algebras in terms of the uniqueness of tensor products, a property first described in [Gro55]. While of historical importance and useful in its own right, the work of [Kir77] and [CE78] showed it was identical to the *completely positive approximation property (CPAP)*. This characterization has found an equal share of attention—thanks in part to its connection to hyperfinite von Neumann algebras—and it is what we investigate here.

Definition 1.1. [BO08, Ch. 2] A contractive completely positive (c.c.p.) map $\pi : A \to B$ between C*-algebras (or to a von Neumann algebra B) is called **nuclear** (resp. **weakly nuclear**) if it admits a certain decomposition: there exists a net (F_n) of finite-dimensional C*-algebras and c.c.p. maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ such that $(\varphi_n \circ \psi_n)$ converges to π_n in the point-norm (resp. point-σ-weak) topology. It is this decomposition to which the "CPA" in CPAP refers. If π is the identity, then we say that A itself is **nuclear**; if π is an inclusion, we say A is **exact**.

The germinal idea of this investigation was the definition of *nuclear dimension* in [WZ09][Def. 2.1.], which sought to quantify the CPAP. (Although this notion goes back even further to [KW04]'s *decomposition rank*, and so on.) Methods for strengthening the CPAP were first explored in [HKW12]. This was synthesized with [BK97]'s results on quasidiagonal nuclear C*-algebras into [BCW17], which was built upon by [CS19]. This part builds further by providing a partial answer to the final question in [CS19]. These experiments are about seeing how far we can stretch these finite-dimensional approximations. As in [BCW17], they hinge on properties related to quasidiagonality.

Definition 1.2. [BO08, Thm. 7.2.1] A C*-algebra $A \subset \mathbb{B}(\mathcal{H})$ is **quasidiagonal** if, for any $\epsilon > 0$ and any finite set $\mathcal{F} \subset A$ and $\mathfrak{F} \subset \mathcal{H}$, there exists a finite-rank projection $p \in \mathbb{B}(\mathcal{H})$ such that

$$\|px - xp\|, \|pv - v\| \leq \epsilon$$

for every $x \in \mathcal{F}$ and $v \in \mathfrak{F}$.

More generally, quasidiagonality is defined by the existence of another kind of finite-dimensional approximation: approximately multiplicative, *approximately isometric* c.c.p. maps. The equivalence of these is due to [Voi91].

Definition 1.3. [BO08, Def. 7.1.1] A C*-algebra A is **quasidiagonal** if, for every $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, there exist c.c.p. maps $\psi : A \to F$ to a finite-dimensional C*-algebra F such that

$$\|\psi(x)\psi(y) - \psi(xy)\|, \big|\|\psi(x)\| - \|x\|\big| < \epsilon$$

for all $x, y \in \mathcal{F}$.

This characterization motivates the following:

Definition 1.4. [Bro06, Def. 3.3.1] A trace (by which we mean a tracial state) τ on a C*-algebra A is **quasidiagonal** if, for every $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, there exist unitary completely positive (u.c.p.) maps $\psi : A \to F$ to a finite-dimensional C*-algebra F with trace tr_F such that

$$\|\psi(x)\psi(y) - \psi(xy)\|, |\tau(x) - \mathrm{tr} \circ \psi(x)| < \epsilon$$

for all $x, y \in \mathcal{F}$.

Our search for multiplicativity-in-the-limit also leads to interest in another approximate form of multiplicativity introduced alongside nuclear dimension, that being the preservation of the multiplication of orthogonal positive elements.

Definition 1.5. [WZ09, Def. 2.3] A $*$-homomorphism $\varphi : A \to B$ between C*-algebras A, B is called **order zero** if $\varphi(a_0)\varphi(a_1) = 0$ for every positive $a_0, a_1 \in A$ such that $a_0 a_1 = 0$.

The CPAP was previously examined for nuclear A in [HKW12, Thm. 1.4], which showed that we may then find $\varphi_n : F_n \to A$ (as in Definition 1.1theorem.1.1) that are convex combinations of finitely many c.c.p. order zero maps. This was further strengthened in [BCW17, Thm. 3.1] by showing the sequence $(\psi_n : A \to F_n)$ may be chosen to be approximately order zero. Furthermore, drawing from [BK97], if both A and all of its traces are quasidiagonal, then (ψ_n) may be chosen to be approximately multiplicative. In fact, [BCW17, Thm. 2.2] shows that the converse is true as well.

Using the result from [HKW12] as their starting point, [CS19] began the process of generalizing these to non-nuclear A: so long as π is order zero, we may find φ_n that are convex combinations of c.c.p. order zero maps [CS19, Th. 1]. Alternatively, approximate multiplicativity of (ψ_n) is attainable for weakly nuclear π with quasidiagonal A [CS19, Prop. 3] .

The question is, then, "What is necessary or sufficient to get approximate multiplicativity for nuclear π?" This part's results are two necessary conditions.

Lemma 2.6theorem.2.6. *Let $\pi : A \to B$ be a $*$-homomorphism between C^*-algebras A, B that admits an approximately multiplicative norm-decomposition (see Definition 2.5theorem.2.5), and τ be a trace on $\pi(A)$. Then $\tau \circ \pi$ is a quasidiagonal trace on A.*

Theorem 2.7theorem.2.7. *Let $\pi : A \to B \subseteq B^{**}$ be a $*$-homomorphism that admits an approximately multiplicative σ-strong*-decomposition (see Definition 2.2theorem.2.2). Then the inclusion $\pi(A) \subseteq B$ is nuclear.*

We also reach a full characterization in the exact case, which stems from a more general sufficient condition given in Theorem 2.12theorem.2.12.

Theorem 2.12theorem.2.12. *Let A be an exact C^*-algebra and $\pi : A \to B$ be a $*$-homomorphism to another C^*-algebra B. Then π admits an approximately multiplicative norm-decomposition iff it is nuclear and quasidiagonal and $\tau \circ \pi$ is a quasidiagonal trace on A for every trace τ on $\pi(A)$.*

We shall require Kirchberg's ϵ-test [Kir06, Lem. A.1], which—given sequences that approximately satisfy a condition arbitrarily closely—allows us to re-index into a sequence that *exactly* satisfies the condition. We state it in slightly less generality so as to make our application clear.

Theorem 1.6. (E. Kirchberg) *Let $O \subset N$ be the unit ball of a von Neumann algebra N, and suppose there is a $\gamma \in \mathbb{N}$ such that for every $n, k \in \mathbb{N}$ there exists a function $f_n^k : O \to [0, \infty)$ with Lipschitz constant less than γ. For fixed ultrafilter ω, define functions $f_\omega^k : \ell^\infty(O) \to [0, \infty)$ by $f_n^k\big((b_n)\big) = \lim_{n\to\omega} f_\omega^k(b_n)$. Finally, suppose that for every $K \in \mathbb{N}$ and $\epsilon > 0$ there is a sequence $W \in \ell^\infty(O)$ such that $f_\omega^k(W) < \epsilon$ for every $k < K$. Then there exists $U \in \ell^\infty(O)$ such that $f_\omega^k(U) = 0$ for every $k \in \mathbb{N}$.*

We also use that the σ-strong* topology on a von Neumann algebra N agrees on bounded subsets with the topology generated by seminorms

$$\|x\|_\rho := \sqrt{\rho\left(\frac{x^*x + xx^*}{2}\right)}$$

for a separating family of normal states ρ on N (see e.g. [Bla06, III.2.2.19]).

For simplicity, we shall assume that our C*-algebras are separable and unital.

2 Results

This part is only possible through use of the following as-yet unpublished result, which is something of a folklore theorem (cf. [CETW20]). We thank the authors of [CCE$^+$21] for allowing us to reproduce the proof.

Theorem 2.1. *Let $\theta, \pi : A \to N$ be weakly nuclear *-homomorphisms from a C*-algebra A to a finite von Neumann algebra N that agree on traces (that is, $\tau \circ \theta = \tau \circ \pi$ for every trace τ on N). Then θ and π are strong* approximately unitarily equivalent.*

Proof. Let normal trace τ_0 on N be given. Replacing N with $\pi_{\tau_0}(N)$ (where π_τ is the GNS representation corresponding to τ_0) if necessary, we may assume τ_0 is faithful. We need to show that θ and π are unitarily equivalent as maps into the tracial ultrapower N^ω_τ of N with respect to τ_0.

Define the weakly nuclear $*$-homomorphism $\mu : A \to \mathbb{M}_2(N^\omega_\tau)$ by

$$\mu(a) = \begin{bmatrix} \theta(a) & 0 \\ 0 & \pi(a) \end{bmatrix}.$$

We claim that $M := \pi(A)''$ is hyperfinite.

Indeed, let $\psi_n : A \to F_n$ and $\varphi_n : F_n \to M$ be c.c.p. maps for finite-dimensional C*-algebras F_n such that $(\varphi_n \circ \psi_n)$ point-weak* converges to μ. Fix a unital normal representation $M \subseteq \mathbb{B}(\mathcal{H})$, and define maps $\eta, \eta_n : A \otimes_{\mathrm{alg}} M' \to \mathbb{B}(\mathcal{H})$ by $\eta(a \otimes b) = \mu(a)b$ and $\eta_n(a \otimes b) = \varphi_n \circ \psi_n(a)b$, so that (η_n) point-weak* converges to η. The η_n are continuous with respect to the minimal tensor product since they factor through $F_k \otimes M'$, hence so is η. A conditional expectation of $\mathbb{B}(\mathcal{H})$ onto

M' is then provided by [BO08, Prop. 3.6.5], confirming that M is injective, hence hyperfinite by Connes' Theorem [Con76, Thm. 6].

Define projections

$$p_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad p_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$. Then $\tau(p_1 x) = \tau(p_2 x)$ for every normal trace τ and every $x \in \pi(A)$, hence also for every $x \in \pi(A)''$. By [CET+20, Lem. 4.5], $\tau(p_1) = \tau(p_2)$ for every trace τ on $\mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$. Thus there is a unitary $u = [u_{i,j}] \in \mathbb{M}_2(N_\tau^\omega) \cap \pi(A)'$ such that $u^* p_1 u = p_2$, hence $u_{1,1} = 0$ and $u_{1,2}^* u_{1,2} = 1_{N_\tau^\omega}$. Therefore $u_{1,2}$ is unitary since N_τ^ω is finite, and $\theta(a) u_{1,2} = u_{1,2} \pi(a)$ since $u \in \pi(A)'$. ∎

Most of the following lemma's proof is borrowed from [BCW17]; improvements are due to mixing in material from [CS19] and Theorem 2.1theorem.2.1. To properly state the theorems, we need (the W* half of) the part's central definition.

Definition 2.2. Let $\pi : A \to N$ be a $*$-homomorphism from a C*-algebra A to a von Neumann algebra N. An **approximately multiplicative σ-strong*-decomposition** of π is a net of u.c.p. maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} N$ for finite-dimensional C*-algebras F_n such that

(i) $\varphi_n \circ \psi_n(x) \to \pi(x)$ in the σ-strong* topology for all $x \in A$,

(ii) $\|\psi_n(x)\psi_n(y) - \psi_n(xy)\| \to 0$ for all $x, y \in A$,

(iii) every φ_n is a $*$-homomorphism.

Lemma 2.3. *Let A be a quasidiagonal C*-algebra and $\pi : A \to N$ be a weakly nuclear $*$-homomorphism such that, for every trace τ on $\pi(A)$, the trace $\tau \circ \pi$ is quasidiagonal. Then π admits an approximately multiplicative σ-strong*-decomposition.*

Remark 2.4. Note that the "quasidiagonal $\tau \circ \pi$" condition is satisfied if either every trace on A or every trace on $\pi(A)$ is quasidiagonal. We shall see that this property is essential.

Proof of Lemma 2.3theorem.2.3. Let $\mathcal{F} \subset A$ be a finite set of contractions, $\mathcal{S}$ a finite set of normal states on N, and $\epsilon > 0$. Define normal state $\rho = \sum_{\rho' \in \mathcal{S}} \rho'/|\mathcal{S}|$, so that $\|b\|_{\rho'}^2 \leq |\mathcal{S}| \|b\|_\rho^2$ for every $\rho' \in \mathcal{S}$ and $b \in N$. Thus we need only make reference to ρ in the proof, rather than the entire set $\mathcal{S}$.

N may be decomposed into $N_1 \oplus N_\infty$ for von Neumann algebras N_1, N_∞ that are respectively finite and properly infinite. Similarly, $\pi = \pi_1 \oplus \pi_\infty$ for weakly nuclear $*$-homomorphisms $\pi_1 : A \to N_1$ and $\pi_\infty : A \to N_\infty$. We shall deal with each summand separately by assuming it comprises all of N.

Suppose N is properly infinite. (cf. proofs of [CS19, Prop. 2] and [BCW17, Lem. 2.4]) By weak nuclearity, there are $k \in \mathbb{Z}^+$ and u.c.p. maps $A \xrightarrow{\psi'} \mathbb{M}_k \xrightarrow{\dot{\varphi}'} N$ such that $\|\dot{\varphi}' \circ \psi'(x) - \pi(x)\|_\rho \leq \epsilon$ for every $x \in \mathcal{F}$. Since A is separable, we may fix a faithful unital representation $A \subseteq \mathbb{B}(\mathcal{H})$ such that $\mathcal{H}$ is separable and A contains no nonzero compact operators. Voiculescu's Theorem [Dav96, Thm. II.5.3] provides an isometry $v : \mathbb{C}^k \to \mathcal{H}$ such that $\|v^* x v - \psi'(x)\| \leq \epsilon$ for every $x \in \mathcal{F}$. Likewise, quasidiagonality provides a finite-rank projection $p \in \mathbb{B}(\mathcal{H})$ such that $\|pv - p\| \leq \epsilon$ and $\|px - xp\| \leq \epsilon$ for every $x \in \mathcal{F}$. Define u.c.p. maps $A \xrightarrow{\psi} \mathbb{B}(p\mathcal{H}) \xrightarrow{\dot{\varphi}} N$ by $\psi(a) = pap$ and $\dot{\varphi}(T) = \dot{\varphi}'(v^* T v)$. Thus, for all $x, y \in \mathcal{F}$,

$$
\begin{aligned}
\|\psi(x)\psi(y) - \psi(xy)\| &\leq \|p\|\|xp - px\|\|yp\| \leq \epsilon, \\
\|\dot{\varphi} \circ \psi(x) - \pi(x)\|_\rho &\leq \|\dot{\varphi}'(v^*\psi(x)v) - \dot{\varphi}' \circ \psi'(x)\| + \|\dot{\varphi}' \circ \psi'(x) - \pi(x)\|_\rho \\
&\leq \|\dot{\varphi}'\|\|v^*\psi(x)v - \psi'(x)\| + \epsilon \\
&\leq \|v^* pxpv - v^* xpv\| + \|v^* xpv - v^* xv\| + \|v^* xv - \psi'(x)\| + \epsilon \\
&\leq \|v^* p - v^*\|\|xpv\| + \|v^* x\|\|pv - v\| + 2\epsilon \leq 4\epsilon.
\end{aligned}
$$

Our properly infinite assumption finally kicks in, allowing us to find a unital embedding $\iota : \mathbb{B}(p\mathcal{H}) \to N$ (a consequence of [Bla06, III.1.3.5]). By [Haa85, Prop. 2.1] there is isometry $w \in N$ such that $\dot{\varphi}(T) = w^* \iota(T) w$ for all $T \in \mathbb{B}(p\mathcal{H})$, and [Haa85][Page 167] shows how w may be approximated by a unitary u so that

$$\| \operatorname{Ad}(u^*) \circ \iota(T) - \dot{\varphi}(T)\|_\rho \leq \|T\|\epsilon \leq \epsilon$$

for every $T \in \psi(\mathcal{F}) \subset \mathbb{B}(p\mathcal{H})$. The $*$-homomorphism $\varphi = \operatorname{Ad}(u^*) \circ \iota$ therefore completes this term of the net.

Now suppose N is finite. (cf. proof of [BCW17, Lem. 2.5]) Then it has a separating family of normal *traces*, hence we may assume ρ is tracial. Moreover, there is central projection $p \in N$ such that the ideal $\{b \in N \mid \rho(b^* b) = 0\}$ is Np. Thus we may identify $\pi_{\rho\circ\pi}(A)''$ (the strong closure of the image of the GNS representation corresponding to the trace $\rho\circ\pi$) with the summand $N(1_N-p)$ and, without loss of generality, with N itself.

Once again, we may split N into factors to be dealt with separately, one of type I and another of type II_1. Also again, we may simply assume that each factor is, in turn, all of N.

First suppose it is type I, ergo can be written $N\cong\bigoplus_{i=1}^{\infty}\mathbb{M}_{n_i}\big(L^\infty(X_i)\big)$, and hence we can decompose $\pi=\bigoplus_{i=1}^{\infty}\pi^{(i)}$. By normality of π, there is a finite sum $\pi_n=\bigoplus_{i=1}^{n}\pi^{(i)}$ such that $\|\pi(x)-\pi_n(x)\|_\rho\le\epsilon$ for every $x\in\mathcal{F}$. Since $\bigoplus_{i=1}^{n}\mathbb{M}_{n_i}\big(L^\infty(X_i)\big)$ is AF, it contains a finite-dimensional C*-subalgebra F such that, for each $x\in\mathcal{F}$, there is a contraction $b_x\in F$ that satisfies $\|\pi_n(x)-b_x\|\le\epsilon$.

Since it is finite-dimensional, F is the direct sum of finitely many matrix algebras F_j. By Arveson's Extension Theorem [BO08, Thm. 1.6.1], for each j the $*$-homomorphism $\bigoplus_k f_k\mapsto f_j$ from F onto F_k may be extended to a u.c.p. map $\psi_{F_k}:\bigoplus_{i=1}^{n}\mathbb{M}_{n_i}\big(L^\infty(X_i)\big)\to F_k$. Putting them together, we get a u.c.p. map $\psi_F:=\bigoplus_j\psi_{F_j}$ which restricts to the identity on F.

Define $\psi:A\to F$ to be $\psi_F\circ\pi_n$, and $\varphi:F\to N$ to simply be the inclusion. Thus, for any $x,y\in\mathcal{F}$,

$$
\begin{aligned}
\|\varphi\circ\psi(x)-\pi(x)\|_\rho &\le \|\psi_F\circ\pi_n(x)-\psi_F(b_x)\|+\|b_x-\pi_n(x)\|+\|\pi_n(x)-\pi(x)\|_\rho\le 3\epsilon,\\
\|\psi(x)\psi(y)-\psi(xy)\| &\le \|\psi(x)\|\,\|\psi_F\circ\pi_n(y)-\psi_F(b_y)\|+\|\psi(x)\psi_F(b_y)-\psi(xy)\|\\
&\le \epsilon+\|\psi_F\circ\pi_n(x)-\psi_F(b_x)\|\,\|b_y\|+\|\psi_F(b_x)b_y-\psi(xy)\|\\
&\le 2\epsilon+\|\psi_F\|\,\|b_x-\pi_n(x)\|\,\|b_y\|+\|\psi_F(\pi_n(x)b_y)-\psi(xy)\|\\
&\le 3\epsilon+\|\psi_F\|\|\pi_n(x)\|\,\|b_y-\pi_n(y)\|\le 4\epsilon.
\end{aligned}
$$

Now suppose N is type II_1. Our strategy for finding $\dot\varphi_n,\psi_n$ that satisfy properties (ii) and (iii) of Definition 2.2theorem.2.2 is to break N into summands Np_i for n-tuples $i\in\{0,\dots,n\}^n$ and central projections p_i. We also use the p_i to find corresponding functions $\dot\varphi_i,\psi_i$ on their respective Np_i that satisfy the desired inequalities. We access approximately multiplicative ψ_i by constructing traces $\rho_i\circ\pi$ on A that are quasidiagonal by assumption. We then define $\dot\varphi_n,\psi_n$ by just putting everything back together again. The point of breaking everything down like this is to show that $\dot\varphi_n\circ\psi_n$ and π approximately agree on *all* normal traces at once; this fact is used in replacing $\dot\varphi_n$ with φ_n to satisfy (i).

N's center is of the form $L^\infty(X)$, where the measure on X is induced by ρ. Let $E : N \to L^\infty(X)$ denote the center-valued trace, and (without loss of generality) assume $\mathcal{F} = \{a_1, \dots, a_n\}$ contains only positive elements and that $3/n < \epsilon$. For each $i = (i_1, \dots, i_n) \in \{0, \dots, n\}^n$, define $p_i \in L^\infty(X)$ to be the characteristic function on

$$\bigcap_{j=1}^{n} \big(E \circ \pi(a_j)\big)^{-1} \left(\left[\frac{i_j}{n}, \frac{i_j + 1}{n} \right) \right) \subseteq X.$$

Ergo $p_i(x) = 1$ for some $x \in X$ iff $i_j \leq E \circ \pi(na_j)(x) < i_j + 1$ for every $j \in [1, n]$; note that such sets partition X.

To avoid division by zero, we define the finite indexing set $I = \big\{ i \in \{0, \dots, n\}^n \;\big|\; \rho(p_i) \neq 0 \big\}$. Since each $0 \leq a_j \leq 1_A$, there exists an $i \in I$ for ρ-almost every $x \in X$ such that $p_i(x) = 1$, and that i is unique by construction. Thus $\sum_{i \in I} p_i = 1_N$, and the projections $p_i, p_{i'}$ are orthogonal if $i, i' \in I$ are distinct. Define states ρ_i on $\pi(A)$ by $\rho_i(b) = \rho(bp_i)/\rho(p_i)$. Since ρ is tracial and p_i is central, ρ_i is also tracial; indeed, $\rho(bb'p_i) = \rho(b'p_i b) = \rho(b'bp_i)$. Thus we have traces $\tau_i := \rho_i \circ \pi$ on A. By our assumption of quasidiagonality, there are matrix algebras F_i and u.c.p. $\psi_i : A \to F_i$ such that, for any integers $j, j_0, j_1 \in [1, n]$,

$$|\mathrm{tr}_{F_i} \circ \psi_i(a_j) - \tau_i(a_j)| \leq 1/n, \tag{1}$$

$$\|\psi_n(a_{j_0})\psi_n(a_{j_1}) - \psi_n(a_{j_0}a_{j_1})\| \leq \epsilon.$$

For every $b \in N$ and $r \in L^\infty(X)$, by the properties of the center-valued trace E, $\rho(br) = \rho \circ E(br) = \rho(E(b)r)$. Moreover, for every integer $j \in [1, n]$, by the construction of the p_i, $\left\| E \circ \pi(a_j) - \sum\limits_{i \in I} \frac{i_j}{n} p_i \right\|_\infty \leq \frac{1}{n}$, thus

$$\left| \tau_i(a_j) - \frac{i_j}{n} \right| = \frac{1}{\rho(p_i)} \left| \rho(\pi(a_j)p_i) - \frac{i_j}{n}\rho(p_i) \right| = \frac{1}{\rho(p_i)} \left| \rho\left(\left(E \circ \pi(a_j) - \sum_{i' \in I} \frac{i'_j}{n} p_{i'} \right) p_i \right) \right| \leq \frac{1}{n}. \tag{2}$$

Let L be the set of positive norm-one absolutely-integrable functions on X. Also recall that each normal trace on N may be written as $b \mapsto \rho(fb)$ for some $f \in L$.Then for any $j \in [1, n]$ and

$f \in L$,

$$\left| \sum_{i \in I} \rho(fp_i) \frac{i_j}{n} - \rho\big(f\pi(a_j)\big) \right| \leq \frac{1}{n}. \tag{3}$$

Since each subalgebra Np_i is itself type II_1, there are unital $*$-homomorphisms $\dot{\varphi}_i : F_i \to Np_i$ (a consequence of [Bla06, III.2.5.4.iii]). We define $F_n = \bigoplus_{i \in I} F_i$, $\psi_n = \bigoplus_{i \in I} \psi_i$, and $\dot{\varphi}_n = \bigoplus_{i \in I} \dot{\varphi}_i$. At this point, we have shown (ii) and (iii) for the type II_1 case. Combining the inequalities (1)ResultsAMS.7, (2)ResultsAMS.9, and (3)ResultsAMS.11 with $\rho\big(f\dot{\varphi}_n(T)\big) = \sum_{i \in I} \rho(fp_i)\mathrm{tr}_{F_i}(T)$, we have also shown that $\dot{\varphi}_n \circ \psi_n$ approximately agrees with π on normal traces:

$$\sup_{f \in L} \left| \rho\big(f\dot{\varphi}_n \circ \psi_n(a_j)\big) - \rho\big(f\pi(a_j)\big) \right| \leq \frac{3}{n} < \epsilon.$$

To get (i), we shall pass to an ultrapower N^ω by taking an increasing sequence $(\{a_1, \dots, a_n\})_{n \in \mathbb{N}}$ of $\mathcal{F}$'s that approach a dense subset of the unit ball. The net $(\dot{\varphi}_n \circ \psi_n)$ induces a $*$-homomorphism $\theta : A \to N^\omega$, which we claim agrees with π^ω on traces (cf. proof of [BBS+19, Lem. 3.21]). Showing this claim is the first step toward using Theorem 2.1theorem.2.1 to deduce strong* approximate unitary equivalence, which will then be replaced using Theorem 1.6theorem.1.6 with strong* *actual* unitary equivalence.

Define $x_{j,n} = \dot{\varphi}_n \circ \psi_n(a_j) - \pi(a_j)$. Since $E\big(x_{j,n} - E(x_{j,n})\big) = 0$, [FdlH80, Thm. 3.2] allows us to write

$$x_{j,n} - E(x_{j,n}) = \sum_{i=1}^{10} [y_{j,n,i}, z_{j,n,i}]$$

where each $y_{j,n,i}, z_{j,n,i} \in N$ satisfy $\|y_{j,n,i}\| \leq 12\|x_{j,n} - E(x_{j,n})\|$ and $\|z_{j,n,i}\| \leq 12$. Thus we may define $y_{j,i} := q\big((y_{j,n,i})_n\big)$ and $z_{j,i} := q\big((z_{j,n,i})_n\big)$, where $q : \ell^\infty(N) \to N^\omega$ is the quotient map.

$$\|E(x_{j,n})\| \leq \sup_{f \in L} \left| \rho\big(f\dot{\varphi}_n \circ \psi_n(a_j)\big) - \rho\big(f\pi(a_j)\big) \right| \overset{n\to\infty}{\longrightarrow} 0,$$

so $q\big((E(x_{j,n}))_n\big) = 0$. Consequently,

$$q\big((x_{j,n})_n\big) = q\left(\left(E(x_{j,n}) + \sum_{i=1}^{10} [y_{j,n,i}, z_{j,n,i}]\right)_n\right) = \sum_{i=1}^{10} [y_{j,i}, z_{j,i}]$$

vanishes on all traces, and the claim is shown.

Note that θ is nuclear by definition: we have finite-dimensional u.c.p. maps $\widetilde{\psi}_n := \bigoplus_{k=1}^n \psi_k : A \to \bigoplus_{k=1}^n F_k$ and $*$-homomorphisms $\widetilde{\dot{\varphi}}_n : \bigoplus_{k=1}^n F_k \to N^\omega$ given by

$$\widetilde{\dot{\varphi}}_n\big((T_1, \ldots, T_n) \big) = q\big((\dot{\varphi}_1(T_1), \ldots, \dot{\varphi}_{n-1}(T_{n-1}), \dot{\varphi}_n(T_n), \dot{\varphi}_n(T_n), \ldots)_n \big).$$

Thus $\theta = \lim_\omega \dot{\varphi}_n \circ \psi_n = \lim_n \widetilde{\dot{\varphi}}_n \circ \widetilde{\psi}_n$.

Let $\{a_k \mid k \in \mathbb{N}\}$ be a dense subset of the unit ball of A, . Define $O \subset N$ to be the unit ball, $f_n^k : O \to [0, \infty)$ by $f_n^k(b) = \|b\dot{\varphi}_n \circ \psi_n(a_k)b^* - \pi(a_k)\|_\rho$ (so they all have Lipschitz constant of 2), and $f_\omega^k : \ell^\infty(O) \to [0, \infty)$ by

$$f_\omega^k\big((b_n)_n\big) = \lim_{n \to \omega} f_\omega^k(b_n) = \left\| q\big((b_n)_n\big)\theta(a_k)q\big((b_n)_n\big)^* - \pi^\omega(a_k) \right\|_{\rho^\omega}.$$

We use Theorem 2.1theorem.2.1 to deduce that for every $K \in \mathbb{N}$, there is a unitary $W \in \ell^\infty(O)$ such that $f_\omega^k\big(W\big) \leq \epsilon$ for every $k < K$. Then by Theorem 1.6theorem.1.6, there is a sequence $U = (u_n) \in \ell^\infty(O)$ such that

$$0 = f_\omega^k\big(U\big) = \lim_\omega \left\| u_n \dot{\varphi}_n \circ \psi_n(a_k)u_n^* - \pi(a_k) \right\|_\rho$$

for every $k \in \mathbb{N}$. Therefore, defining $\varphi_n = \mathrm{Ad}(u_n) \circ \dot{\varphi}_n$ and passing to a subsequence, we conclude

$$\left\| \varphi_n \circ \psi_n(a) - \pi(a) \right\|_\rho \to 0$$

for every $a \in A$. ◻

We introduced this form of W*-decomposition solely to relate it to the following C*-counterpart:

Definition 2.5. Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras A, B. An **approximately multiplicative norm-decomposition** of π is a sequence of u.c.p. maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ for finite-dimensional C*-algebras F_n such that

(i) $\|\varphi_n \circ \psi_n(x) - \pi(x)\| \to 0$ for all $x \in A$,

(ii) $\|\psi_n(x)\psi_n(y) - \psi_n(xy)\| \to 0$ for all $x, y \in A$,

(iii) every φ_n is a convex combination of u.c.p. order zero maps.

We have the following two necessary conditions for the existence of such decompositions:

Lemma 2.6. *Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras A, B that admits an approximately multiplicative norm-decomposition, and τ be a trace on $\pi(A)$. Then $\tau \circ \pi$ is a quasidiagonal trace on A.*

Proof. The composition of a trace with an order zero map is again (rescalable into) a trace [WZ09, Cor. 3.4], so each $\tau \circ \varphi_n$ is a trace on the finite-dimensional C*-algebra F_n. By assumption, the ψ_n are approximately multiplicative, and $\tau \circ \varphi_n \circ \psi_n \to \tau \circ \pi$ in the weak* topology. Therefore $\tau \circ \pi$ is quasidiagonal. ∎

Theorem 2.7. *Let $\pi : A \to B \subseteq B^{**}$ be a $*$-homomorphism that admits an approximately multiplicative σ-strong*-decomposition. Then the inclusion $\pi(A) \subseteq B$ is nuclear.*

Proof. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B^{**}$ be an approximately multiplicative σ-strong*-decomposition. By Arveson's Extension Theorem (as in the finite case of the proof of Lemma 2.3theorem.2.3), each inclusion $\varphi_n(F_n) \subset B^{**}$ extends to a conditional expectation $E_n : B^{**} \to \varphi_n(F_n)$. Moreover, by Alaoglu's Theorem we may pass (E_n) to a subsequence that converges to a linear map $E : B^{**} \to B^{**}$ in the point-σ-weak topology (cf. [BO08, Thm. 1.3.7]).

Let $\epsilon > 0$, $a \in A$, and normal functional $\eta \in B^*$ all be given. Then $\eta \circ E$ is also a normal functional, so by the previous paragraph there exists $m \in \mathbb{N}$ such that $n > m$ implies each of the following hold:

$$\begin{aligned}
\big|\eta\big(E_n\big(\pi(a)\big) - E\big(\pi(a)\big)\big)\big| &\leq \epsilon/4, \\
\big|\eta \circ E\big(\pi(a) - \varphi_n \circ \psi_n(a)\big)\big| &\leq \epsilon/4, \\
\big|\eta\big(E\big(\varphi_n \circ \psi_n(a)\big) - E_n\big(\varphi_n \circ \psi_n(a)\big)\big)\big| &\leq \epsilon/4, \\
\big|\eta\big(\varphi_n \circ \psi_n(a) - \pi(a)\big)\big| &\leq \epsilon/4.
\end{aligned}$$

Combined, we get

$$\big|\eta\big(E_n \circ \pi(a) - \pi(a)\big)\big| \leq \epsilon,$$

so the inclusion $\pi(A) \subseteq B^{**}$ is weakly nuclear.

We now use a Hahn-Banach argument akin to the proof (as in [BO08, Prop. 2.3.6]) that semidiscrete A^{**} implies nuclear A. Let X be the set of all maps from $\pi(A)$ to B^{**} of the form

$\varphi \circ \psi$ for u.c.p. $\psi : \pi(A) \to F$ and $\varphi : F \to B^{**}$ and finite-dimensional F. Then X is convex.

Indeed, let $w \in (0,1)$ and $\varphi_0 \circ \psi_0, \varphi_1 \circ \psi_1 \in X$. Then $\psi := \psi_0 \oplus \psi_1 : \pi(A) \to F_0 \oplus F_1$ is u.c.p., as is the map $\varphi : F_0 \oplus F_1 \to B^{**}$ given by $\varphi\big((M_0, M_1)\big) = w\varphi_0(M_0) + (1-w)\varphi_1(M_1)$. Thus $w\varphi_0 \circ \psi_0 + (1-w)\varphi_1 \circ \psi_1 = \varphi \circ \psi \in X$.

Let $\mathcal{F} = \{b_i \mid i \in [1,k]\} \subset \pi(A)$ be given. By weak nuclearity, the k-tuple $(b_i)_{i=1}^k \in \bigoplus_{i=1}^k B^{**}$ is in the weak-closure of the set $\left\{ \big(\varphi \circ \psi(b_i)\big)_{i=1}^k \;\middle|\; \varphi \circ \psi \in X \right\}$, hence also in its norm-closure by the Hahn-Banach Theorem. Therefore there is a map $\varphi \circ \psi \in X$ such that $\max_i \|b_i - \varphi \circ \psi(b_i)\| = \left\| (b_i)_{i=1}^k - \big(\varphi \circ \psi(b_i)\big)_{i=1}^k \right\| < \epsilon$. ◻

Proposition 2.8. *Let $\pi : A \to B$ be a $*$-homomorphism between C*-algebras A, B. Then π admits an approximately multiplicative norm-decomposition iff $\pi : A \to B^{**}$ admits an approximately multiplicative σ-strong*-decomposition.*

Proof. $\Rightarrow$: We retread the proof of Lemma 2.3theorem.2.3, beginning by letting normal state ρ, finite $\mathcal{F} \subset A$, and $\epsilon > 0$ be given.

We may skip the first paragraph of the properly infinite case, instead using approximately multiplicative norm-decomposition to provide $A \xrightarrow{\psi} F \xrightarrow{\dot{\varphi}} B$ that satisfy the necessary inequalities. We now need only use [Haa85] to find a unitary $u \in N$ so that $\varphi = \mathrm{Ad}(u^*) \circ \iota$ approximates $\dot{\varphi}$ for a unital embedding $\iota : F \to B^{**}$.

The finite case does not require that A be quasidiagonal. Of course, property (i) of our norm-decomposition witnesses the nuclearity of π, hence $\pi : A \to B^{**}$ is weakly nuclear. Therefore Lemma 2.6theorem.2.6 allows us finish this case, and this direction.

$\Leftarrow$: This is a perturbation of [HKW12, Thm. 1.4]. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\dot{\varphi}_n} B^{**}$ be an approximately multiplicative σ-strong*-decomposition. Using [HKW12, Lem. 1.1], we can find order zero u.c.p. maps $\varphi_n : F_n \to B$ (note the range) such that $\varphi_n \circ \psi_n(x) \to \pi(x)$ in the σ-weak topology for every $x \in A$.

We once again use a Hahn-Banach argument. Let $\mathcal{F} = \{a_i \mid i \in [1,k]\} \subset A$ and $\epsilon > 0$ be given. Then there is an index m such that $n > m$ implies $\|\psi_n(x)\psi_n(y) - \psi_n(xy)\| < \epsilon$ for every $x, y \in \mathcal{F}$. By assumption, $\big(\pi(a_i)\big)_{i=1}^k \in \bigoplus_{i=1}^k A$ is in the weak closure of the set $\left\{ \big(\varphi_n \circ \psi_n(a_i)\big)_{i=1}^k \;\middle|\; n > m \right\}$, hence also in the norm-closure of its convex hull.

Ergo, there are $n_j > m$ and finitely-many $w_j \in [0,1]$ such that $\sum_j w_j = 1$ and $\left\| \left(\pi - \sum_j w_j \varphi_{n_j} \circ \psi_{n_j}\right)(x) \right\| < \epsilon$ for every $x \in \mathcal{F}$. Define $\psi = \bigoplus_j \psi_{n_j} : A \to \bigoplus_j F_{n_j}$, the projections $q_{j'} : \bigoplus_j F_{n_j} \to F_{n_{j'}}$, and $\varphi = \sum_j w_j \varphi_{n_j} \circ q_j : \bigoplus_j F_{n_j} \to A$. Therefore φ is a convex combination of u.c.p. order zero maps, and for every $x, y \in \mathcal{F}$,

$$\|\pi(x) - \varphi \circ \psi(x)\| = \left\| \pi(x) - \sum_j w_j \varphi_{n_j} \circ \psi_{n_j}(x) \right\| < \epsilon,$$
$$\|\psi(x)\psi(y) - \psi(xy)\| = \max_j \|\psi_{n_j}(x)\psi_{n_j}(y) - \psi_{n_j}(xy)\| < \epsilon. \qquad \square$$

With the equivalence of these decompositions, we hope to find a way to characterize their existence. An important component seems to be that π is a quasidiagonal $*$-homomorphism.

Definition 2.9. A $*$-homomorphism $\pi : A \to B$ is **quasidiagonal** if it factors through a quasidiagonal C*-algebra D. That is, there exist $*$-homomorphisms $\pi_1 : A \to D$ and $\pi_2 : D \to B$ such that π_1 is surjective and $\pi = \pi_2 \circ \pi_1$.

Definition 2.10. An extension

$$0 \longrightarrow \ker \pi \longrightarrow A \longrightarrow B \longrightarrow 0$$

is called **locally split** if every finite subset $\mathcal{G} \subset B$ of contractions admits a u.c.p. **local lifting** $\lambda : \mathrm{span}(\mathcal{G}) \to A$ such that $\pi \circ \lambda(b) = b$ for every $b \in \mathrm{span}\, \mathcal{G}$.

Locally split is a weaker condition than exactness (see eg. [BO08, Prop. 9.1.4]). It is of note that the full power of exactness is not required for the following result:

Proposition 2.11. *Let A be a C*-algebra, $\pi : A \to B$ be a nuclear, quasidiagonal $*$-homomorphism to another C*-algebra B, and D, π_1, π_2 be as in Definition 2.10theorem.2.10. Further suppose that the trace $\tau \circ \pi$ is quasidiagonal for every trace τ on $\pi(A)$, and that the extension*

$$0 \longrightarrow \ker \pi_1 \longrightarrow A \longrightarrow D \longrightarrow 0$$

is locally split. Then π admits an approximately multiplicative decomposition.

Proof. Let $\epsilon > 0$ and finite subset $\mathcal{G} \subset D$ of contractions be given. Let $\lambda : \mathrm{span}(\mathcal{G}) \to A$ be a local lifting. By nuclearity, there are $i \in \mathbb{N}$ and u.c.p. maps $A \xrightarrow{\psi} \mathbb{M}_i \xrightarrow{\dot{\varphi}} B$ such that $\|\dot{\varphi} \circ \psi(a) - \pi(a)\| < \epsilon$ for every $a \in \lambda(\mathcal{G})$. Arveson's Extension Theorem then provides u.c.p.

$\psi' : D \to \mathbb{M}_i$ that agrees with $\psi \circ \lambda$ on $\operatorname{span}(\mathcal{G})$. Thus, for any $d \in \mathcal{G}$,

$$\left\|\dot{\varphi} \circ \psi'(d) - \pi_2(d)\right\| = \left\|\dot{\varphi} \circ \psi\big(\lambda(d)\big) - \pi\big(\lambda(d)\big)\right\| < \epsilon.$$

From this we conclude that π_2 is nuclear. This allows us to apply the properly infinite case of Lemma 2.3theorem.2.3 to get approximately multiplicative decompositions of $\pi_2 : D \to B^{**}$. This case may then be extended to work for A itself by replacing the resultant ψ_n with $\psi_n \circ \pi_1$. The finite case goes through for A unmodified, resulting in approximately multiplicative decompositions of $\pi : A \to B^{**}$. ∎

However, exactness does provide us with the converse statement.

Theorem 2.12. *Let A be an exact C*-algebra and $\pi : A \to B$ be a $*$-homomorphism to another C*-algebra B. Then π admits an approximately multiplicative decomposition iff it is nuclear and quasidiagonal and $\tau \circ \pi$ is a quasidiagonal trace on A for every trace τ on $\pi(A)$.*

Proof. As mentioned, Proposition 2.11theorem.2.11 already provides the backward direction, so we need now only address the forward one. Let $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} B$ be the approximately multiplicative norm-decomposition. The decomposition itself witnesses the nuclearity of π. The quasidiagonality of every $\tau \circ \pi$ was shown in Lemma 2.6theorem.2.6. The quasidiagonality of π itself is a consequence of [Dad97, Thm. 4.8]; for the convenience of the reader, we present a distillation of the proof.

Since A is exact, we may treat it as a C*-subalgebra of some C*-algebra C such that the inclusion is nuclear. Using Arveson's Extension Theorem, we may treat the ψ_n's domains as C. They induce a u.c.p. map Ψ from C to an ultraproduct $\prod_\omega F_n$, where $(\psi_n(a))_n$ is a representative sequence of $\Psi(a)$. Note that the approximate multiplicativity of (ψ_n) makes $\Psi|_A$ a $*$-homomorphism. Likewise, we have a c.c.p. map Φ from $\prod_\omega F_n$ to the ultrapower B^ω given by $\Phi\big((T_n)_n\big) = \big(\varphi_n(T_n)\big)_n$. Thus $\Phi \circ \Psi(a) = \big(\varphi_n \circ \psi_n(a)\big)_n = \big(\pi(a)\big)_n$, which we may identify with $\pi(a)$ itself by treating B as a C*-subalgebra of B^ω through constant sequences. Thus $\Phi|_{\Psi(A)}$ must also be a $*$-homomorphism.

Let $\epsilon > 0$ and finite subset $\mathcal{G} \subset \Psi(A)$ of contractions be given. Also let $\lambda : \operatorname{span}(\mathcal{G}) \to A$ be a local lifting of Ψ. By nuclearity of $A \subseteq C$, there exist a finite-dimensional C*-algebra G and c.c.p. maps $A \xrightarrow{\theta} G \xrightarrow{\xi} C$ such that, for every $d \in \mathcal{G}$,

$$\|(\Psi \circ \xi) \circ (\theta \circ \lambda)(d) - d\| = \|\Psi \circ \xi \circ \theta \circ \lambda(d) - \Psi \circ \lambda(d)\| \leq \|\xi \circ \theta \circ \lambda(d) - \lambda(d)\| < \epsilon.$$

Another use of Arveson's Extension Theorem yields $\theta' : \Psi(A) \to G$ with restriction $\theta'|_{\mathcal{G}} = \theta \circ \lambda$.

Thus the inclusion $\Psi(A) \subseteq \prod_\omega F_n$ is nuclear. By the Choi-Effros Lifting Theorem [BO08, Thm. C.3], said inclusion lifts to a c.c.p. map to $\prod_n F_n$, therefore $\Psi(A)$ is quasidiagonal (see eg. [BO08, Exc. 7.1.3]). $\square$

Remark 2.13. It seems probable to the author that this theorem may be strengthened to show that π admits an approximately multiplicative decomposition iff it factors through a quasidiagonal C*-algebra D via $A \xrightarrow{\pi_1} D \xrightarrow{\pi_2} B$ such that π_2 admits an approximately multiplicative decomposition. All that is needed is to show that the trace $\tau \circ \pi_2$ is quasidiagonal for every trace τ on $\pi_2(D) = \pi(A)$. This is of course satisfied if every trace on B is quasidiagonal, but the common theme of these results has been moving requirements away from the C*-algebras and onto the $*$-homomorphism itself.

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VITA

Douglas Augustine Wagner was born on December 14, 1990. Growing up in Mobile, AL, Douglas overcame neurological disabilities to earn a Presidential Scholarship to the University of South Alabama. There he graduated *summa cum laude* in 2013 with a Bachelor of Science in Mathematics and Statistics and a minor in Physics.

He went on to earn a graduate assistantship at USA as he worked on his Master of Science in Mathematics until 2015. For his thesis he studied the Riesz potential of Leja points distributed on a circle.

From there he moved to Fort Worth, TX, where he attended Texas Christian University in pursuit of a Doctorate of Philosophy in Mathematics. While at TCU he also took a leadership role in many student organizations, including giving talks to the Math Club and becoming president of both the Asian Media Association and the Gamer's Guild.

He is the first Wagner with a doctorate, narrowly beating out his brother, Will.

ABSTRACT

C*-ALGEBRAS OF ORBIT-CLOSURES,
THEN DECOMPOSITION OF NUCLEAR MAPS

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This dissertation is split into two parts. In the first part we expand upon work by Gábor Elek on C*-algebras of Uniformly Recurrent Subgroups. We construct a dynamical system from the set of subgroups of a finitely-generated discrete group. This has a nice correspondence with a Cayley-like graph of a subgroup's cosets. From these structures we construct a C*-algebra. We then apply techniques from other constructions to reveal properties of the new C*-algebra and relate them to properties of the graph, the dynamical system, and the subgroup itself. In the second we expand upon work from many hands on the decomposition of nuclear maps. Such maps can be characterized by their ability to be approximately written as the composition of maps to and from matrices. Under certain conditions (such as quasidiagonality), we can find a decomposition whose maps behave nicely, such as preserving multiplication up to an arbitrary degree of accuracy. We investigate these conditions and relate them to a W*-analog.