AN INVESTIGATION INTO RIEMANNIAN MANIFOLDS OF POSITIVE SCALAR CURVATURE

by

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OF POSITIVE SCALAR CURVATURE

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ABSTRACT

In the field of Riemannian geometry, the condition on the Riemannian metric so that a manifold has positive scalar curvature (PSC) is important for a number of reasons. Many famous researchers have contributed gradually to this area of geometry, and in this project, we study more about PSC metrics on such manifolds. Specifically, we refine and provide some details to the proof of Gromov and Lawson that the connected sum of 2 n-dimensional manifolds will admit a PSC metric, provided each of the manifolds has a metric with the same condition. We then derive some useful formulas related to the Riemann curvature tensor, the Ricci tensor, and the scalar curvature in many different scenarios. We compute the quantities for a manifold equipped with an orthonormal frame and its dual coframe, namely the connection one-form and the curvature two-form. Then, we observe the change in the structure functions, defined as a function that determines the Lie derivative of the orthonormal frame, under a nearly conformal change of the said frame. The aim of these calculations is that, by expressing the scalar curvature of a manifold M entirely in terms of the structure functions, we can determine a condition on the conformal factor so that when dividing the tangent bundle of M into two sub-bundles, then the scalar curvature restricted to one sub-bundle will “dominate” that of the other one so that if we know the scalar curvature of the former sub-bundle is positive, we can be assured that the scalar curvature of M as a whole is also positive

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LIST OF NOTATION AND SYMBOLS

M , N	Riemannian manifolds
u , v , w	tangent vectors on a manifold
U , V , W	tangent vector fields (or sometimes, open sets)
$\mathfrak{X}(M)$	the set of smooth vector fields on M
$\mathfrak{F}(M)$	the set of smooth functions on M
$T_p(M)$	the tangent space of a manifold M at a point p
TM	the tangent bundle
X , Y	sub-bundles of the tangent bundle
x^1 , x^2 , $\cdots$	the coordinates on a manifold
∂_1 , ∂_2 , $\cdots$	the coordinate vector fields
g	the metric tensor
g_{ij}	the (i, j)-th component of the metric tensor
g^{ij}	the (i, j)-th component of the inverse metric tensor
$\nabla_V W$	the covariant derivative of W in the direction of V
Γ^k_{ij}	the Christoffel symbols with respect to the coordinate vector fields
$[U, V]$	the Lie bracket of vector fields V and W.
$R(U, V)W$	the Riemann curvature tensor acting on the vector fields U , V and W
$R^i{}_{jkl}$	the components of the Riemann tensor
Ric_{ij}	the components of the Ricci tensor
K	the sectional curvature of a manifold (or the Gaussian curvature of a surface)
S	the scalar curvature of a manifold
α , β , γ	curves on a manifold (not in Chapter 7 and 8)
G	a group (in this case, a Lie group)
$GL(n, \mathbb{R})$	the general linear group of $n \times n$ matrices over $\mathbb{R}$

$SU(n)$	the $n \times n$ special unitary group
$\mathfrak{g}$	the Lie algebra of a Lie group
C_{ij}^k	the structure constants of a Lie group
$\oplus$	the direct sum
$\lambda_1 , \lambda_2, \cdots$	the principal curvatures of a submanifold
II	the shape tensor
tan and nor	the tangential and normal components of a vector
$e_1 , e_2 \cdots$	an orthonormal frame of the manifold
$e^1 , e^2 , \cdots$	the dual coframe to an orthonormal frame
$\mathbf{\Gamma}_{ij}^k$	the Christoffel symbols of an orthonormal frame
c_{ij}^k	the structure functions of an orthonormal frame
$\omega^i{}_j$	the (i, j)-th entry of the connection form
$\Omega^i{}_j$	the (i, j)-th entry of the curvature form

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CHAPTER 1. INTRODUCTION

In this report, we will investigate the question of whether a Riemannian manifold has a positive scalar curvature metric under certain special conditions. We will start in Chapter 2 by providing some background knowledge on the subject of Riemannian geometry, and specifically, how various curvature quantities are defined. We will also mention some geometric consequences of the study of curvature on manifolds, namely, the exponential map and geodesics on manifolds. In Chapter 3, we will prove a nice result concerning the geometric meaning of the scalar curvature of a Riemannian manifold as the volume deviation of geodesic balls living in the manifold as opposed to volumes of balls of the same radius but live in flat space. In Chapter 4, we use Lie group to construct a metric of negative scalar curvature on the 3-sphere $\mathbb{S}^3$. In general, every n-dimensional manifold, where $n \geq 3$, admits a metric of negative scalar curvature, but the same statement with positive scalar curvature is still an open question within the research community. Hence, in Chapter 5 and 6, we will investigate two constructions of manifolds in which we know the answer: the Cartesian product and the connected sum (the latter of which was first proven by Gromov and Lawson). In Chapter 7, we will lay out our approach used in this project, namely, the approach using local orthonormal frame and the dual coframe. This is a segue into Chapter 8, where we state the biggest result that we have come up with (Theorem 14 below) and some corollaries related to the additional geometric properties of a Riemannian manifold so that in our project, we can make the resulting scalar curvature of M to be positive as a whole.

CHAPTER 2. PRELIMINARY RESULTS IN RIEMANNIAN GEOMETRY

In this chapter, we will lay out some of the basic results of Riemannian geometry, which we will use throughout the report. All of the definitions, theorems, and formulas below can be found in [1] and [2]. Also, unless otherwise specified, in this report, we will use the Einstein summation convention. Furthermore, in all of the definitions, theorems, and lemmas below, assume that the objects considered (vector fields, tensor fields, functions, etc.) are smooth.

2.1 Riemannian Manifolds and the Metric Tensor:

2.1.1 The Tangent Space

First of all, we have a few definitions.

Definition 1 ([1]). *Let p be a point on a manifold M. A* ***(tangent) vector field*** *to M at p is a real-valued function $v : \mathfrak{F}(M) \to \mathbb{R}$ such that*

- *v is $\mathbb{R}$-linear: $v(af + bg) = av(f) + bv(g)$*
- *v satisfies Leibnitz's Rule: $v(fg) = v(f)g(p) + f(p)v(g)$ for all $f, g \in \mathfrak{F}(M) \to \mathbb{R}$ and $a, b \in \mathbb{R}$*

At first, this definition seems scary. However, differential geometers rarely think of tangent vector fields this way. Rather, one should think of the tangent vector as a differential operator that takes in a function and returns the directional derivative of the function in the direction of the said vector. Put another way, this definition is an axiomatization of the notion of directional derivative in Calculus 3.

Definition 2 ([1]). *The set of all tangent vectors at a point $p \in M$ is a vector space under usual addition and scalar multiplication, and is called the* ***tangent space*** *of M at p, denoted by $T_p(M)$.*

If $x^1, x^2, \cdots, x^n$ are coordinates of a coordinate chart on the manifold M, then a basis for

*$T_p(M)$ is the set $\{\partial_1, \partial_2, \cdots, \partial_n\}$, where n is the dimension of M and each $\partial_i = \frac{\partial}{\partial x^i}\Big|_p$ is called a **coordinate vector** in a coordinate chart of M at p.*

*The **coordinate vector fields** for a coordinate neighborhood $U \subset M$ are obtained by assigning to each point $p \in U$ a set of coordinate vectors in the sense of the above definitions.*

For instance, in $\mathbb{R}^3$, the coordinate vectors are the vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ that are seen commonly in Calculus 3. Furthermore, it can be proven that in the case of a parametric surface $\mathbf{x}(u, v)$, the vectors $\mathbf{x}_u, \mathbf{x}_v$ are the coordinate vectors of the surface.

2.1.2 Riemannian Metrics and Riemannian Manifolds

Now, we will come to the most important object in the study of Riemannian geometry.

Definition 3 ([1]). *Let M be an n-dimensional manifold and $p \in M$. A symmetric positive definite bilinear form g assigning at p an inner product g_p on the tangent space $T_p(M)$ is called a **metric tensor** (or **Riemannian metric**) on M. The pair (M, g) is then called a **Riemannian manifold***

In particular, since g is a symmetric bilinear form, it is enough to record the action of g on the coordinate vectors of a neighborhood by the theory of linear algebra. Hence, the (i, j) component of g is $g_{ij} = \langle \partial_i, \partial_j \rangle$.

Hence, we can record g in a matrix form, and since the inner product is nondegenerate, the matrix g is invertible, and we will denote the (i, j) component of the matrix g^{-1} by g^{ij}. In other words, $g_{ik}g^{kj} = \delta_{ij}$, where δ_{ij} is the Kronecker-delta symbol.

Intuitively, the metric tensor provides us with a way to measure distances, lengths, and angles on curved manifolds. Put in another way, it is a generalization of the usual dot product (or inner product) commonly seen in linear algebra.

2.2 The Levi-Civita Connection:

2.2.1 Connections

Definition 4 ([1])**.** *A **connection** on a smooth manifold M is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ so that it obeys the properties below*

- $\nabla_V W$ *is* $\mathfrak{F}(M)$*-linear in* V*,*
- $\nabla_V W$ *is* $\mathbb{R}$*-linear in* W*,*
- $\nabla_V W$ *obeys Leibnitz's Rule:* $\nabla_V(fW) = f\nabla_V W + V(f)W$ *for* $f \in \mathfrak{F}(M)$*.*

*Furthermore, the object $\nabla_V W$ is called the **covariant derivative** of W in the direction of V.*

The connection as described earlier is really just an axiomatization of the directional derivative of each of the components of the vector field W in the direction of V, in the sense that in $\mathbb{R}^3$, if $W = W^i \partial_i$, then $\nabla_V W = V(W^i)\partial_i$.

2.2.2 The Levi-Civita Connection

The following theorem is one of the greatest results in the subject of Riemannian geometry.

Theorem 1 ([1])**.** *[The fundamental theorem of Riemannian geometry] On a Riemannian manifold M, there exists a unique connection that satisfies*

- $[V, W] = \nabla_V W - \nabla_W V$*, and*
- $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$*,*

*where $[V, W]$ is the **Lie bracket** between V and W, defined as a vector field so that $[V, W](f) = V(W(f)) - W(V(f))$, and $\langle \cdot, \cdot \rangle$ is the metric tensor of M. This connection is called the **Levi-Civita connection** of M, and is characterized by **Koszul's formula**:*

$$2\langle \nabla_V W, X\rangle = V\langle W, X\rangle + W\langle V, X\rangle - X\langle V, W\rangle - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle.$$

From this point on, unless otherwise stated, we will use the Levi-Civita connection in our calculations.

2.2.3 The Christoffel Symbols

From this particular connection, we can calculate various important quantities for geometry. We begin with

Definition 5 ([1]). *The* ***Christoffel symbols*** *(or* ***connection coefficients****) of a coordinate chart of a Riemannian manifold M are functions Γ^k_{ij} so that*

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

Notice that, since partial derivatives commute, $[\partial_i, \partial_j] = 0$, and so from the first property of the Levi-Civita connection, we have that $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$, and so comparing the k-th component, we get that $\Gamma^k_{ij} = \Gamma^k_{ji}$. This is an important symmetry of the Christoffel symbol.

Furthermore, the symbols above can be calculate explicitly by

Theorem 2 ([1]). *Let g be the metric tensor on a Riemannian manifold M, then*

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_i(g_{jm}) + \partial_j(g_{im}) - \partial_m(g_{ij}))$$

The proof of this theorem follows from applying Koszul's formula to the vector fields ∂_i , ∂_j , ∂_m.

2.3 The Curvature Tensors:

2.3.1 The Riemann Tensor

Another very important quantity to be calculated with the Levi-Civita connection is the Riemann curvature tensor, defined as follows:

Definition 6 ([1]). *Let M be a Riemannian manifold. The* ***Riemann curvature tensor*** *of M is the* *multilinear function* $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ *defined by*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Note that depending on the author, the sign of the Riemann tensor above can be flipped, as in the case of [1].

Expanding the term above, we can calculate the components of the Riemann tensor as follows:

Theorem 3 ([1])**.** *On a coordinate neighborhood of a Riemann manifold* M*, we have*

$$R(\partial_k, \partial_l)\partial_j = R^i{}_{jlk}\partial_i,$$

where

$$R^i{}_{jlk} = \partial_k(\Gamma^i_{lj}) - \partial_l(\Gamma^i_{kj}) + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}.$$

This tensor, as it is, provides all in information about the curvature of M, but it is very hard to study. However, as seen below, it can be used to derive more useful objects to study the curvature of M.

2.3.2 The Sectional Curvature

A **tangent plane** Π to a Riemannian manifold M at the point p is a 2-dimensional subspace of the tangent space $T_p(M)$.

Now, we have

Definition 7 ([1])**.** *Let* Π *be a tangent plane to* M *at* p*. The quantity*

$$K(U,V) = \frac{\langle R(U,V)V, U\rangle}{\langle U,U\rangle\langle V,V\rangle - \langle U,V\rangle^2}$$

where U *and* V *are two basis vectors of* Π*, is called the* ***sectional curvature*** *of* Π*.*

The sectional curvature $K(U,V)$ as defined above is *independent of* the choice of the basis vectors U and V.

Roughly speaking, the sectional curvature is the curvature of a 2-dimensional submanifold of M that has $\{U, V\}$ as a basis for its tangent plane. This number proves to be important in understanding the two concepts below.

2.3.3 The Ricci Tensor

Definition 8 ([1])**.** *The* ***Ricci tensor*** *Ric of a Riemannian manifold M is a (0,2)-tensor field on M whose components are defined by $Ric_{ij} = R^m_{imj}$*

The summation over the upper and the lower indices is equivalent to taking the trace of the Riemann tensor. Hence, the Ricci tensor is another way to condense the information contained in the Riemann tensor to be more manageable.

Using an *orthonormal basis* of the tangent space $\beta = \{e_1, e_2, \cdots, e_n\}$, we have

$$Ric(u, u) = \sum_{k=1}^{n-1} K(u, e_k)$$

Hence, the Ricci tensor acting on a vector u is the sum of the sectional curvatures of all the mutually orthogonal tangent planes of M that have u as one of its basis vectors.

2.3.4 The Scalar Curvature

Definition 9 ([1])**.** *The* ***scalar curvature*** *of M is the function S defined by*

$$S = g^{ij} Ric_{ij} = g^{ij} R^m_{imj}$$

Another interpretation of S can be obtained by working with an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_p(M)$

$$S = 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} K(e_i, e_j)$$

Hence, the scalar curvature is the sum of all possible sectional curvatures of all the mutually orthogonal tangent planes to M at p.

In classical surface theory, we have that $S = 2K$, where K is the Gaussian curvature of a surface S.

2.4 Geodesics on a manifold:

Definition 10 ([1])**.** *Let $\alpha : I \to M$ be a curve on a manifold M, where $I \subset \mathbb{R}$ is an open interval containing 0. α is called a* ***geodesic*** *of M if $\nabla_{\alpha'}\alpha' = 0$.*

Intuitively, geodesics are curves on the manifold which are consider "straightest" in the sense that a particle on a manifold must travel in a geodesic to get to another point with the shortest distance covered. This is analogous to straight lines on planes, in that a straight line is the shortest path between any 2 points on a plane.

Another example would be great circles on spheres, where the great circles are defined as the intersections of the planes passing through the center of the sphere and the sphere itself.

2.5 The Exponential Map and Geodesic Normal Coordinates:

The existence and uniqueness theorem for ordinary differential equations implies

Theorem 4 ([1])**.** *Let M be a Riemannian manifold and $p \in M$. If $v \in T_p(M)$, then there exists an interval $J \subset I$ containing 0 and a unique geodesic $\gamma : J \to M$ of M so that $\gamma'(0) = v$*

2.5.1 The Exponential Map

From this theorem, we can define the exponential map as follows.

Definition 11 ([1])**.** *Let M be a Riemannian manifold and $p \in M$. Let G_p be the set of vectors in $T_p(M)$ so that the inextendible geodesic γ_v where $\gamma_v'(0) = v$ is defined at least on $[0, 1]$ for all $v \in G_p$. The* ***exponential map*** *of M at p is the map $\exp : G_p \to M$ so that $\exp_p(v) = \gamma_v(1)$*

Intuitively, the exponential map creates a "geodesic region" around a neighborhood of $p \in M$ in a sense that, centering at p, it will propagate outward in the direction of all possible geodesics of M having initial velocity lying in $T_p(M)$.

2.5.2 Geodesic Normal Coordinates

Proposition 1 ([1]). *For each point $p \in M$, there exists a neighborhood U' of p in $T_p(M)$ on which the exponential map $\exp_p$ is a diffeomorphism onto a neighborhood U of p in M (meaning that $\exp_p : U' \to U$ is differentiable and has a differentiable inverse).*

Now, a region $V \subset T_p(M)$ is *star-shaped* around p if $v \in V$ implies that $tv \in V$ for $0 \leq t \leq 1$. If U and U' is as in Proposition 1 and U' is star-shaped around p, then U is called a **normal neighborhood** of p.

The introduction of a normal neighborhood of p allows us to introduce a new coordinate system that has many beneficial properties for calculations.

Definition 12 ([1]). *Let $U \subset M$ be a normal neighborhood of p and $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for $T_p(M)$. The **geodesic normal coordinate system** (or **Riemann normal coordinate system**) $\beta' = (x^1, x^2, \cdots, x^n)$ determined by β assigns to each point $q \in U$ the vector coordinates relative to β of the point $\exp_p^{-1}(q)$*

The geodesic normal coordinates are extremely useful in computations because of the following proposition

Proposition 2 ([1]). *Let M be a Riemannian manifold and $p \in M$. If $x^1, x^2, \cdots, x^n$ is a geodesic normal coordinate system at p then $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$*

Hence, the metric is flat at p and consequently, all the Christoffel symbols vanish at that point.

CHAPTER 3. THE RELATIONSHIP BETWEEN THE SCALAR CURVATURE AND THE VOLUME DEVIATION OF GEODESIC BALLS

In this chapter, we will discuss more the geometric meaning of the scalar curvature of a Riemannian manifold. Specifically, we will prove the following theorem relating the volume of a geodesic ball of a Riemannian manifold and the volume of the ball with the same radius but in flat Euclidean space.

Theorem 5. *Let M be an n-dimensional Riemannian manifold and $a \in M$. Consider the geodesic ball $B_r(a)$ centered at a, of radius r (r is small). Then,*

$$V(B_r(a)) = V_f \left(1 - \frac{r^2}{6(n+2)} S(p) + O(r^3)\right),$$

where V_f denotes the volume of a ball of the same radius in $\mathbb{R}^n$ and S is the scalar curvature of M at p.

First, we need some preliminary results.

3.1 The Taylor Expansion of the Metric Tensor in a Normal Neighborhood:

In this section, we have the first result considering the Taylor expansion of the metric in a geodesic normal coordinate neighborhood of a point a in a Riemannian manifold M. The result can be found in [3].

Proposition 3 ([3])**.** *Let a be a point in an n-dimensional Riemmanian manifold M. Then, the Taylor expansion of the metric in a normal neighborhood of a, with a being the origin ($a = 0$), is:*

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\delta_{iu} x^p x^q R^u{}_{pqj}(0) + O(\|x\|^3)$$

3.2 The Square Root of the Determinant of the Metric Tensor - an Expansion:

With the result on the Taylor expansion of the metric tensor in a normal neighborhood established, we have

Lemma 1. *In a geodesic normal coordinate system near a, we have:*

$$\det g = 1 - \frac{1}{3}Ric_{pq}(0)x^p x^q + O(|x|^3).$$

Proof. Notice that by the result in Section 3.1, the matrix g in a normal neighborhood of a has the form $g = I + B$, where B is the term involving the Riemann tensor and the higher order terms. Hence, let $A = \ln g = \ln(I + B)$, and using the expansion for the log function, we have:

$$\begin{aligned} A &= \ln(I + B) \approx B - \frac{B^2}{2} + \cdots \\ &= B + O(\|B\|^2). \end{aligned}$$

Since we will account for the higher-order terms in the result (by $O(|x|^3)$), we can approximate A by the matrix B. Then, note that

$$\begin{aligned} tr(A) &= tr(B) + O(|x|^2) \\ &= \frac{1}{3}R^u{}_{pqu}x^p x^q + O(B^2) \\ &= -\frac{1}{3}x^p x^q Ric_{pq}(0) + O(B^2), \end{aligned}$$

where we used the formula quoted in Section 3.1 and taking the trace of the first and the lower last indices. Also, we used the definition of the Ricci tensor and the fact that the Riemann tensor is antisymmetric in its lower last two indices.

Now, we will use a matrix identity, which asserts that $det(e^A) = e^{trA}$

This is easy to see in the diagonal case, and since the metric tensor is symmetric and hence diagonalizable, $A = \ln(g)$ is diagonalizable too (using the definition as the series expansion of the log function, and if X is diagonalizable, then $p(X)$, where p is a polynomial, is also diagonalizable). And, since the determinant and trace are invariant under a similarity transformation, the identity holds in this case too.

Using this, we will get that:

$$\det(e^A) = \det g = \exp\left(-\frac{1}{3}x^p x^q Ric_{pq}(0) + O(|x|^3)\right).$$

Now, using the Taylor expansion for the exponential function to estimate this, we will get

$$\det g = 1 - \frac{1}{3}Ric_{pq}(0)x^p x^q + O(|x|^3).$$

as claimed. □

From this lemma, since $\sqrt{1+t} = 1 + \frac{1}{2}t + O(t^2)$, we have the following approximation of $\sqrt{\det g}$.

Corollary 1. *In a normal neighborhood near a, we have*

$$\sqrt{\det g} = 1 - \frac{1}{6}Ric_{pq}(0)x^p x^q + O(|x|^3).$$

3.3 Proof of Theorem 5:

We are now ready to prove Theorem 5.

Proof of Theorem 5. Without loss of generality (and by the existence of a normal neighborhood of p as determined by the exponential map), we can choose a geodesic polar coordinate system x of M at a so that $x(a) = 0$ (a aligns with the origin of such system). Thus, by definition, we can calculate the volume of a geodesic ball $B_r(a)$ centered at a of a small radius r by

$$V(B_r(a)) = \int\limits_{B_r(a)} \sqrt{\det g}\, dx^1 \cdots dx^n.$$

However, by Corollary 1, we can replace the integrand by its Taylor expansion up to the second degree term. When doing that, we have:

$$\begin{aligned} V(B_r(a)) &= \int\limits_{B_r(p)} \left(1 - \frac{1}{6}Ric_{pq}(0)x^p x^q + O(r^3)\right) dx^1 \cdots dx^n \\ &= \int\limits_{B_r(a)} dx^1 \cdots dx^n - \frac{1}{6}Ric_{pq}(0) \int\limits_{B_r(a)} x^p x^q dx^1 \cdots dx^n + O(r^{n+3}). \end{aligned}$$

Notice that the first term in the sum above is not dependent on any curvature quantities, so we may as well denote this quantity the volume of the ball of radius r in flat space V_f.

Now, consider the second integral. Notice that in the integrand, if $p \neq q$, the integral over the ball will be 0 (since, for instance, on $B_r(a)$, there is one half of it where $x^p > 0$ and the other half where $x^p < 0$. Now, since the expression in the formula is symmetric, the two portions will evaluate to equal value but opposite sign, hence will cancel out). Thus, we will only consider the case where $p = q$, in which case, the integral turns to

$$\int\limits_{B_r(a)} (x^i)^2 dx^1 \cdots dx^n = \frac{1}{n} \int\limits_{B_r(a)} r^2 dx^1 \cdots dx^n,$$

since the sum of the squares is symmetric in each variable, so

$$\int\limits_{B_r(a)} n(x^i)^2 dx^1 \cdots dx^n = \int\limits_{B_r(a)} r^2 dx^1 \cdots dx^n.$$

Changing the integral on the right hand side to spherical coordinates, we get:

$$\begin{aligned} \int\limits_{B_r(a)} r^2 dx^1 \cdots dx^n &= \int\limits_0^r \int\limits_{S^{n-1}(1)} u^2 u^{n-1} du dA \\ &= A(S^{n-1}(1)) \int\limits_0^r u^{n+1} du \\ &= A(S^{n-1}(1)) \frac{r^{n+2}}{n+2}. \end{aligned}$$

Now, notice that we have $A(S^{n-1}(r)) = r^{n-1}A(S^{n-1}(1))$. Hence, now solving for the volume V_f of the n-ball of radius r in flat space, we get:

$$\begin{aligned} V_f &= \int_0^r A(S^{n-1})(u))du \\ &= A(S^{n-1})(1)\int_0^r r^{n-1}dr \\ &= \frac{r^n A(S^{n-1}(1))}{n}. \end{aligned}$$

Solving for the area and plugging back in the integral, we have:

$$\begin{aligned} \int_{B_r(a)} r^2 dx^1 \cdots dx^n &= \frac{nV_f r^2}{n+2} \\ \Rightarrow \frac{1}{n}\int_{B_r(a)} r^2 dx^1 \cdots dx^n &= \frac{V_f r^2}{n+2}. \end{aligned}$$

Now, since the original integral is in terms of x^p and x^q, when plugging back the formula above into the integral (remember that the above is exactly equal to the original integral, just adding zeros when $p \neq q$), we can multiply by a δ_{pq} to indicate the distinction between the two cases.

Plugging the formula above back in the original formula for $V(B_r(a))$, we get

$$\begin{aligned} V(B_r(p)) &= V_f - \frac{1}{6}\delta^{pq}(0)Ric_{pq}(0)V_f\frac{r^2}{n+2} + O(r^{n+3}) \\ &= V_f - \frac{1}{6}g^{pq}(0)Ric_{pq}(0)V_f\frac{r^2}{n+2} + O(r^{n+3}) \\ &= V_f\left(1 - \frac{r^2 S(a)}{6(n+2)} + O(r^3)\right), \end{aligned}$$

by the definition of S, and by the property of geodesic normal coordinates that $g^{pq} = \delta^{pq}$. Hence, the theorem is proven. □

CHAPTER 4. THE CONSTRUCTION OF NEGATIVE SCALAR CURVATURE METRICS ON THE 3-SPHERE

In this chapter, we will consider the case of the 3-sphere $\mathbb{S}^3$. We will prove the following theorem regarding its scalar curvature.

Theorem 6 ([4])**.** *The sphere $\mathbb{S}^3$ is diffeomorphic to $SU(2)$. It admits a left-invariant metric of negative scalar curvature.*

However, first, we need to lay out some basic knowledge of the theory of Lie groups in order to understand the construction of such metrics.

4.1 Lie Groups and Lie Algebra:

Definition 13. *A **Lie group** is a C^∞ manifold with a group structure so that the group multiplication in G and the inverse map $i : G \to G$ by $i(g) = g^{-1}$ are smooth as maps between manifolds.*

This adds another level of structure to a general group G (commonly seen in algebra). Since a Lie group is both a group and a smooth manifold, we can use both algebraic and analytical methods to study these objects.

Example 1. *$GL(n, \mathbb{R})$ is a Lie group under matrix multiplication and the inverse map is $i(A) = A^{-1}$ for every matrix $A \in GL(n, \mathbb{R})$*

Example 2. *$SU(n)$, the special unitary group over the complex numbers, is also a Lie group (it is in fact a Lie subgroup of $SL(n, \mathbb{C})$) under matrix multiplication and the inverse map.*

Specifically, we will consider mainly the Lie group $SU(2)$, which is defined as $SU(2) = \{A \in M_n(\mathbb{C}) | AA^* = I; \det A = 1\}$. After expanding out the definition and finding the constraint on the entries of A, we will have a general form of the matrices in this group. Specifically, $SU(2)$ is the set of matrices of the form:

$$A = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & z_1 \end{bmatrix}$$

where $z_1, z_2 \in \mathbb{C}$ and $|z_1|^2 + |z_2|^2 = 1$. Also, we can identify each pair of points (z_1, z_2) with a point on the sphere $\mathbb{S}^4 \subset \mathbb{C}^2$.

Definition 14. *A* ***Lie algebra*** $\mathfrak{g}$ *is a vector space* $\mathfrak{g}$ *over a field* $\mathbb{F}$ *endowed with a bilinear map* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(a Lie bracket) that sends* (x, y) *to* $[x, y]$ *so that:*

$$[x, y] = -[y, x]$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

A very important example in this thesis is the Lie algebra $\mathfrak{su}(2)$ consisting of traceless skew-Hermitian matrices with the bracket operator $[A, B] = AB - BA$ for $A, B \in \mathfrak{su}(2)$. Specifically, the set $\mathfrak{su}(2)$ is the set of all matrices of the form

$$X = \begin{bmatrix} ia & z \\ -\bar{z} & -ia \end{bmatrix}$$

for some $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

There is one final definition that we need to address.

Definition 15. *Let* G *be a Lie group and* $g \in G$ *be fixed. Define a map* $L_g : G \to G$ *by* $L_g(a) = ga$ *for all elements* a *of* G*. Then, this map is called the* ***left multiplication map****.*

From group theory, we know that the map above is an action of the group G on itself.

4.2 The Geometry of Lie Groups:

Definition 16. *The* ***tangent space of a Lie group*** G ***at the identity****, denoted* $T_e(G)$*, is the set of all vectors of forms* $\gamma'(0)$ *where* $\gamma : \mathbb{R} \to G$ *is a curve in the group* G *so that* $\gamma(0) = e$*.*

In the case of $G = SU(2)$, one can prove that the tangent space of G at the identity is isomorphic to the Lie algebra $\mathfrak{su}(2)$ defined above.

A vector field V on a Lie group is a function that assigns to each point p a vector V_p in $T_p(G)$ as usual (The definition of $T_p(G)$ is just as the definition of $T_e(G)$, only that the base point now is p instead of the identity). Now, since G is a smooth manifold, we can endow G with a smooth Riemannian metric g, which is simply a symmetric bilinear form on $T_p(G)$.

Now, we have

Definition 17. *A metric* $\langle \cdot, \cdot \rangle$ *on a Lie group* G *is* ***left-invariant*** *if and only if*

$$\langle u, v \rangle_p = \langle dL_g(u), dL_g(v) \rangle_{gp}$$

for all $g, p \in G$*,* $u, v \in T_p(G)$*, and* dL_g *is the differential of the left multiplication map.*

4.3 The Sectional Curvature of Lie Groups:

Let $\beta = \{e_1, \cdots, e_n\}$ be an orthonormal basis for the tangent space $T_p(G)$ of a Lie group G. In this section, we will calculate the sectional curvature of the tangent plane of G spanned by two arbitrary vectors $e_i, e_j \in \beta$ with $i \neq j$. First, recall the definition of the sectional curvature:

$$K(e_i, e_j) = \frac{\langle R(e_i, e_j)e_j, e_i \rangle}{\langle e_i, e_i \rangle \langle e_j, e_j \rangle - \langle e_i, e_j \rangle^2}$$

However, since β is an orthonormal basis, the quantity in the denominator is exactly 1, so

$$K(e_i, e_j) = \langle R(e_i, e_j)e_j, e_i \rangle \tag{4.1}$$

Before working with the formula above, we have some observations. First of all, in order to easily calculate the covariant derivative of one vector field in the direction of another in the setting of a Lie group, it will be more advantageous to use Koszul's formula:

$$2\langle\nabla_V W, X\rangle = V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle$$

on the vector fields in β. Furthermore, since all of the calculations below are applied to left-invariant metrics on G, the derivative of the metric in the first three terms above are zero, and we get

$$2\langle\nabla_V W, X\rangle = -\langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle$$

Next, observe also that $[W, X] = WX - XW = -(XW - WX) = -[X, W]$. Hence, the Lie bracket is antisymmetric, and thus, using the bilinearity of the metric, we can rewrite the above once more to get

$$2\langle\nabla_V W, X\rangle = \langle V, [X, W]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle \tag{4.2}$$

Definition 18. *The* ***structure constants*** *of a Lie group G are constants C_{ij}^k so that for every $e_i\,, e_j\,, e_k$ in $T_e(G)$,*

$$[e_i, e_j] \;=\; \sum_k C_{ij}^k e_k$$

In the case of an orthonormal frame (as it is in our hypothesis), it is easy to see that $C_{ij}^k = \langle[e_i, e_j], e_k\rangle$. Furthermore, by the antisymmetry of the Lie bracket as above, we have

$$C_{ij}^k \;=\; -C_{ji}^k\,(\dagger)$$

This symmetry of the structure constants will be of vital importance later on.

Finally, since β is an orthonormal basis, we have

$$\nabla_{e_i} e_j \;=\; \sum_k \langle \nabla_{e_i} e_j, e_k \rangle e_k \,(*)$$

We can now return to Equation (4.1). Using the definition of the Riemann tensor and the bilinearity of the metric, we can expand the equation as

$$K(e_i, e_j) = \langle \nabla_{e_i} \nabla_{e_j} e_j, e_i \rangle - \langle \nabla_{e_j} \nabla_{e_i} e_j, e_i \rangle - \langle \nabla_{[e_i, e_j]} e_j, e_i \rangle \tag{4.3}$$

We now focus on the first term of (4.3) above. First of all, we have:

$$\begin{aligned}
\nabla_{e_j} e_j &= \sum_k \langle \nabla_{e_j} e_j, e_k \rangle e_k \\
&= \frac{1}{2} \sum_k (\langle e_j, [e_k, e_j] \rangle + \langle e_j, [e_k, e_j] \rangle + \langle e_k, [e_j, e_j] \rangle) e_k \\
&= \frac{1}{2} \sum_k 2 C^j_{kj} e_k \\
&= \sum_k C^j_{kj} e_k
\end{aligned}$$

where we have used the orthonormal expansion of the covariant derivative in (*), Koszul's formula (4.2) and the fact that $[V, V] = 0$.

Hence,

$$\begin{aligned}
\nabla_{e_i} \nabla_{e_j} e_j &= \nabla_{e_i} \left(\sum_k C^j_{kj} e_k \right) \\
&= \sum_k C^j_{kj} \nabla_{e_i} e_k) \\
&= \sum_k C^j_{kj} \cdot \frac{1}{2} \sum_l \langle \nabla_{e_i} e_k, e_l \rangle e_l \\
&= \frac{1}{2} \sum_{k,l} C^j_{kj} (\langle e_i, [e_l, e_k] \rangle + \langle e_k, [e_l, e_i] \rangle + \langle e_l, [e_i, e_k] \rangle e_l \\
&= \frac{1}{2} \sum_{k,l} C^j_{kj} (C^i_{lk} + C^k_{li} + C^l_{ik}) e_l
\end{aligned}$$

where we used the $\mathbb{R}$-linearity of the connection and Koszul's formula (4.2).

Therefore, taking the inner product via the metric with e_i, we get:

$$\begin{aligned} \langle \nabla_{e_i} \nabla_{e_j} e_j, e_i \rangle &= \frac{1}{2} \sum_{k,l} C_{kj}^j (C_{lk}^i + C_{li}^k + C_{ik}^l) \langle e_l, e_i \rangle \\ &= \frac{1}{2} \sum_{k} C_{kj}^j (C_{ik}^i + C_{ii}^k + C_{ik}^i) \\ &= \sum_{k} C_{kj}^j C_{ik}^i \end{aligned} \tag{4.4}$$

since the basis is orthonormal.

Expanding the second term in (4.3) in terms of the structure constants as above, we have:

$$\begin{aligned} \nabla_{e_i} e_j &= \sum_{k} \langle \nabla_{e_i} e_j, e_k \rangle e_k \\ &= \frac{1}{2} \sum_{k} (\langle e_i, [e_k, e_j] \rangle + \langle e_j, [e_k, e_i] \rangle + \langle e_k, [e_i, e_j] \rangle) e_k \\ &= \frac{1}{2} \sum_{k} (C_{kj}^i + C_{ki}^j + C_{ij}^k) e_k \end{aligned}$$

where we used Koszul's formula (4.2).

Hence,

$$\begin{aligned} \nabla_{e_j} \nabla_{e_i} e_j &= \nabla_{e_j} \left(\frac{1}{2} \sum_{k} (C_{kj}^i + C_{ki}^j + C_{ij}^k) e_k \right) \\ &= \frac{1}{2} \sum_{k} (C_{kj}^i + C_{ki}^j + C_{ij}^k) \nabla_{e_j} e_k \\ &= \frac{1}{2} \sum_{k} (C_{kj}^i + C_{ki}^j + C_{ij}^k) \cdot \frac{1}{2} \sum_{l} (C_{lk}^j + C_{lj}^k + C_{jk}^l) e_l \\ &= \frac{1}{4} \sum_{k,l} (C_{kj}^i + C_{ki}^j + C_{ij}^k)(C_{lk}^j + C_{lj}^k + C_{jk}^l) e_l \end{aligned}$$

by the $\mathbb{R}$-linearity of the connection and a re-indexing of the covariant derivative $\nabla_{e_j} e_k$.

Therefore, taking the inner product via the metric with e_i, we get:

$$\begin{aligned}\langle\nabla_{e_j}\nabla_{e_i}e_j, e_i\rangle &= \frac{1}{4}\sum_{k,l}(C^i_{kj}+C^j_{ki}+C^k_{ij})(C^j_{lk}+C^k_{lj}+C^l_{jk})\langle e_l, e_i\rangle \\ &= \frac{1}{4}\sum_{k}(C^i_{kj}+C^j_{ki}+C^k_{ij})(C^j_{ik}+C^k_{ij}+C^i_{jk}) \qquad (4.5)\end{aligned}$$

since the basis is orthonormal.

Last but not least, we will expand the last term of (4.3). Expanding the term, we get

$$\begin{aligned}\langle\nabla_{[e_i,e_j]}e_j, e_i\rangle &= \left\langle\nabla_{\sum_k C^k_{ij}e_k}e_j, e_i\right\rangle \\ &= \left\langle\sum_k C^k_{ij}\nabla_{e_k}e_j, e_i\right\rangle \\ &= \sum_k C^k_{ij}\left\langle\nabla_{e_k}e_j, e_i\right\rangle \\ &= \sum_k C^k_{ij}\left\langle\frac{1}{2}\sum_l(C^k_{lj}+C^j_{lk}+C^l_{kj})e_l, e_i\right\rangle \\ &= \frac{1}{2}\sum_{k,l}C^k_{ij}(C^k_{lj}+C^j_{lk}+C^l_{kj})\left\langle e_l, e_i\right\rangle \\ &= \frac{1}{2}\sum_{k}C^k_{ij}(C^k_{ij}+C^j_{ik}+C^i_{kj}) \qquad (4.6)\end{aligned}$$

where we use, again, Koszul's formula (4.2), the bilinearity of the metric, and the fact that the basis is orthonormal.

Now, taking (4.4)-(4.5)-(4.6), we get that the sectional curvature of the plane spanned by e_i and e_j will be

$$\begin{aligned}K(e_i, e_j) &= \sum_k C^j_{kj}C^i_{ik} \\ &- \frac{1}{4}\sum_{k}(C^i_{kj}+C^j_{ki}+C^k_{ij})(C^j_{ik}+C^k_{ij}+C^i_{jk}) \\ &- \frac{1}{2}\sum_{k}C^k_{ij}(C^k_{ij}+C^j_{ik}+C^i_{kj}) \qquad (4.7)\end{aligned}$$

Now, expanding the terms above, we have

$$\begin{aligned} K(e_i,e_j) &= \sum_k C^j_{kj}C^i_{ik} \\ &- \frac{1}{4}\sum_k C^i_{kj}C^j_{ik} + C^i_{kj}C^k_{ij} + C^i_{kj}C^i_{jk} + C^j_{ki}C^j_{ik} + C^j_{ki}C^k_{ij} + C^j_{ki}C^i_{jk} + C^k_{ij}C^j_{ik} + C^k_{ij}C^k_{ij} + C^k_{ij}C^i_{jk} \\ &- \frac{1}{2}\sum_k C^k_{ij}C^k_{ij} + C^k_{ij}C^j_{ik} + C^k_{ij}C^i_{kj} \end{aligned}$$

Using the symmetry outlined in (†) to simplify the above, we get

$$\begin{aligned} K(e_i,e_j) &= -\sum_k C^j_{jk}C^i_{ik} \\ &- \frac{1}{4}\sum_k(-C^i_{jk}C^j_{ik} - C^i_{jk}C^k_{ij} - C^i_{jk}C^i_{jk} - C^j_{ik}C^j_{ik} - C^j_{ik}C^k_{ij} - C^j_{ik}C^i_{jk} + C^k_{ij}C^j_{ik} + C^k_{ij}C^k_{ij} + C^k_{ij}C^i_{jk}) \\ &- \frac{1}{2}\sum_k C^k_{ij}C^k_{ij} - C^k_{ij}C^j_{ki} + C^k_{ij}C^i_{kj} \end{aligned}$$

Hence,

$$K(e_i,e_j) = \sum_k\left(\frac{-3}{4}(C^k_{ij})^2 + \frac{1}{4}(C^j_{ik})^2 + \frac{1}{4}(C^i_{jk})^2 - C^j_{jk}C^i_{ki} + \frac{1}{2}C^k_{ij}(C^j_{ki} - C^i_{kj}) + \frac{1}{2}C^i_{jk}C^j_{ik}\right) \tag{4.8}$$

4.4 Return to the Proof:

Now, we are ready to prove Theorem 6.

Proof of Theorem 6: Notice that by definition, we have $\mathbb{S}^3 = \{(a,b,c,d) \in \mathbb{R}^4 | a^2+b^2+c^2+d^2 = 1\}$. However, we can identify the 3-sphere as the set $S = \{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\}$.

Now, let $\phi : S \to SU(2)$ by

$$\phi(z,w) = \begin{bmatrix} z & w \\ -\bar{w} & z \end{bmatrix}$$

Viewing ϕ as the restriction of a smooth map ϕ' from $\mathbb{C}^2 \cong \mathbb{R}^4$ to $M_2(\mathbb{C}) \cong \mathbb{R}^8$, we can see that ϕ' is a smooth map between the two parent spaces. Furthermore, ϕ is a bijection from S to $SU(2)$.

Thus, since $S = \mathbb{S}^3$ and $SU(2)$ are submanifolds of $\mathbb{C}^2$ and $M_2(\mathbb{C})$, ϕ restricted to S must also be a diffeomorphism ($\phi = \phi'$ but restricted to S, so ϕ must also be smooth and has smooth inverse. Along with being bijective, we get the diffeomorphism claim.)

Now, we know from the above discussion that the tangent space at the identity of $SU(2)$ is precisely $\mathfrak{su}(2)$. Hence, a basis for the tangent space $T_e(SU(2))$ comprises of the matrices

$$e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

By a simple check, one can see that these form an orthonormal basis for the tangent space with respect to the modified Frobenius inner product $\langle A, B\rangle = -\frac{1}{2}\text{tr}(AB)$. Let g be the left-invariant metric determined by this inner product. (The full construction is long, but the rough idea is that we define the inner product above at $\mathfrak{su}(2) \cong T_e(SU(2))$, and propagate that inner product by the left action of G on itself. More precisely, for every pair of vectors $u, v \in T_g(SU(2))$, where $g \in SU(2)$ is arbitrary, define $\langle u, v\rangle_g = \langle dL_{g^{-1}}(u), dL_{g^{-1}}(v)\rangle_e$, where $dL_{g^{-1}}$ denotes the differential of the left multiplication map by g^{-1}).

By another calculations, using the Lie bracket $[A, B] = AB - BA$, we see that $[e_1, e_2] = 2e_3$, $[e_2, e_3] = 2e_1$, $[e_1, e_3] = -2e_2$.

Now, define on $\mathfrak{su}(2)$ the scaled left-invariant metric g_μ given by

$$g_\mu(e_i, e_j) = \begin{cases} \mu_1 & i = j = 1 \\ \mu_2 & i = j = 2 \\ \mu_3 & i = j = 3 \\ 0 & \text{otherwise} \end{cases}$$

Then, note that the basis $\{e_i\}$ for $i = 1, 2, 3$ above is no longer orthonormal with respect to g_μ. However, if we set $\epsilon_i = \frac{e_i}{\sqrt{\mu_i}}$, then the set $\{\epsilon_i\}$ for $i = 1, 2, 3$ is an orthonormal basis for the tangent space.

Let us calculate the sectional curvature $K(\epsilon_1, \epsilon_2)$ of $SU(2)$, which is now identified with the 3-sphere!

Expanding (4.8) above, taking $n = 3$ in the case of $\mathbb{S}^3$, we have:

$$
\begin{aligned}
K(\epsilon_1, \epsilon_2) &= -\frac{3}{4}\left((C_{12}^1)^2 + C_{12}^2)^2 + (C_{12}^3)^2\right) + \frac{1}{4}\left((C_{11}^2)^2 + (C_{12}^2)^2 + (C_{13}^2)^2\right) \\
&+ \frac{1}{4}\left((C_{21}^1)^2 + (C_{22}^1)^2 + (C_{23}^1)^2\right) - (C_{11}^1 C_{21}^2 + C_{12}^1 C_{22}^2 + C_{13}^1 C_{23}^2) \\
&+ \frac{1}{2}\left[C_{12}^1(C_{11}^2 - C_{12}^1) + C_{12}^2(C_{21}^2 - C_{22}^1) + C_{12}^3(C_{31}^2 - C_{32}^1)\right] + \frac{1}{2}(C_{21}^1 C_{11}^2 + C_{22}^1 C_{12}^2 + C_{23}^1 C_{13}^2)
\end{aligned} \tag{4.9}
$$

Note that by our the symmetries of the structure constants, we have the terms with the two matching lower indices will vanishes, for instance $C_{22}^1 = 0$. Now, we also have the following

$$
\begin{aligned}
C_{ij}^k &= \langle \epsilon_k, [\epsilon_i, \epsilon_j] \rangle \\
&= \left\langle \frac{e_k}{\sqrt{\mu_k}}, \left[\frac{e_i}{\sqrt{\mu_i}}, \frac{e_j}{\sqrt{\mu_j}}\right] \right\rangle \\
&= \frac{1}{\sqrt{\mu_i \mu_j \mu_k}} \langle e_k, [e_i, e_j] \rangle
\end{aligned} \tag{4.10}
$$

Furthermore, by the above calculations, note that the bracket here is cyclic. Hence, if $k = i$ or $k = j$, then the bracket operator will yield a vector different from e_k, which will turn to 0 when taking the product with e_k by definition of g_μ. Consequently, all of the terms with an upper index matching one of the lower indices, for instance, C_{12}^1 will also turn to 0.

Hence, the sectional curvature will simplify to

$$
K(\epsilon_1, \epsilon_2) = -\frac{3}{4}(C_{12}^3)^2 + \frac{1}{4}(C_{13}^2)^2 + \frac{1}{4}(C_{23}^1)^2 + \frac{1}{2}\left[C_{12}^3(C_{31}^2 - C_{32}^1)\right] + \frac{1}{2}C_{23}^1 C_{13}^2
$$

After plugging in to (4.10), we can see that

$$\begin{aligned} C_{12}^3 &= \frac{2\sqrt{\mu_3}}{\sqrt{\mu_1\mu_2}} \\ C_{13}^2 = -C_{31}^2 &= \frac{-2\sqrt{\mu_2}}{\sqrt{\mu_1\mu_3}} \\ C_{23}^1 = -C_{32}^1 &= \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2\mu_3}} \end{aligned}$$

Finally, plug the coefficients back into the formula for K in (4.9) and simplify, we will get

$$K(\epsilon_1, \epsilon_2) = -3\frac{\mu_3}{\mu_1\mu_2} + \frac{\mu_2}{\mu_1\mu_3} + \frac{\mu_1}{\mu_2\mu_3} + \frac{2}{\mu_1} + \frac{2}{\mu_2} - \frac{2}{\mu_3}$$

Similarly, the other scalar curvatures can be similarly derived, and we have

$$\begin{aligned} K(\epsilon_1, \epsilon_3) &= -3\frac{\mu_2}{\mu_1\mu_3} + \frac{\mu_3}{\mu_1\mu_2} + \frac{\mu_1}{\mu_2\mu_3} + \frac{2}{\mu_1} + \frac{2}{\mu_3} - \frac{2}{\mu_2} \\ K(\epsilon_2, \epsilon_3) &= -3\frac{\mu_1}{\mu_2\mu_2} + \frac{\mu_3}{\mu_1\mu_2} + \frac{\mu_2}{\mu_1\mu_3} + \frac{2}{\mu_2} + \frac{2}{\mu_3} - \frac{2}{\mu_1} \end{aligned}$$

Taking twice the sum of the above 3 equations, we will get the scalar curvature of $\mathbb{S}^3$. It is

$$\begin{aligned} S &= 2\sum_{i<j} K(\epsilon_i, \epsilon_j) \\ &= -2\left(\frac{\mu_1}{\mu_2\mu_3} + \frac{\mu_2}{\mu_3\mu_1} + \frac{\mu_3}{\mu_1\mu_2}\right) + 4\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}\right) \end{aligned}$$

Since we are dealing with a Riemannian metric, the parameters μ_1, μ_2, μ_3 must all be positive. Now, let $\mu_1 = k, \mu_2 = k^2, \mu_3 = k^3$ for a parameter $k > 0$, then

$$S = -2(1 + k^{-2} + k^{-4}) + 4(k^{-1} + k^{-2} + k^{-3})$$

We can choose k so that k is small (let k approach 0). In that case, the k^{-4} term will dominate, and so the scalar curvature will tends towards $-2k^{-4} < 0$. Hence, g_μ is the metric desired, and the proof is done. □

4.5 Manifolds of Dimension 3 or Higher:

In fact, it is known that any manifolds of dimension 3 or higher always admit a metric of negative scalar curvature. However, whether there exists a metric of positive scalar curvature on these manifolds is still an open question within the research community. In this thesis, we aim to get a better understanding of this question through an analysis of a special case of a Riemannian manifold, outlined in the chapters to come.

One may ask, how about manifolds of dimension 1 or 2? In fact, for a 2-dimensional manifold, the Gauss-Bonnet theorem dictates the possible sign of the Gaussian curvature (and hence, the scalar curvature) of the manifold. For instance, applying the Gauss-Bonnet theorem to the 2-sphere $\mathbb{S}^2$, we have

$$\int K dA = 2\pi\chi(\mathbb{S}^2),$$

where $\chi(\mathbb{S}^2)$ is the Euler characteristic of the sphere, which is 2, and K is its Gaussian curvature. Hence, $\int K dA = 4\pi > 0$. Since K is a continuous function on the sphere, it must be the case that $K > 0$ at some point on $\mathbb{S}^2$. Hence, there does not exist a Riemannian metric for which $\mathbb{S}^2$ has negative Gaussian curvature (which is half of its scalar curvature), since by saying that $\mathbb{S}^2$ has negative Gaussian curvature, we mean that it has negative Gaussian curvature everywhere.

CHAPTER 5. SCALAR CURVATURE AND CARTESIAN PRODUCTS OF MANIFOLDS

After having laid out the basics of the scalar curvature in the case of $\mathbb{S}^2$, in this chapter, we will prove the first known result considering positive scalar curvature metrics in a special construction of Riemannian manifolds. In particular, we will prove the following theorem

Theorem 7. *Let M and N be compact Riemannian manifolds, and suppose that M admits a metric of posiitve scalar curvature (PSC). Then, there exists a metric on $M \times N$ that is a PSC metric.*

However, before going to the proof of the theorem, we need to prove two other results.

5.1 The Relationship between the Scalar Curvature of the Product Manifold to the Scalar Curvature of the Component Manifolds:

First, we have

Lemma 2. *Let M and N be Riemannian manifolds with metrics g_1 and g_2 and scalar curvatures S_1 and S_2 respectively. Then the scalar curvature of $M \times N$, the product of M and N, will be $S = S_1 + S_2$.*

Proof. Let M and N be Riemannian manifolds with local coordinate functions $\{x^1, \cdots, x^m\}$ and $\{y^1, \cdots, y^n\}$ respectively and $p \in M$, $q \in N$. Then, by the result in page 4 of [1], the product $M \times N$ has local coordinates $\{x^1, \cdots, x^m, y^1, \cdots, y^n\}$.

Also, the tangent space of the product $T_{(p,q)}(M \times N)$ is, by Lemma 1.43 in [1], the direct sum of the tangent spaces of the individual component. In other words, $T_{(p,q)}(M \times N) = T_p(M) \oplus T_q(N)$. Hence, every tangent vector to the product manifold can be decomposed uniquely as the sum of two tangent vectors of the two constituent manifolds.

Now, the metric on the product manifold is defined as follows. For $X, Y \in \mathfrak{X}(M \times N)$, by above, we can write $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ where $X_1, Y_1 \in \mathfrak{X}(M)$ and $X_2, Y_2 \in \mathfrak{X}(N)$. By Lemma 3.5 in [1], with π and σ are the projections of $M \times N$ to M and N respectively:

$$\begin{aligned} g(X,Y) &= g_1(X_1,Y_1) + g_2(X_2,Y_2) \\ &= g_1 \oplus g_2(X,Y), \end{aligned}$$

where we have used the definition of the pullback of a tensor and the definition of the projection map.

Specifically, this definition implies that the components of g will be as follows.

On the product manifold, the local coordinate vector fields will be $\partial_{i_1}, \cdots, \partial_{i_m}, \partial_{j_1}, \cdots, \partial_{j_n}$, the union of the coordinate fields on M and N, where the first m coordinate fields are vector fields on M and the last n fields are on N (pg. 7 of [1]). Hence, for g, we will have that on the first m coordinate vector fields, g is nothing but g_1 (since g_2 does not work on the last n coordinate fields). Similarly, on the last n coordinate vector fields, $g = g_2$. However, on the cross terms (one from the first m coordinates and one from the last n), $g = 0$ (Since, for instance, $g(\partial_{i_u}, \partial_{j_v}) = g(\partial_{i_u} + 0, 0 + \partial_{j_v}) = g_1(\partial_{i_u}, 0) + g_2(0, \partial_{j_v}) = 0$).

Hence, in these coordinates, the matrix for the metric is

$$g = \left[\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right],$$

a partitioned $(m+n) \times (m+n)$ matrix.

Now, notice that an implication of Corollary 3.58 in [2] is that the Riemann tensor (and consequently the Ricci tensor) on $M \times N$ will be related by $R_{XY}(Z) = (R_1)_{X_1Y_1}(Z_1) + (R_2)_{X_2Y_2}(Z_2)$, where $X = X_1 + X_2$, $Y = Y_1 + Y_2$ and $Z = Z_1 + Z_2$ with $X_1, Y_1 Z_1 \in \mathfrak{X}(M)$ and $X_2, Y_2, Z_2 \in \mathfrak{X}(N)$ and R_1,R_2 are the Riemann tensor in M and N respectively (in the Lemma, the right-hand side of part 1 and 2 is our proposed $R_{XY}(Z)$).

In particular, the components $R^i{}_{jlk}$ of the Riemann tensor of $M \times N$ matches that of R_1 for

$i, j, k, l \leq m$, matches that of R_2 for $m+1 \leq i, j, k, l \leq m+n$ and is zero otherwise.

Finally, the scalar curvature S of $M \times N$, is

$$\begin{aligned} S &= \sum_{i,j,k=1}^{m+n} g^{ij} R^k{}_{ikj} \\ &= \sum_{i,j,k=1}^{m} g^{ij} R^k{}_{ikj} + \sum_{i,j,k=m+1}^{m+n} g^{ij} R^k{}_{ikj} \\ &= S_1 + S_2. \end{aligned}$$

We used the fact that the inverse metric of the product manifold behaves exactly like g. All the indices have to start either at 1 or at $m+1$ because the "cross terms" are all zero. Either the value of g or of the Riemann tensor are zero at those values (for instance, if i starts at 1 but j starts at $m+1$, then $g^{ij} = 0$. If $i, j \leq m$ but $k \geq m+1$, then $R^k_{ikj} = 0$).

Hence, we get that $S = S_1 + S_2$, as desired. □

5.2 The Change in the Scalar Curvature of a Manifold under a Scaling of the Riemannian Metric:

The second result that we need to prove Theorem 5 is

Lemma 3. *Let M be a Riemannian manifold with metric g, and let $t > 0$ be arbitrary. Denote S the scalar curvature of M with respect to g. Then, tg is another metric on M, and the scalar curvature S' of M with respect to tg will be $S' = \frac{1}{t}S$*

Proof. The fact that tg is a metric on M is easy to see from the definition, since scaling a tensor by a nonzero constant does not affect its symmetry or its nondegeneracy.

For all the computations below, denote the quantities with a prime the quantities corresponding to the scaled metric, and the ones without a prime the quantities with respect to the old metric.

Notice that the scalar curvature depends on the Ricci tensor, which in turn depends on the Riemann tensor, which ultimately depends on the Christoffel symbols. Hence, we will need to investigate how the Christoffel symbol changes under a scaling of the metric. First, notice that if g is scaled by a factor of t, then $g'_{uv} = tg_{uv}$, which means that $g'^{uv} = \frac{1}{t}g^{uv}$ for all u and v, since if this was the case, then $g'^{uv}g'_{uv} = \frac{1}{t}g^{uv}tg_{uv} = \delta_{uv}$, which is what we want.

We have, by the formula of the Christoffel symbols:

$$\begin{aligned}
\Gamma'^{a}_{bc} &= \frac{1}{2}g'^{ad}\left(\partial_b(g'_{cd}) + \partial_c(g'_{bd}) - \partial_d(g'_{bc})\right) \\
&= \frac{1}{2}\left(\frac{1}{t}g^{ad}\left(\partial_b(tg_{cd}) + \partial_c(tg_{bd}) - \partial_d(tg_{bc})\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{t}\cdot tg^{ad}\left(\partial_b(g_{cd}) + \partial_c(g_{bd}) - \partial_d(g_{bc})\right)\right) \\
&= \frac{1}{2}g^{ad}\left(\partial_b(g_{cd}) + \partial_c(g_{bd}) - g\partial_d(g_{bc})\right) \\
&= \Gamma^{a}_{bc}
\end{aligned}$$

Hence, the Christoffel symbols are invariant under a scaling of the metric.

Therefore, the component of the Riemann tensor does not change as well. In other words, $R'^{i}_{jlk} = R^{i}_{jlk}$, since the formula of the components of the tensor only involves the Christoffel symbols and their derivatives and by the above.

Now, the Ricci tensor of the new metric will be, by definition

$$Ric'_{jk} = R'^{e}{}_{jek} = R^{e}{}_{jek} = Ric_{jk}$$

by the invariance of the Riemann tensor above. Finally, we have

$$S' = g'^{jk}Ric'_{jk} = \frac{1}{t}g^{jk}Ric_{jk} = \frac{1}{t}S$$

by the inverse metric scaling as above. Hence, $S' = \frac{1}{t}S$, as claimed. □

5.3 Proof of Theorem 7:

Now, we are in the position to prove Theorem 7.

Proof of Theorem 7. Let M and N be compact Riemannian manifolds with metric g_1 and g_2 respectively. Furthermore, assume g_1 is a PSC metric on the manifold M.

The definition of the metric on the product manifold is defined in Lemma 2. Now, denote S_1 be the scalar curvature of M under g_1 and likewise for S_2. Hence, by Lemma 1, we have the scalar curvature S of $M \times N$ is $S = S_1 + S_2$.

Now, proceed by scaling the metric g_1 on M by the real function $f(t) = t$ (we demand that $f(t) > 0$). First, note that this function is actually a constant with respect to the coordinates of M. Hence, the calculations in Lemma 3 is still valid.

When we scale the metric of M by t, the resulting scaling curvature of M will be S_1', and thus, it induces a new scalar curvature S' of the product $M \times N$. However, by Lemma 2, we have:

$$\begin{aligned} S' &= S_1' + S_2 \\ &= \frac{1}{t} S_1 + S_2 (\text{By Lemma 3}) \end{aligned} \tag{5.1}$$

Now, notice that depending on the position of the point $(p, q) \in M \times N$ (where $p \in M$ and $q \in N$) and because S_1 is positive by the assumption that g_1 is a PSC metric on M, we can choose a sufficiently small t so that in an open neighborhood of (p, q), the first term in (1) dominates the second term, making S locally positive (since S is a continuous function on M).

So how can we make the construction global?

For each point $(p_i, q_i) \in M \times N$, choose a t_i so that the scalar curvature S is locally positive in an open neighborhood U_i of (p_i, q_i). Then, it is easy to see that the collection $\bigcup_i U_i$ is an open cover of $M \times N$. However, $M \times N$ is compact (since M and N are, and Cartesian products of compact sets are compact).

Hence, by definition of compactness, there is a finite collection $U = \bigcup_i^n U_i$ that also covers $M \times N$.

Notice that in each of the set U_i in U, there is a corresponding t_i that makes S locally positive in that open set. Now, let $k = \min\{t_i : i = 1, 2, \cdots, n\}$ (Since n is finite, the minimum actually exists, and since the scaling function t is mandated to be nonzero anywhere, k is also nonzero).

Then, notice that the scaling factor k on g_1 will make the scalar curvature positive globally on $M \times N$, since it will make S positive for every U_i in the cover U of the product!

Hence, $g = kg_1 \oplus g_2$ is the desired positive scalar curvature metric on the product manifold, as claimed. □

5.4 Remark:

This theorem provides an inspiration for us with this project. It basically asserts that if a certain Riemannian manifold M admits a metric of positive scalar curvature, then when we do the Cartesian product withanother Riemannian manifold N, we can always ensure that there is a metric for which the scalar curvature of M "dominates" the scalar curvature of N, making the overall scalar curvature of the Cartesian product $M \times N$ positive.

In our project, we will consider a similar situation as in the theorem, but on a single manifold. In other words, given a manifold M, we can split the tangent bundle into the direct sum of two sub-bundles, and we aim to find out a condition on one of the sub-bundles and the geometric properties of M so that this sub-bundle will have a "dominating" scalar curvature and hence, making the entire manifold admit a metric of positive scalar curvature. The detailed investigation will be in Chapter 7 below.

CHAPTER 6. SCALAR CURVATURE AND CONNECTED SUMS OF MANIFOLDS

In this chapter, we will provide some details to the proof of Gromov and Lawson in their paper [5] about the relationship between the existence of positive scalar curvature metrics on two Riemannian manifolds and the existence of such a metric on their connected sum. However, before going to the actual proof, we need to discuss the theory of Riemannian submanifolds, especially the shape tensor of a Riemannian submanifold.

6.1 Riemannian Submanifolds:

All of the following results can be found in Chapter 4 of [1].

Definition 19 ([1]). *Let M be a submanifold of a Riemannian manifold (N, g), $p \in M$ and $i : M \to M$ be the inclusion map. If $i^*(g) : T_p(M) \times T_p(M) \to \mathbb{R}$ by $i^*(g)(u, v) = g(di(u), di(v))$ is a metric on M, then M is called a* ***Riemannian submanifold*** *of N. A Riemannian hypersurface is a submanifold M of N so that* $\dim(T_p(M))^{\perp} = 1$.

6.1.1 Tangent and Normal Vectors

Let M be a Riemannian submanifold of N. Then, the tangent space $T_p(M)$ is a subspace of $T_p(N)$. Hence,$T_p(N)$ can be decomposed as

$$T_p(N) = T_p(M) \oplus T_p(M)^{\perp}$$

where $T_p(M)^{\perp}$ is the orthogonal complement of $T_p(M)$. As a consequence, a vector $x \in T_p(N)$ can be decomposed as $x = \tan x + \operatorname{nor} x$, where $\tan x \in T_p(M)$ is called **the tangent vector** to M and $\operatorname{nor} x = T_p(M)^{\perp}$ is called **the normal vector** to M.

6.1.2 The Induced Connection and the Shape Tensor

If M is a Riemannian submanifold of N, then the Levi-Civita connection ∇^N on N induces a natural connection on M by restricting the smooth vector fields U and V on N to the vector fields U' and V' smoothly on N, then the covariant derivative $\nabla^M_{U'}V'$ is just the covariant derivative $\nabla^N_U V$ orthogonally projected to $T_p(M)$.

Definition 20 ([1])**.** *Let M be a Riemannian submanifold of N. The function $II : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M, by $II(U,V) = \mathrm{nor}(\nabla^N_U V)$ is called the* ***shape tensor*** *(or the* ***second fundamental form tensor****) of M.*

The shape tensor is very useful to us because it gives rise to an essential ingredient in the proof, the *Gauss equation*.

Theorem 8. *Let M be a Riemannian submanifold of N, and $u, v \in T_p(M)$. Let $K(u,v)$ and $K'(u,v)$ be the sectional curvatures of the plane spanned by u and v of M and N respectively. Then*

$$K(u,v) = K'(u,v) + \frac{\langle II(u,u), II(v,v)\rangle - \langle II(u,v), II(u,v)\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}$$

The eigenvalues of the matrix representation of the shape tensor in any orthonormal basis of the tangent space are called the **principal curvatures** of M, denoted by λ. Each corresponding eigenvector v is a **principal direction** of M, and a curve whose tangent vectors are principal directions is called a **principal curve** of M.

6.2 The Existence of Positive Scalar Curvature Metric on the Connected Sum of Two Riemannian Manifolds:

After laying out the foundation, we will prove the main theorem in this chapter.

Theorem 9 (Gromov-Lawson)**.** *If (X_1, g_1) and (X_2, g_2) are compact n-manifolds, with $n \geq 3$, having positive scalar curvature metrics, then their connected sum also has a positive scalar curvature metric.*

Proof. Throughout this proof, when referring to the principal curvature of a hypersurface, it is understood that we refer to its principal curvature with respect to the unit normal vector which respects the orientation of the parent manifold, since the principal curvature depends on the direction of the normal vector.

Given (X_1, g_1) with scalar curvature $S > 0$ and $p \in X_1$, consider a normal coordinate ball D centered at p of radius r. The ball is defined as $D = \{x_1e_1 + \cdots + x_ne_n = x : \|x\| \leq r\}$ where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_p(X_1)$.

Then, let $D' = exp_p(D)$ be a geodesic ball of radius r_1. By the same argument, on X_2, given $q \in X_2$, there exists a geodesic ball B centered at q of radius r_2. Let $r = \min(r_1, r_2)$. We will proceed with the construction below using r. For the sake of simplicity, assume that $r = r_1$ (if otherwise, for the construction below, we can always reduce D' to another geodesic ball of a smaller radius, which is contained in D', so the exponential map is still a diffeomorphism).

Now, we aim to change the metric on D' so that the new metric agrees with the old one at $\partial D'$ and near p it resembles the product metric of $\mathbb{R} \times S^{n-1}$, where S^{n-1} is a sphere of a certain radius.

Let $r(x) = \|x\|$ be the distance of a point in D' to the origin p, and set $S^{n-1}(\epsilon) = \{x \in D' : \|x\| = \epsilon\}$.

Now, we have the following lemma:

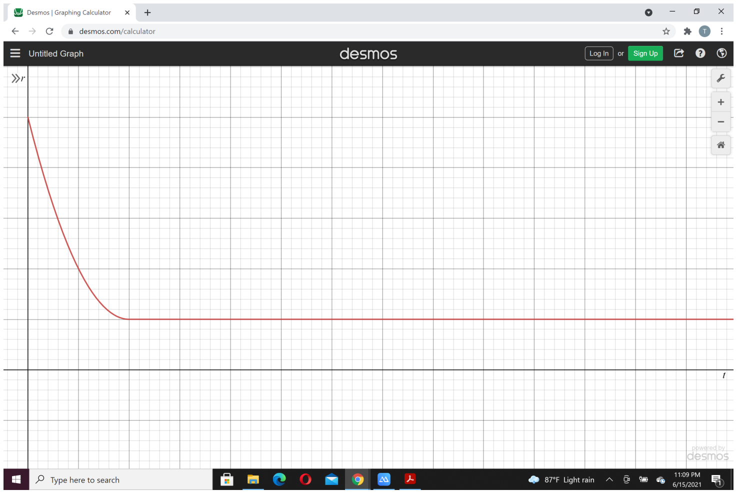
Lemma 4. *All principal curvatures of the hypersurface $S^{n-1}(\epsilon)$ are of the form $\frac{-1}{\epsilon} + O(\epsilon)$ for ϵ small. Furthermore, let g_ϵ be the induced metric on $S^{n-1}(\epsilon)$ and $g_{0,\epsilon}$ be the standard Euclidean metric on the sphere (sectional curvature $\frac{1}{\epsilon^2}$). Then, as ϵ approaches 0, g_ϵ approaches $g_{0,\epsilon}$*

Using this lemma, we can proceed with the proof as follows:

Consider the product $D' \times \mathbb{R}$ and define a hypersurface $M \subset D' \times \mathbb{R}$ by the relation

$$M = \{(x, t) : (\|x\|, t) \in \gamma\}$$

where γ is a curve in the (r, t)-plane that starts on the positive r-axis and ends parallel to the t-axis. This is the picture of γ.



Note that the metric on M is induced from the metric on D starting at its boundary and ends with a metric of $S^{n-1}(\epsilon) \times \mathbb{R}$ near the origin. If ϵ is small, then by the convergence condition in Lemma 4, we can alter the metric near the origin so that it is similar to a metric of positive scalar curvature which is a product of the ϵ-sphere Euclidean metric with $\mathbb{R}$.

Now, we will choose γ in more detail so that the curvature $k(s)$ of γ is always less than or equal to 1.

Let ℓ be a geodesic ray in D' emanating from the origin p.

Then, notice that the velocity of the line ℓ, which is in $T_p(D' \times \mathbb{R})$ is tangent to the hypersurface $\ell \times \mathbb{R}$. Furthermore, the geodesic ℓ with that velocity initially lies totally in $\ell \times \mathbb{R}$. Hence, $\ell \times \mathbb{R}$ is totally geodesic in $D' \times \mathbb{R}$ by Proposition 4.13 in [1].

Furthermore, the normal field of M along the intersection $\alpha = M \cap (\ell \times \mathbb{R})$ lies tangent to $\ell \times \mathbb{R}$.

It follows that α will be a principal curve of M.

Now, notice that γ above can be viewed as the "cross-section" of $M \cap (\ell \times \mathbb{R})$ when identifying the r-axis as the ray ℓ and the t-axis as $\mathbb{R}$. Hence, γ is exactly the curve α above, and so the curvature of α is the curvature of γ.

By Lemma 4 above and the construction of M, the other principal curvatures will be of form $\left(\frac{-1}{r} + O(r)\right) \sin\theta$, where θ is the angle between the normal to M and the t-axis.

Let $q \in \alpha$ be arbitrary. Let $v_1, \cdots, v_n$ be an orthonormal basis for $T_q(M)$ consisting of principal directions so that v_1 is tangent to α (since the second fundamental form is symmetric, this is possible, by the spectral theorem in linear algebra) and let $\lambda_1, \cdots, \lambda_n$ be the associated principal curvatures.

In the Gauss equation above, if we replace the vectors v and w by two vectors in the above orthonormal basis, we immediately get $K_{ij} = K'_{ij} + \lambda_i \lambda_j$ where K_{ij} and K'_{ij} are the sectional curvatures of the plane spanned by v_i and v_j of M and $D' \times \mathbb{R}$.

Note that by the above observation, $\lambda_1 = k$ and $\lambda_p = \left(\frac{-1}{r} + O(r)\right) \sin\theta$ for $p = 2, 3, \cdots, n$.

Now, let $\frac{\partial}{\partial r}$ be the direction of the velocity of the geodesic ℓ above. Observe that since $D' \times \mathbb{R}$ has the product metric, it follows that

$$K'_{1j} = K^{D'}_{\frac{\partial}{\partial r}, j} \cos^2\theta$$
$$K'_{ij} = K^{D'}_{ij}$$

for $i, j = 2, \cdots, n$, where $K^{D'}$ is the sectional curvature of D'.

Why is that the case?

The second equation is clear from the structure of $D' \times \mathbb{R}$ (since in the directions 2,3,...,n, $D' \times \mathbb{R}$ is essentially just D' shifted along the $\mathbb{R}$-axis)

Now, we will prove the first equation:

Recall that by the above, we have that $\lambda_1 = k$ (the curvature of α, which is isometric to γ) and the corresponding principal direction is v_1 along α. Since we want to find a relationship between the sectional curvature of D' and $D' \times \mathbb{R}$, we need to project v_1 onto the manifold D'.

Notice that $\frac{\partial}{\partial r} = \partial_r$, the radial geodesic direction, is orthogonal to $\frac{\partial}{\partial t} = \partial_t$, a vector in the t-axis. The set $\{\partial_r, \partial_t\}$ is an orthonormal set.

Hence, when v_1 is projected back onto the plane spanned by the two vectors above, we get that $v_1 = (-\cos\theta)\partial_r + (\sin\theta)\partial_t$ (Note that v_1 is perpendicular to the outward normal and ∂_r is perpendicular to ∂_t, so the angle between v_1 and ∂_r is exactly θ.).

Now, let R' and $R^{D'}$ be the Riemann tensor of $D' \times \mathbb{R}$ and D', respectively. Also, let j correspond to the vector v_j above (which is tangent to M and also $S^{n-1}(r)$).

By the sectional curvature formula, we get:

$$\begin{aligned} K'_{1j} &= K'(v_1, v_j) \\ &= \langle R'_{v_1,v_j}(v_1), v_j \rangle \\ &= \left\langle R'_{(-\cos\theta)\partial_r + (\sin\theta)\partial_t, v_j}((-\cos\theta)\partial_r + (\sin\theta)\partial_t), v_j \right\rangle \text{ (By above)} \\ &= \cos^2\theta \langle R'_{\partial_r,v_j}(\partial_r), v_j\rangle \text{(By the multilinearity of } R' \text{ and the product metric)} \\ &= \cos^2\theta \langle R^{D'}_{\partial_r,v_j}(\partial_r), v_j\rangle \\ &= \cos^2\theta K^{D'}(\partial_r, v_j) = K^{D'}_{\frac{\partial}{\partial r},j}\cos^2\theta \end{aligned}$$

whereby the restriction of R' to the 2 vectors above coincides with $R^{D'}$ acting on those vectors. Hence, the first equation follows.

Now, using Gauss equation above and the formula, the scalar curvature of M at the point (x,t) will be

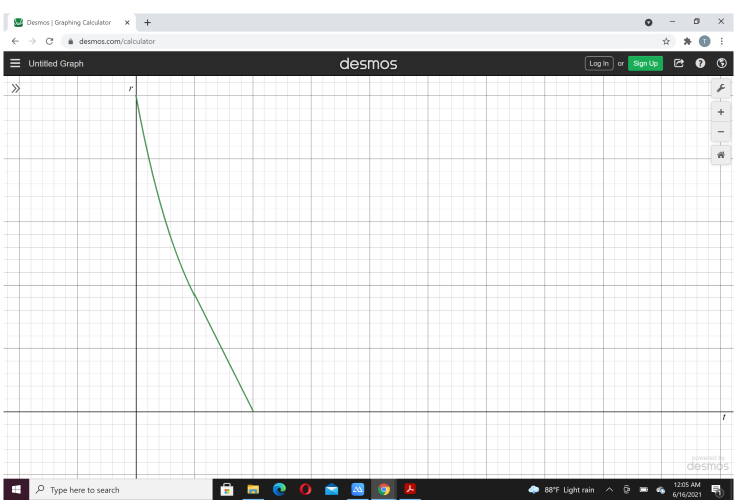
$$\begin{aligned} S &= \sum_{i\neq j} K_{ij} \\ &= \sum_{j=2}^{n} K_{1j} + \sum_{i=2}^{n} K_{i1} + \sum_{i\neq 1, j\neq 1, j\neq i} K_{ij} \\ &= \sum_{j=2}^{n}\left(K^{D'}_{\frac{\partial}{\partial r},j}\cos^2\theta + k\left(-\frac{1}{r} + O(r)\right)\sin\theta\right) + \sum_{i=2}^{n}\left(K^{D'}_{i,\frac{\partial}{\partial r}}\cos^2\theta + k\left(-\frac{1}{r} + O(r)\right)\sin\theta\right) \\ &+ \sum_{i\neq 1, j\neq 1, j\neq i}\left(K^{D'}_{ij} + \left(\frac{1}{r^2} + O(1)\right)\sin^2\theta\right) \\ &= 2\sum_{j=2}^{n} K^{D'}_{\frac{\partial}{\partial r},j}\cos^2\theta + \sum_{i\neq 1, j\neq 1, j\neq i} K^{D'}_{ij} + 2\sum_{j=2}^{n} k\left(-\frac{1}{r} + O(r)\right)\sin\theta \\ &+ \sum_{i\neq 1, j\neq 1, j\neq i}\left(\frac{1}{r^2} + O(1)\right)\sin^2\theta \text{(Rearrange terms)} \end{aligned}$$

Use $\cos^2\theta + \sin^2\theta = 1$ and combining the first two terms, and taking the sum of the last two terms, we will get

$$
\begin{aligned}
S &= 2\sum_{j=2}^{n} K^{D'}_{\frac{\partial}{\partial r},j}\cos^2\theta + \sum_{i\neq 1, j\neq 1, j\neq i} K^{D'}_{ij} - 2(n-1)\left(\frac{1}{r}+O(r)\right)k\sin\theta \\
&+ (n-1)(n-2)\left(\frac{1}{r^2}+O(1)\right)\sin^2\theta \\
&= 2\sum_{j=2}^{n} K^{D'}_{\frac{\partial}{\partial r},j} + \sum_{i\neq 1, j\neq 1, j\neq i} K^{D'}_{ij} - 2\sum_{j=2}^{n} K^{D'}_{\frac{\partial}{\partial r},j}\sin^2\theta - 2(n-1)\left(\frac{1}{r}+O(r)\right)k\sin\theta \\
&+ (n-1)(n-2)\left(\frac{1}{r^2}+O(1)\right)\sin^2\theta \\
&= S^{D'} - 2\,\mathrm{Ric}^{D'}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)\sin^2\theta - 2(n-1)\left(\frac{1}{r}+O(r)\right)k\sin\theta \\
&+ (n-1)(n-2)\left(\frac{1}{r^2}+O(1)\right)\sin^2\theta \\
&= S^{D'} - \left[2\,\mathrm{Ric}^{D'}(\partial_r,\partial_r)\sin\theta - (n-1)(n-2)\left(\frac{1}{r^2}+O(1)\right)\sin\theta\right]\sin\theta \\
&- \left[2(n-1)\left(\frac{1}{r}+O(r)\right)k\right]\sin\theta \qquad (6.1)
\end{aligned}
$$

where $S^{D'}$ and $Ric^{D'}$ are the scalar curvature and the Ricci tensor of D'.

As can be seen from the formula (6.1) above, we will prove that there exists a $\theta = \theta_0 > 0$ so that the resulting "bending" of the curve γ will give an M with positive scalar curvature. See the picture below.



Here, the θ_0 is the angle with which the straight line makes with the r-axis

The reasoning can be seen as follows:

First of all, assume that $\rho \leq 1$, and consider the portion of M corresponding to the portion of γ, where, say $\frac{R}{2} \leq r \leq R$.

Notice that $\mathrm{Ric}^{D'}(v, v)$, where v is a unit vector on the tangent bundle of D', is a bounded function on the closure of D'. Hence, when considering $\mathrm{Ric}^{D'}(\partial_r, \partial_r)$, there is a positive constant C_1 so that $\mathrm{Ric}^{D'}(\partial_r, \partial_r) \leq C_1$.

Notice that by our hypothesis above, we have chosen γ so that $k(s) \leq 1$. Also, by above, the Ricci curvature is bounded above by a constant. Hence, the Ricci term in the bracket is bounded above by a constant as well. Now, since the error term $O(r)$ and $O(1)$ are bounded by constants on $\left[\frac{R}{2}, R\right]$, it follows that the two bracket terms in (1) above, when combined, is bounded above by a constant K (can be positive or negative) on the above interval. Hence, from (1), we can immediately deduce that $S \geq S^{D'} - K \sin\theta$ for some constant K.

Now, since $\sin\theta$ tends to 0 as θ gets smaller and smaller, there is an angle θ_0 so that the $\sin\theta_0$ is sufficiently small so that $K \sin\theta_0 < S^{D'}$. On the interval $\theta \in (0, \theta_0]$, then, we will have that $S \geq S^{D'} - K \sin\theta > 0$.

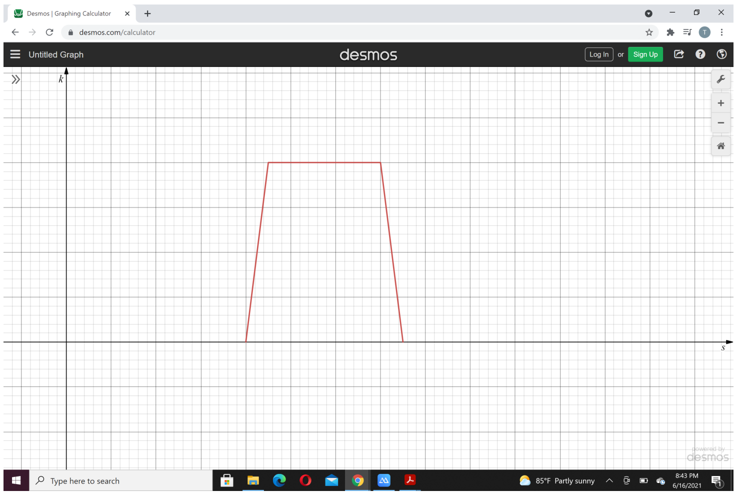
It only remains to construct k and show what the curvature will stay positive when we decrease r below our threshold $\frac{R}{2}$.

Now, let γ_0 be the straight line indicated in the figure. Hence, $k = 0$ on γ_0, so as r becomes small, the scalar curvature of M will, by (1), turns to

$$S = S^{D'} + \frac{(n-1)(n-2)\sin^2\theta_0}{r^2} + O(1)$$

since the Ricci term is bounded

Now, proceed by choosing a point $(r_0, t_0) \in \gamma_0$ so that r_0 is small. Now, bend γ_0 starting at this point with a curvature function that has the form of a “smooth” trapezoid with the “base” of $\frac{1}{2}r_0$ and “height” of $\frac{\sin\theta_0}{2r_0}$. See the picture below.



Note that since $n \geq 3$, by the formula above, S will continue to be positive corresponding to this line segment. Also, the total amount of bend will be $\Delta\theta_0 = \int k ds \approx \frac{\sin\theta_0}{4}$.

Notice that we will prove that the scalar curvature of M will be positive from the point r_0 to the start of the second bend. Then we can apply the argument for subsequent bends (the curvature function of γ on which depends on the "starting angle" of such a bend) until we get to the point where the angle $\theta = \frac{\pi}{2}$. Also, the curve resulting from the bend cannot get down below the line $r = \frac{r_0}{2}$, since the length of the bend is $\frac{r_0}{2}$, and it starts at r_0.

Note that on the first bend, we have that $k(s) \leq \frac{\sin\theta_0}{2r_0}$. Hence, notice that when $r \leq r_0$, when we group every term in (1) (except the terms involving $\frac{1}{r}$ and $\frac{1}{r^2}$) together, we have:

$$\begin{aligned} S &= S^{D'} - \left[2\,\mathrm{Ric}^{D'}(\partial_r, \partial_r)\sin\theta + 2(n-1)O(r)k - (n-1)(n-2)O(1)\sin\theta\right]\sin\theta \\ &- \frac{2(n-1)k\sin\theta}{r} + \frac{(n-1)(n-2)\sin^2\theta}{r^2} \end{aligned}$$

When $r < r_0$ is small, we can see that the combination of the first two terms above is bounded below by a positive constant (since $O(r)$ and $O(1)$ are bounded by constants, and by changing the angle θ_0 to be smaller if necessary).

Furthermore, since $k(s) \leq \frac{\sin\theta_0}{2r_0}$, we have that

$$\begin{aligned} &-\tfrac{2(n-1)k\sin\theta}{r} + \tfrac{(n-1)(n-2)}{r^2}\sin^2\theta \\ &\geq -\tfrac{2(n-1)\sin\theta\sin\theta_0}{2r_0 r} + \tfrac{(n-1)(n-2)}{r^2}\sin^2\theta \\ &\geq -\tfrac{(n-1)\sin^2\theta_0}{r^2} + \tfrac{(n-1)(n-2)}{r^2}\sin^2\theta_0 \\ &= \tfrac{(n-1)\sin^2\theta_0}{r^2}(n-3) \geq 0 \end{aligned}$$

Hence, overall, during the first bend up to the start of the second bend, we have that $S \geq M > 0$, where M is a positive constant. In other words, during the above process of bending, S will stay positive.

Notice that when finishing the first bend, we will finish off with another straight line γ_1 going from the endpoint of the bend to the t-axis (corresponding to the portion of the $k(s)$-graph to the right of the "trapezoid", which has the value of $k(s) = 0$) and by that time, the angle change will be $\theta_1 \approx \theta_0 + \Delta\theta \approx \theta_0 + \frac{\sin\theta_0}{4}$.

Now, since γ_1 is a straight line, by the same argument as above, the scalar curvature of M will be

$$S = S^{D'} + \frac{(n-1)(n-2)\sin^2\theta_1}{r^2} + O(1)$$

Now, proceed, as above, by choosing a point $(r_1, t_1) \in \gamma_1$ so that r_1 is small and bend γ_1 with a similar curvature function, only now, we adjust the "base" of the trapezoid to be $\frac{1}{2}r_1$ and the "height" to be $\frac{\sin\theta_1}{2r_1}$(notice that we can adjust the curvature function $k(s)$ by adjusting the "height" to be smaller if necessary so that the angle change $\Delta\theta$ can range from 0 to $\frac{\sin\theta_0}{4}$. This must be necessary in order for the last bend in the process to be able to reach a final angle of $\frac{\pi}{2}$, since within the last bend, we may adjust the angle change to reach the exact desired angle).

Hence, the angle change now would be $\Delta\theta_1 \approx \frac{\sin\theta_1}{4}$. Also, the curve resulting from the bend cannot get down below the line $r = \frac{r_1}{2}$, since the length of the bend is $\frac{r_1}{2}$, and the bend starts at r_1.

Notice that by the same inequality argument as above (but now, replacing r_0 by r_1 and θ_0 by θ_1 in every instances, noticing that now $r \leq r_1$), we see that $S > 0$ during this bend, also

Moreover, after the second bend, we will have another straight line γ_2 and another angle $\theta_2 = \theta_1 + \Delta\theta_1$. Repeat the process exactly as above, changing the curvature function according to the starting angle of each bend, and choosing an r_2 suitably (since the curve do not get down to $\frac{r_1}{2}$), we will have that $S > 0$ on the second bend

Repeat the process above for subsequent bend, remembering to change the curvature accordingly until we get to an angle of $\theta = \frac{\pi}{2}$ by building up the $\Delta\theta$ accordingly, we will get that $S > 0$ when r is below our threshold of $\frac{R}{2}$

Hence, M, which helps D' to achieve a metric very close to the form $S^{n-1}(\epsilon) \times \mathbb{R}$ near the origin (by Lemma 4), has a positive scalar curvature metric.

During the course of doing this operation on X_1, do a similar deformation on X_2 (namely, with a geodesic ball $B' \subset B$ of radius ρ_1 and a hypersurface $N \subset B' \times \mathbb{R}$).

By the exact same process as above, we know that N, which helps B' to achieve a metric very close to the form $S^{n-1}(\epsilon) \times \mathbb{R}$ near the origin (by Lemma 34), has a positive scalar curvature metric. Hence, connecting the two manifolds via the extension of M and N along the portion corresponding to the straight line of γ in the very first picture (and correspondingly, along the portion corresponding to the straight line of a curve γ' when constructing N in X_2), using a homotopy of metrics from the metric on X_1 to that on X_2 we will form the connected sum of the manifolds, which admits a positive scalar curvature metric.

In the part of the construction above, we used this result, whose proof is in Section 6.4, concerning the linear homotopy of two metrics.

Lemma 5. *If g is a metric on $M \subset D' \times \mathbb{R}$ and h is a metric on $N \subset B' \times \mathbb{R}$, each inherits the metrics of D' and B' at their boundary and ends with metrics which converge to the Euclidean metric of $S^{n-1}(\epsilon) \times \mathbb{R}$ near the origin, then the linear homotopy $g'(t) = (1-t)g + th$ for $0 < t < 1$ will help the connected sum of X_1 and X_2 admit a positive scalar curvature metric overall.*

(We should note that to properly join the two hypersurfaces M and N together, we will need to compare the two constructions above on X_1 and X_2, and take the minimum of the quantities employed when constructing the hypersurfaces (i.e., the minimum s of r_0 above and its counterparts when constructing N, etc.) and form the connection according to that minimum)

Hence, the proof is done. □

6.3 The Proof of Lemma 4:

There are two claims in the proof that we postponed until this section and the next one. In this section, we will prove Lemma 4 that we saw being used extensively throughout the proof. Also, in this section, the metric g used here is totally irrelevant to the metric g used in Lemma 5 above.

Proof of Lemma 4. Notice that by the formula quoted in Chapter 3, the metric on the sphere D' near a normal coordinate neighborhood has the form:

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\delta_{iu}x^p x^q R^u_{\ pqj}(0) + O(\|x\|^3) = \delta_{ij} + O(\|x\|^2) = \delta_{ij} + O(\epsilon^2)$$

Notice that since $S = S^{n-1}(\epsilon)$ is a submanifold of D' (and of X), the tangent space $T_p(S)$ is an (n-1)-dimensional subspace of $T_p(X)$. Hence, by the theory of basis from linear algebra, $T = T_p(S)$ will have an orthonormal basis consisting of (n-1) vectors in the basis for the tangent space of X.

Without loss of generality, we can rename the vectors so that the orthonormal basis for T is spanned by $e_2, \cdots, e_n$. Hence, since e_1 is orthogonal to these vectors, we must have that e_1 is a normal vector to S. Furthermore, assume that $e_1 = (1, 0, \cdots, 0)$ in $T_p(X)$ with standard coordinates. (we can do this by arranging the axes accordingly)

Now, we claim that the curve $\gamma(s) = (\epsilon\cos\left(\frac{s}{\epsilon}\right), \epsilon\sin\left(\frac{s}{\epsilon}\right), 0, \cdots, 0)$ is on S.

Notice that using the metric on D as above, we can see that

$$\|\gamma(x)\|^2 = g_{11}\epsilon^2\cos^2\left(\frac{x}{\epsilon}\right) + g_{22}\epsilon^2\sin^2\left(\frac{x}{\epsilon}\right) = \epsilon^2 + O(\epsilon^4)$$

because the cross terms will have the Kronecker delta equals 0, and have terms involving ϵ^2 and when multiplied with the two components, will result in terms bounded by ϵ^4. Also, we are using the delta portion of the metric to carry out the computations, because the error terms have exactly an analogous reasoning as above.

Then, when ϵ is small as in question, the error term is negligible, so we eventually have that $\|\gamma(x)\| = \epsilon$ for all points on the curve. Hence, the curve is on S, as claimed.

Now, the covariant derivative of the velocity vector $v = \frac{d\gamma}{ds}(0)$ along itself, when expanded, will have components according to the formula

$$(\nabla_v v)^k = \frac{d^2\gamma^k}{ds^2}(0) + \Gamma^k_{ij}\frac{d\gamma^i}{ds}(0)\frac{d\gamma^j}{ds}(0).$$

where we used the summation convention for the indices i and j. Now, note that taking two derivatives of γ, we have:

$$\gamma'(s) = \left(-\sin\left(\frac{s}{\epsilon}\right), \cos\left(\frac{s}{\epsilon}\right), 0, \cdots, 0\right),$$
$$\gamma''(s) = \left(-\frac{1}{\epsilon}\cos\left(\frac{s}{\epsilon}\right), -\frac{1}{\epsilon}\sin\left(\frac{s}{\epsilon}\right), 0, \cdots, 0\right).$$

Evaluating at 0, we immediately have that $\gamma'(0) = e_2$ and $\gamma''(0) = -\frac{1}{\epsilon}e_1$.

Since by the above setup, e_1 is normal to S, we have that the second fundamental form of S acting on the vector v above will be given by the tensor metrically equivalent to the shape operator as defined in O'Neill([1]), or equal to $\langle II(v,v), e_1\rangle$ (where II is the shape tensor of S). Calculating this using the metric on D, and notice that only the first component of $\mathrm{II}(v,v)$ is important (since the inner product is with e_1, whose components are all 0 except for the first), we have

$$\begin{aligned}\langle II(v,v),e_1\rangle &= \langle \text{nor}\,\nabla_v v, e_1\rangle \\ &= \langle \nabla_v v, e_1\rangle \\ &= g_{11}(x)\left(\frac{d^2\gamma^1}{ds^2}(0)+\Gamma^1_{ij}\frac{d\gamma^i}{ds}(0)\frac{d\gamma^j}{ds}(0)\right) \\ &= g_{11}(x)\left(-\frac{1}{\epsilon}+\Gamma^1_{22}\right) \\ &= (1+O(\epsilon^2))\left(-\frac{1}{\epsilon}+\Gamma^1_{22}\right) \\ &= -\frac{1}{\epsilon}+O(\epsilon)\end{aligned}$$

where we have used the decomposition of $\nabla_v v$ to its tangential and normal components to the hypersurface $S^{n-1}(\epsilon)$ and the formulas for $\gamma\,,\gamma'$, the metric tensor and the Christoffel symbols (where in the formula for the Christoffel symbol Γ^1_{22}, which involves the derivative of the metric, by an estimation of the metric as bounded by terms involving ϵ^2, we can see that the expansion for Γ^1_{22} has terms bounded by ϵ).

Note that by the above, we have that the second fundamental form acting on the unit vector $v=\frac{d\gamma}{ds}$ has form $-\frac{1}{\epsilon}+O(\epsilon)$. By a change of coordinates, we get that this identity is true for all tangent vectors to S. Hence, the principal curvature of S is $-\frac{1}{\epsilon}+O(\epsilon)$, as claimed.

For the second claim, note that g_ϵ can be induced via inducing g to $S^{n-1}(1)$, then push it forward via the map $f: S^{n-1}(1)\to S$ by $f(x)=\epsilon x$. When applying the pullback of g_ϵ via f, we have, at the points x where $\|x\|=1$:

$$\begin{aligned}f^*(g_\epsilon)_x &= \sum g_{ij}(\epsilon x)(\text{Pullback definition}) \\ &= \sum_{i,j}\left(\delta_{ij}+\sum_{p,q}\epsilon^2\frac{1}{3}\delta_{iu}x^p x^q R^u_{ipq}(0)+O(\epsilon^3)\right)(\text{Formula for the metric})\end{aligned}$$

It follows that as ϵ approaches 0, $f^*(g_\epsilon)_x$ also approaches the usual Euclidean metric (since the terms involving ϵ vanishes).

Looking at the terms after the first equal sign above, we can conclude the same thing for g_ϵ. Hence, the second claim follows. $\square$

6.4 The Proof of Lemma 5:

The next claim that we will prove is Lemma 5 above, which helps to connect the two manifolds together via a connected sum and using the linear homotopy of metric.

Proof of Lemma 5. Note that g and h are both represented by $n \times n$ symmetric matrices. Hence, now, consider the manifold U formed by the connected sum of M and N via there straight line portion and the space Ω of all $n \times n$ symmetric matrices with entries being in $\mathfrak{F}(U)$, the space of smooth functions on U. We can endow Ω with the norm

$$\|A\| = \max_{x \in U} \max_{i,j,k,l} \{|a_{ij}|, |\partial_k a_{ij}, |\partial_k \partial_l a_{ij}|, |a^{ij}|, |\partial_k a^{ij}|\}.$$

where the upper indices denote the entries of the inverse matrix.

Notice that by Lemma 4, we know that as the radius ϵ (which corresponds to the straight line portion of γ on M and γ' on N) goes to 0, the metric g and h in the hypothesis both go to the usual Euclidean metric e of the sphere of sectional curvature $\frac{1}{\epsilon^2}$.

Hence, since the metrics are matrices, we must have the convergence follows entry-wise. In other words, $\lim_{\epsilon \to 0} g_{ij} = \lim_{\epsilon \to 0} h_{ij} = e_{ij}$ for all i, j. Hence, by the sum law of limits, we get that $\lim_{\epsilon \to 0} (h_{ij} - gij) = 0$.

Using the definition of limits, the above expression means that given a $u > 0$, there exists a number $v > 0$ so that whenever we have $|\epsilon| < v$, then $|h_{ij} - g_{ij}| < u_1$.

Now, using the Taylor expansion of the metric in Section 6.3, we have that

$$g_{ij}(x) = \delta_{ij} + \sum_{p,q} \delta_{iu} \epsilon^2 \frac{1}{3} x^p x^q R^u{}_{pqj}(0) + O(\epsilon^3).$$

Taking two derivatives of the metric above, we will get

$$\begin{aligned} \partial_k g_{ij} &= \partial_k \delta_{ij} + O(\epsilon^2) = 0 + O(\epsilon^2), \\ \partial_k \partial_l g_{ij} &= 0 + O(\epsilon^2). \end{aligned}$$

Hence, again, when ϵ goes to 0, both partial derivatives of g_{ij} go to 0. A similar argument shows the same thing for the partial derivatives of h_{ij}. Therefore, by the same argument as above (adjusting the bound v for $|\epsilon|$ if necessary by going back to our construction and make the s, the minimum of r_0 and its counterpart on N as noted in the last part of the proof in Section 6.2, smaller if necessary so that $s < v$), we have that given $u_1, u_2, u_3 > 0$, there is a $v > 0$ so that when $|\epsilon| < v$, $|h_{ij} - g_{ij}| < u_1$, $|\partial_k h_{ij} - \partial_k g_{ij}| < u_2$, $|\partial_k \partial_l h_{ij} - \partial_k \partial_l g_{ij}| < u_3$

The argument above can be extended to all indices $i, j = 1, 2, \cdots, n$. Now, we will consider $g'(t) = (1-t)g + th$ for $0 \leq t \leq 1$. Note that when $|\epsilon| < v$, we have the following computations:

$$\begin{aligned} |g'_{ij}(t) - g_{ij}| &= |(1-t)g_{ij} + th_{ij} - g_{ij}| = |t(h_{ij} - g_{ij})| < u_1 \\ |\partial_k g'_{ij}(t) - \partial_k g_{ij}| &= |(1-t)\partial_k g_{ij} + t\partial_k h_{ij} - \partial_k g_{ij}| = |t(\partial_k h_{ij} - \partial_k g_{ij})| < u_2 \\ |\partial_k \partial_l g'_{ij}(t) - \partial_k \partial_l g_{ij}| &= |(1-t)\partial_k \partial_l g_{ij} + t\partial_k \partial_l h_{ij} - \partial_k \partial_l g_{ij}| = |t(\partial_k \partial_l h_{ij} - \partial_k \partial_l g_{ij})| < u_3 \end{aligned}$$

by the above assertion and the fact that $t \in (0, 1)$.

Also, g'^{ij} will be a rational function of the entries of g (using the formula for matrix inverses), so the same reasoning as above (again, adjusting s if necessary) shows that given $u_4, u_5 > 0$, there is a v so that when $|\epsilon| < v$, $|g'^{ij}(t) - g^{ij}| < u_4$ and $|\partial_k g'^{ij}(t) - \partial_k g^{ij}| < u_5$.

Hence, by all of the argument above, we have proven that when ϵ is sufficiently small, then $\|g'(t) - g\|$ is small where $\| \cdot \|$ is defined at the beginning of the proof.

Now, notice that the scalar curvature S is a continuous function in terms of the metric, its two derivatives, the inverse metric and its derivative. Moreover, the domain U in question is compact. Hence, S is actually uniformly continuous on U.

By contradiction, assume that $S_{g'(t)} < 0$ for some $t_0 \in (0,1)$. Notice that since S_g is assumed to be positive by the PSC construction in Section 1, this implies that the quantity $|S_{g'(t)} - S_g| > K > 0$. However, by above, we have that $\|g'(t) - g\|$ is small when ϵ is small. Hence, given a $C > 0$, we can always have the condition that $\|g'(t) - g\| < C$ implies $|S_{g'(t)} - S_g| > K$. Using the negation of the definition of uniform continuity, we can see that S is not uniformly continuous on U, a contradiction.

Hence, our assumption is wrong, so $S_{g'(t)} > 0$. Therefore, on U, there exists a PSC metric.

Now, since the connected sum of X_1 and X_2 is a compact set, let $V = \bigcup_{\alpha=1}^{\infty} V_\alpha$ be an open cover and $V' = \bigcup_{\alpha=1}^{m} V_\alpha$ be its finite subcover. Let $\{\tau_\alpha\}_{\alpha=1}^{m}$ be a partition of unity subordinate to this subcover.

Note that in each set in the subcover, there always exists a metric of PSC (g if the subset considered is in X_1, h if it is in X_2, $g'(t)$ if it is in U, and if the subset happens to contain both portion, we use another set of bump function to "connect" the metrics smoothly).

Hence, let $\overline{g} = \sum_{i=1}^{m} g_i \tau_i$, where g_i is a metric of PSC on the subset $V_i \in V'$ and $\tau_i \in \{\tau_\alpha\}_{\alpha=1}^{m}$. Then, $\overline{g}$ is a metric on the connected sum which helps it to have positive scalar curvature, and the proof is done. $\square$

CHAPTER 7. USING ORTHONORMAL FRAMES IN RIEMANNIAN GEOMETRY

In this chapter, we will lay out some prerequisite knowledge and formulas for our research results, which we will discuss in Chapter 8. The approach which we will use in our project lies mainly with the other viewpoint of Riemannian geometry, a viewpoint first developed by the French mathematician Elie Cartan. But first, we need some definitions and results.

7.1 Some Useful Constructions and Quantities:

7.1.1 The Tangent Bundle and Local Orthonormal Frame

Definition 21 ([2])**.** *Let M be a n-dimensional smooth manifold. The set $TM = \bigsqcup_{p \in M} T_p(M)$ is called* ***the tangent bundle*** *of M.*

We can impose the structure of a smooth $2n$-dimensional manifold on TM. See[2] for explicit details.

Definition 22 ([2])**.** *Given the tangent bundle TM of a manifold, a* ***subbundle*** *X of rank r of the tangent bundle is an embedded submanifold of TM, so that for any $p \in M$, $X|_p$ is a subspace of $T_p(M) = TM|\,p$ of dimension r.*

Given a Riemannian manifold (M, g), one can ask whether there is a local basis for the tangent space TM at p in which it is easy to calculate. The answer is given in the following definition.

Definition 23. *Let (M, g) be a Riemannian manifold. A set $\beta = \{e_1, e_2, \cdots, e_n\}$ of vector fields defined on a neighborhood U of $p \in M$ is a* ***local orthonormal frame*** *if $\{e_1(x), e_2(x), \cdots, e_n(x)\}$ is an orthonormal basis of $T_x(M)$ for all $x \in U$.*

It follows from the local construction of a coordinate vector field and the Gram-Schmidt process that for all $p \in M$, we can construct a local orthormal frame field on a neighborhood of p.

Recall from linear algebra that given a basis for a vector space, we may be interested in its dual basis for the dual space. It turns out that in our approach, the dual basis is particularly important, so we may have the following definition.

Definition 24. *Given a local orthonormal frame $\beta = \{e_1, e_2, \cdots, e_n\}$ of $T_p(M)$, the dual basis $\{e^1, e^2, \cdots, e^n\}$ for the dual space $(T_pM)^*$ (sometimes called **the cotangent space**) is called **the dual coframe** .*

Notice that from the relationship established in linear algebra, we have $e^i(e_j) = \delta^i_j$ for all $1 \leq i, j \leq n$. Also, it turns out that each e^i for $1 \leq i \leq n$ is a differential 1-form, so the algebra and calculus of differential forms applies in our case. For a more in-depth treatment of differential forms, see [6].

7.1.2 Useful Geometric Quantities

From this point on, unless stated otherwise, let $\beta = \{e_1, e_2, \cdots, e_n\}$ be a local orthonormal frame field around p of a Riemannian manifold M and all calculations will be with respect to this basis. Also, we assume that all of our calculations occur in a neighborhood U of p.

Given a Riemannian manifold (M, g), one nice property of an orthonormal frame $\beta = \{e_1, e_2, \cdots, e_n\}$ is that $g(e_i, e_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. However, because of not being the coordinate frame fields, in general, the Lie bracket $[e_i, e_j] \neq 0$. However, we can use the Lie bracket to define a new set of functions useful for our calculations.

Definition 25. *The **structure functions** of β are the functions c^k_{ij} so that*

$$[e_i, e_j] = c^k_{ij} e_k$$

Notice that in the case of M being a Lie group and the vector fields e_i being left-invariant for all $1 \leq i \leq n$, the structure functions are constant, so we recover the definition of the structure constants discussed in Chapter 4. Also, the antisymmetry of the Lie bracket implies that $c^k_{ij} = -c^k_{ji}$.

Definition 26. *The **Christoffel symbols** with respect to the orthonormal frame β are the functions*

$\mathbf{\Gamma}_{ij}^k$ *so that*

$$\nabla_{e_i} e_j = \mathbf{\Gamma}_{ij}^k e_k.$$

Notice that the Christoffel symbol with respect to the orthonormal frame cannot be computed by the formula in Chapter 2, since these are not coordinate vector fields. Hence, we use the boldfaced symbol to distinguish the two cases: the orthonormal frame field and the coordinate frame field. However, the following proposition tells us the formula for the symbols in this case.

Proposition 4 ([2])**.** *Let (M, g) be a Riemannian manifold with local orthonormal frame β. Given the structure functions as above, then*

$$\mathbf{\Gamma}_{ij}^k = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j).$$

With the Christoffel symbols calculated, we can define the Riemann tensor as in Chapter 2. However, in this case, since β is an orthonormal frame, we have to account for the nonvanishing Lie bracket in our definition of its component.

Definition 27. *The component of the Riemann tensor in the local orthonormal frame is*

$$R^i{}_{jlk} e_i = \nabla_{e_k} \nabla_{e_l} e_j - \nabla_{e_l} \nabla_{e_k} e_j - \nabla_{[e_k, e_l]} e_j.$$

However, we can study the curvature quantities of a Riemannian manifold M under the local orthonormal frame in an easier way by introducing two new objects.

Definition 28 ([2])**.** *The* ***connection forms*** *of a Riemannian manifold M under β are differential 1-forms $\omega^j{}_i$ so that for all $X \in \mathfrak{X}(M)$*

$$\nabla_X (e_i) = \omega^j{}_i (X) e_j.$$

It turns out that the collection of all the connection forms form an $n \times n$ matrix ω of 1-forms, with components calculated by

Theorem 10. *Let (M, g) be a Riemannian manifold with local orthonormal frame β. Then*

$$\omega^j{}_i = \mathbf{\Gamma}^j_{pi} e^p.$$

This is immediate, since if we plug the formula above to the definition of the connection form above, we get the equality holds true.

Definition 29. *The **curvature form** of a Riemannian manifold M with respect to β is the matrix of differential 2-forms Ω given by*

$$\Omega = d\omega + \omega \wedge \omega,$$

where the exterior derivative is taken entry-wise and the wedge product is taken just like usual matrix multiplication.

It turns out that there is a nice relationship between the entry of the curvature form matrix and the Riemann tensor components calculated above. The relationship is

Theorem 11. *Let (M, g) be a Riemannian manifold with local orthonormal frame β and the curvature form matrix Ω. Then*

$$\Omega^k{}_j = \frac{1}{2} R^k{}_{jiq} e^i \wedge e^q$$

With these quantities introduced, we can easily calculate the scalar curvature of M (defined as in Chapter 2, except now, $g_{ij} = \delta_{ij}$ due to β being orthonormal).

Observe that

$$\begin{aligned}
\sum_{k,j} \Omega^k{}_j(e_k, e_j) &= \sum_{k,j} \frac{1}{2} R^k{}_{jiq} e^i \wedge e^q(e_k, e_j) \\
&= \sum_{k,j} \left(\frac{1}{2} R^k{}_{jkj} - \frac{1}{2} R^k{}_{jjk} \right) \\
&= \sum_{j,k} R^k{}_{jkj} \\
&= S
\end{aligned}$$

where we used Theorem 11, the identity $dx \wedge dy(u,v) = dx(u)dy(v) - dx(v)dy(u)$, the relationship between the orthonormal frame vectors and their dual co-vectors, the symmetry of the Riemann tensor, and the definition of the scalar curvature S respectively. Hence, all in all, $S = \sum_{i,j} \Omega^k{}_j(e_k, e_j)$

7.2 The Scalar Curvature of a Manifold in terms of the Structure Functions:

With the above formulas established, we can continue to derive one of the most important formulas in our project, namely, the scalar curvature of a Riemannian manifold totally in terms of the structure functions defined above.

7.2.1 The Curvature Form in terms of the Structure Functions

Let (M, g) be a Riemannian manifold with local orthonormal frame β and corresponding structure functions c_{ij}^k.

First, notice that the Christoffel symbols of M with respect to the frame β are, by Proposition 4,

$$\mathbf{\Gamma}_{ij}^k = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j)$$

Hence, the connection form of M will be a matrix ω with entries

$$\omega^k{}_j = \mathbf{\Gamma}_{ij}^k e^i = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j)e^i$$

by Theorem 10.

Now, the curvature form of M is defined as a matrix of 2-forms Ω so that $\Omega = d\omega + \omega \wedge \omega$.

Hence, we have the following computations:

$$
\begin{aligned}
\Omega^k{}_j &= d\omega^k{}_j + (\omega \wedge \omega)^k{}_j \\
&= d\omega^k{}_j + \omega^k{}_l \wedge \omega^l{}_j \\
&= \frac{1}{2}d[(c^k_{ij} - c^i_{jk} + c^j_{ki})e^i] + \frac{1}{4}[(c^k_{ml} - c^m_{lk} + c^l_{km})e^m] \wedge [(c^l_{pj} - c^p_{jl} + c^j_{lp})e^p] \\
&= \frac{1}{2}d(c^k_{ij} - c^i_{jk} + c^j_{ki}) \wedge e^i + \frac{1}{2}(c^k_{ij} - c^i_{jk} + c^j_{ki})de^i \\
&+ \frac{1}{4}(c^k_{ml} - c^m_{lk} + c^l_{km})(c^l_{pj} - c^p_{jl} + c^j_{lp})e^m \wedge e^p \\
&= \frac{1}{2}e_q(c^k_{ij} - c^i_{jk} + c^j_{ki})e^q \wedge e^i + \frac{1}{2}(c^k_{ij} - c^i_{jk} + c^j_{ki})de^i \\
&+ \frac{1}{4}(c^k_{ml} - c^m_{lk} + c^l_{km})(c^l_{pj} - c^p_{jl} + c^j_{lp})e^m \wedge e^p,
\end{aligned}
$$

where we use Leibnitz's Rule and the linearity of the exterior derivative.

7.2.2 The scalar curvature calculations

In this section, we will derive the formula for the scalar curvature of a Riemannian manifold M entirely in terms of the structure functions of its local orthonormal frame β. First of all, notice that by the result at the end of Section 7.1, we have the scalar curvature S of M in terms of the curvature form of β is given by $S = \sum_{k,j} \Omega^k{}_j(e_k, e_j)$.

Before performing our calculations, we will simplify the notation in this subsection by suppressing the summation symbols, with an implicit understanding that the indices i and j are summed over from 1 to n. Hence, continuing our calculations with the curvature form found in Section 7.2.1, we have

$$
\begin{aligned}
S &= \Omega^k{}_j(e_k, e_j) \\
&= \frac{1}{2}e_q(c^k_{ij} - c^i_{jk} + c^j_{ki})e^q \wedge e^i(e_k, e_j) + \frac{1}{2}(c^k_{ij} - c^i_{jk} + c^j_{ki})de^i(e_k, e_j) \\
&+ \frac{1}{4}(c^k_{ml} - c^m_{lk} + c^l_{km})(c^l_{pj} - c^p_{jl} + c^j_{lp})e^m \wedge e^p(e_k, e_j) \\
&= \frac{1}{2}e_q(c^k_{ij} - c^i_{jk} + c^j_{ki})(\delta_{qk}\delta_{ij} - \delta_{qj}\delta_{ik}) + \frac{1}{2}(c^k_{ij} - c^i_{jk} + c^j_{ki})(-c^i_{kj}) \\
&+ \frac{1}{4}(c^k_{ml} - c^m_{lk} + c^l_{km})(c^l_{pj} - c^p_{jl} + c^j_{lp})(\delta_{mk}\delta_{pj} - \delta_{mj}\delta_{pk}),
\end{aligned}
$$

where we used the two following facts. First, $dx \wedge dy(u, v) = dx(u)dy(v) - dx(v)dy(u)$. Secondly,

$$
\begin{aligned}
de^i(e_k, e_j) &= e_k(e^i(e_j)) - e_j(e^i(e_k)) - e^i([e_k, e_j]) \\
&= -e^i([e_k, e_j]) \\
&= -e^i(c^m_{kj}e_m) \\
&= -c^m_{kj}e^i(e_m) = -c^i_{kj},
\end{aligned}
$$

whereby we used the formula of de^i, the relationship between the orthonormal frame vectors and their co-vectors, the definition of the structure functions, and the linearity of differential 1-forms respectively.

Continue simplifying by using the Kronecker delta symbol, we get

$$
\begin{aligned}
S &= \frac{1}{2}e_k(c^k_{jj} - c^j_{jk} + c^j_{kj}) - \frac{1}{2}e_j(c^k_{kj} - c^k_{jk} + c^j_{kk}) - c^i_{kj}(c^k_{ij} - c^i_{jk} + c^j_{ki}) \\
&+ \frac{1}{4}(c^k_{kl} - c^k_{lk} + c^l_{kk})(c^l_{jj} - c^j_{jl} + c^j_{lj}) - \frac{1}{4}(c^k_{jl} - c^j_{lk} + c^l_{kj})(c^l_{kj} - c^k_{jl} + c^j_{lk}).
\end{aligned}
$$

Notice that the structure functions are anti-symmetric in their lower 2 indices. In other words, $c_{ij}^k = -c_{ji}^k$ and hence, $c_{ii}^k = 0$. Simplifying the above formula, we get

$$\begin{aligned}
S &= \frac{1}{2}e_k(c_{kj}^j + c_{kj}^j) - \frac{1}{2}e_j(c_{kj}^k + c_{kj}^k) - \frac{1}{2}c_{kj}^i(c_{ij}^k - c_{jk}^i + c_{ki}^j) \\
&+ \frac{1}{4}(c_{kl}^k + c_{kl}^k)(-c_{jl}^j - c_{jl}^j) - \frac{1}{4}(c_{jl}^k - c_{lk}^j + c_{kj}^l)(c_{kj}^l - c_{jl}^k + c_{lk}^j) \\
&= e_k(c_{kj}^j) - e_j(c_{kj}^k) - \frac{1}{2}c_{kj}^i(c_{ij}^k - c_{jk}^i + c_{ki}^j) \\
&- c_{kl}^k c_{jl}^j - \frac{1}{4}(c_{jl}^k c_{kj}^l - (c_{jl}^k)^2 + c_{jl}^k c_{lk}^j - c_{lk}^j c_{kj}^l + c_{lk}^j c_{jl}^k - (c_{lk}^j)^2) \\
&- \frac{1}{4}((c_{kj}^l)^2 - c_{kj}^l c_{jl}^k + c_{kj}^l c_{lk}^j) \\
&= e_k(c_{kj}^j) - e_j(c_{kj}^k) - \frac{1}{2}c_{kj}^i(c_{ij}^k - c_{jk}^i + c_{ki}^j) \\
&- c_{kl}^k c_{jl}^j + \frac{1}{4}((c_{jl}^k)^2 + (c_{lk}^j)^2 - (c_{kj}^l)^2) + c_{lj}^k c_{lk}^j \\
&= e_k(c_{kj}^j) - e_j(c_{kj}^k) + \frac{1}{4}((c_{jl}^k)^2 + (c_{lk}^j)^2 - (c_{kj}^l)^2) \\
&+ \frac{1}{2}c_{lj}^k c_{lk}^j - c_{kl}^k c_{jl}^j - \frac{1}{2}c_{kj}^i c_{ij}^k - \frac{1}{2}(c_{kj}^i)^2 - \frac{1}{2}c_{jk}^i c_{ik}^j,
\end{aligned}$$

where we sum over all indices k, l to get the scalar curvature.

There are more simplifications to make. First of all, rearranging the terms of the above, we have

$$\begin{aligned}
S &= e_k(c_{kj}^j) - e_j(c_{kj}^k) + \frac{1}{4}(c_{jl}^k)^2 + \frac{1}{4}(c_{lk}^j)^2 - \frac{1}{4}(c_{kj}^l)^2 \\
&+ \frac{1}{2}c_{lj}^k c_{lk}^j - \frac{1}{2}c_{jk}^i c_{ik}^j - c_{kl}^k c_{jl}^j - \frac{1}{2}c_{kj}^i c_{ij}^k - \frac{1}{2}(c_{kj}^i)^2,
\end{aligned}$$

Since i, j, k, l are dummy indices, we can re-name them to simplify the calculations. Hence, in the second term, we rename j to k and k to j. Moreover, in the third term, we rename k to l, j to k, and l to j. Finally, in the sixth term, we keep j as is, but rename k to i and l to k. Doing the renaming, we get

$$\begin{aligned} S &= e_k(c^j_{kj}) - e_k(c^j_{jk}) + \frac{1}{4}(c^l_{kj})^2 + \frac{1}{4}(c^j_{lk})^2 - \frac{1}{4}(c^l_{kj})^2 \\ &+ \frac{1}{2}c^i_{kj}c^j_{ki} - \frac{1}{2}c^i_{jk}c^j_{ik} - c^k_{kl}c^j_{jl} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{2}(c^i_{kj})^2 \\ &= e_k(c^j_{kj}) + e_k(c^j_{kj}) + \frac{1}{4}(c^j_{lk})^2 \\ &+ \frac{1}{2}c^i_{jk}c^j_{ik} - \frac{1}{2}c^i_{jk}c^j_{ik} - c^k_{kl}c^j_{jl} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{2}(c^i_{kj})^2 \\ &= 2e_k(c^j_{kj}) + \frac{1}{4}(c^j_{lk})^2 - c^k_{kl}c^j_{jl} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{2}(c^i_{kj})^2. \end{aligned}$$

In the second term, renaming j to i, l to k and k to j, then simplifying, we get,

$$\begin{aligned} S &= 2e_k(c^j_{kj}) + \frac{1}{4}(c^i_{kj})^2 - c^k_{kl}c^j_{jl} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{2}(c^i_{kj})^2 \\ &= 2e_k(c^j_{kj}) - c^k_{kl}c^j_{jl} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{4}(c^i_{kj})^2. \end{aligned}$$

Changing l to i in the second term, we get

$$S = 2e_k(c^j_{kj}) - c^k_{ik}c^j_{ij} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{4}(c^i_{kj})^2, \qquad (7.1)$$

which is the desired equation. This formula will be very important for our later analysis of the scalar curvature of M.

CHAPTER 8. RESULTS AND DISCUSSION

With all the foundational knowledge established in Chapter 7, in this chapter, we will get to our research result. We consider the following case:

Consider a n-dimensional Riemannian manifold (M, g) and its tangent bundle TM. Suppose we can split TM into two orthogonal subbundle X and Y, with ranks r and s respectively. In other words, $TM = X \oplus Y$. Declare $g' = (P^{-1})^T g P^{-1}$ to be a new metric on M, where $P = I_{r\times r} \oplus (aI)_{s\times s}$, where a is a positive constant. In other words, $g' = g_X \oplus \left(\frac{1}{a^2} g_Y\right)$, where g_X is the metric restricted to X, g_Y is the metric restricted to Y. Also, let $\beta = \{e_1, e_2, \cdots, e_r, e_{r+1}, \cdots, e_n\}$ be an **adapted** local orthonormal frame with respect to g. Denote S the scalar curvature of M with respect to g', S_1 the scalar curvature of M restricted to X and S_2 the scalar curvature of M restricted to Y (both with respect to g). Also, for our calculations below, denote the indices corresponding to the first r coordinates with Roman letters and the last s coordinates by Greek letters, which, for now, we may denote by (††).

We want to investigate what conditions on a, M, S_1 and/ or S_2 will ensure that $S > 0$.

8.1 The Change in the Adapted Frame under the Specified Change in the Metric:

We will start by proving

Theorem 12. *For the rescaled metric as specified above, if $\beta' = \{e'_1, e'_2, \cdots, e'_n\}$ is a local orthonormal frame with respect to g', then $e'_i = Pe_i$ for all $1 \leq i \leq n$.*

Proof. First, notice that if β and β' are two local basis for a subset of TM, then there is a unique linear operator T that sends β to β'. Let the matrix for that linear operator be X. In other words, $T(e_m) = Xe_m = e'_m$ for all $m = 1, 2, \cdots, n$.

Now, since β is an orthonormal basis with respect to g, we have that $g(e_i, e_j) = \delta_{ij}$ for all pair of vectors in β. However, we can rewrite the above relation as

$$e_i^T g e_j = \delta_{ij}. \tag{8.1}$$

Similarly, since β' is an orthonormal basis with respect to g', we have that $g'(e'_i, e'_j) = \delta_{ij}$ for all pair of vectors in β', or

$$e_i'^T g' e'_j = \delta_{ij}. \tag{8.2}$$

by the same reasoning.

Combining (8.1) and (8.2), we get

$$\begin{aligned} e_i^T g e_j &= e_i'^T g' e'_j \\ \Rightarrow \quad e_i^T g e_j &= (Xe_i)^T((P^{-1})^T g(P^{-1}))(Xe_j) \\ \Rightarrow \quad e_i^T g e_j &= e_i^T(X^T(P^{-1})^T)g((P^{-1})X)e_j. \end{aligned}$$

Notice that the last equation in the above calculations must hold true for every indices $i, j = 1, 2, \cdots, n$. Hence, the only case for that to happen is if $g = (X^T(P^{-1})^T)g((P^{-1})X) = (P^{-1}X)^T g(P^{-1}X)$.

Notice that we can choose $P^{-1}X = I_n$, the identity matrix, to make the above equality valid. Solving the equation above, we'll get that $X = P$.

Hence, to find β', we will multiply every vector in β by the matrix P. In other words, the basis $\beta' = \{Pe_1, \cdots, Pe_n\}$ will be an orthonormal frame with respect to g', and Theorem 12 is shown. $\square$

8.2 The Change in the Structure Functions under the Specified Change in the Metric:

Now, we will continue to investigate the change of the structure functions of β' induced by the above change of the metric. However, first, we will relax the form of P a little bit.

Theorem 13. *Let c_{mn}^p and $\widetilde{c_{mn}^p}$ be the structure functions of β and β' respectively. If, in the form* *of P as above, $a = f$ is a function that changes according to the location on M, then we have the following relationship, written using* $(\dagger\dagger)$*:*

$$\begin{aligned}
\widetilde{c^k_{ij}} &= c^k_{ij}, \\
\widetilde{c^\alpha_{ij}} &= \frac{1}{f} c^\alpha_{ij}, \\
\widetilde{c^k_{i\alpha}} &= f c^k_{i\alpha}, \\
\widetilde{c^\beta_{i\beta}} &= \frac{1}{f} e_i(f) + c^\beta_{i\beta}, \\
\widetilde{c^\beta_{i\alpha}} &= c^\beta_{i\alpha}\,(\textit{when } \beta \neq \alpha), \\
\widetilde{c^k_{\alpha\beta}} &= f^2 c^k_{\alpha\beta}, \\
\widetilde{c^\gamma_{\gamma\beta}} &= f c^\gamma_{\gamma\beta} - e_\beta(f)\,(\textit{when } \gamma \neq \beta), \\
\widetilde{c^\gamma_{\alpha\beta}} &= f c^\gamma_{\alpha\beta}.
\end{aligned}$$

Proof. In the calculations below, we are considering the components of the bracket expressed by the subscript of the respective frame vector. In other words, for instance, in Case 1, when we write $[e'_i, e'_j]$, what we mean is the k-th component of the said bracket (or $([e'_i, e'_j])_k$, in matrix language). Also, by Theorem 12, we have the following change in the frame field vectors:

$$\begin{aligned}
e'_i &= e_i \\
e'_\alpha &= f e_\alpha
\end{aligned}$$

There are 8 cases to consider.

Case 1: If $1 \leq m, q, p \leq r$, then in this case, we may use the convention $(\dagger\dagger)$ to write $m = i\,, q = j\,, p = k$. Hence, in this case, $[e'_i, e'_j] = \widetilde{c^k_{ij}} e'_k$, while $[e_i, e_j] = c^k_{ij} e_k$. However, since $1 \leq i, j \leq r$, $[e'_i, e'_j] = [e_i, e_j]$ and $e'_k = e_k$, so

$$\widetilde{c^k_{ij}} e_k = c^k_{ij} e_k. \tag{8.3}$$

Case 2: If $1 \leq m, q \leq r, r+1 \leq p \leq n$, then in this case, we may use the convention $(\dagger\dagger)$ to write $m = i\,, q = j\,, p = \alpha$. Hence, in this case, $[e'_i, e'_j] = \widetilde{c^{\alpha}_{ij}} e'_\alpha = \widetilde{c^{\alpha}_{ij}} f e_\alpha$. On the other hand, $[e_i, e_j] = c^{\alpha}_{ij} e_\alpha$. However, by the same reasoning as above, we get

$$f\widetilde{c^{\alpha}_{ij}} e_\alpha \;=\; c^{\alpha}_{ij} e_\alpha. \tag{8.4}$$

Case 3: If $1 \leq m, p \leq r, r+1 \leq q \leq n$ then in this case, we may use the convention $(\dagger)$ to write $m = i\,, q = \alpha\,, p = k$. Hence, in this case, $[e'_i, e'_\alpha] = \widetilde{c^{k}_{i\alpha}} e'_k = \widetilde{c^{k}_{i\alpha}} e_k$. On the other hand,

$$\begin{aligned}
[e'_i, e'_\alpha] &= [e_i, f e_\alpha] \\
&= e_i(f e_\alpha) - f e_\alpha e_i \\
&= e_i(f) e_\alpha + f e_i e_\alpha - f e_\alpha e_i \\
&= e_i(f) e_\alpha + f[e_i, e_\alpha] \\
&= e_i(f) e_\alpha + f c^{k}_{i\alpha} e_k.
\end{aligned}$$

Hence,

$$\widetilde{c^{k}_{i\alpha}} e_k \;=\; e_i(f) e_\alpha + f c^{k}_{i\alpha} e_k. \tag{8.5}$$

Case 4: If $1 \leq m \leq r, r+1 \leq q, p \leq n$ then in this case, we may use the convention $(\dagger\dagger)$ to write $m = i\,, q = \alpha\,, p = \beta$. Hence, in this case, $[e'_i, e'_\alpha] = \widetilde{c^{\beta}_{i\alpha}} e'_\beta = \widetilde{c^{\beta}_{i\alpha}} f e_\beta$. On the other hand,

$$\begin{aligned}
[e'_i, e'_\alpha] &= [e_i, f e_\alpha] \\
&= e_i(f e_\alpha) - f e_\alpha e_i \\
&= e_i(f) e_\alpha + f e_i e_\alpha - f e_\alpha e_i \\
&= e_i(f) e_\alpha + f[e_i, e_\alpha] \\
&= e_i(f) e_\alpha + f c^{\beta}_{i\alpha} e_\beta.
\end{aligned}$$

Hence,

$$\widetilde{c_{i\alpha}^{\beta}} f e_{\beta} = e_i(f)e_{\alpha} + f c_{i\alpha}^{\beta} e_{\beta}. \tag{8.6}$$

Case 5: If $r+1 \leq m \leq n, 1 \leq q, p \leq r$ then in this case, we may use the convention $(\dagger\dagger)$ to write $m = \alpha\,, q = i\,, p = k$. Hence, in this case, $[e'_\alpha, e'_i] = \widetilde{c_{\alpha i}^{k}} e'_k = \widetilde{c_{\alpha i}^{k}} e_k$. On the other hand,

$$\begin{aligned}
[e'_\alpha, e'_i] &= [fe_\alpha, e_i] \\
&= fe_\alpha e_i - e_i(fe_\alpha) \\
&= fe_\alpha e_i - fe_i e_\alpha - e_i(f)e_\alpha \\
&= f[e_\alpha, e_i] - e_i(f)e_\alpha \\
&= fc_{\alpha i}^{k} e_k - e_i(f)e_\alpha.
\end{aligned}$$

Hence,

$$\widetilde{c_{\alpha i}^{k}} e_k = f c_{\alpha i}^{k} e_k - e_i(f)e_\alpha. \tag{8.7}$$

Case 6: If $r+1 \leq m, p \leq n, 1 \leq q \leq r$ then in this case, we may use the convention $(\dagger\dagger)$ to write $m = \alpha\,, q = i\,, p = \beta$. Hence, in this case, $[e'_\alpha, e'_i] = \widetilde{c_{\alpha i}^{\beta}} e'_\beta = \widetilde{c_{\alpha i}^{\beta}} f e_\beta$. On the other hand,

$$\begin{aligned}
[e'_\alpha, e'_i] &= [fe_\alpha, e_i] \\
&= fe_\alpha e_i - e_i(fe_\alpha) \\
&= fe_\alpha e_i - fe_i e_\alpha - e_i(f)e_\alpha \\
&= f[e_\alpha, e_i] - e_i(f)e_\alpha \\
&= fc_{\alpha i}^{\beta} e_\beta - e_i(f)e_\alpha.
\end{aligned}$$

Hence,

$$\widetilde{c^{\beta}_{\alpha i}} f e_{\beta} \;=\; f c^{\beta}_{\alpha i} e_{\beta} - e_i(f) e_{\alpha}. \tag{8.8}$$

Case 7: If $r+1 \leq m, q \leq n, 1 \leq p \leq r$ then in this case, we may use the convention $(\dagger\dagger)$ to write $m=\alpha\,, q=\beta\,, p=k$. Hence, in this case, $[e'_{\alpha}, e'_{\beta}] = \widetilde{c^{k}_{\alpha\beta}} e'_k = \widetilde{c^{k}_{\alpha\beta}} e_k$. On the other hand,

$$\begin{aligned}
[e'_{\alpha}, e'_{\beta}] &= [f e_{\alpha}, f e_{\beta}] \\
&= f e_{\alpha}(f e_{\beta}) - f e_{\beta}(f e_{\alpha}) \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 e_{\alpha} e_{\beta} - f^2 e_{\alpha} e_{\beta} \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 [e_{\alpha}, e_{\beta}] \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 c^{k}_{\alpha\beta} e_k.
\end{aligned}$$

Hence,

$$\widetilde{c^{k}_{\alpha\beta}} e_k \;=\; f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 c^{k}_{\alpha\beta} e_k. \tag{8.9}$$

Case 8: If $r+1 \leq m, q, p \leq n$ then in this case, we may use the convention $(\dagger\dagger)$ to write $m=\alpha\,, q=\beta\,, p=\gamma$. Hence, in this case, $[e'_{\alpha}, e'_{\beta}] = \widetilde{c^{\gamma}_{\alpha\beta}} e'_{\gamma} = \widetilde{c^{\gamma}_{\alpha\beta}} f e_{\gamma}$. On the other hand,

$$\begin{aligned}
[e'_{\alpha}, e'_{\beta}] &= [f e_{\alpha}, f e_{\beta}] \\
&= f e_{\alpha}(f e_{\beta}) - f e_{\beta}(f e_{\alpha}) \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 e_{\alpha} e_{\beta} - f^2 e_{\alpha} e_{\beta} \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 [e_{\alpha}, e_{\beta}] \\
&= f e_{\alpha}(f) e_{\beta} - f e_{\beta}(f) e_{\alpha} + f^2 c^{\gamma}_{\alpha\beta} e_{\gamma}.
\end{aligned}$$

Hence,

$$\widetilde{c^{\gamma}_{\alpha\beta}} f e_\gamma = f e_\alpha(f) e_\beta - f e_\beta(f) e_\alpha + f^2 c^{\gamma}_{\alpha\beta} e_\gamma. \tag{8.10}$$

Now, notice that in Case 1 and 2, the expansion of $[e'_i, e'_j]$ must be the same on both sides of the equality. Hence, we must have matching coefficients when adding (8.3) and (8.4). Therefore, matching the coefficients of e_k and e_α respectively in (8.3) and (8.4) yields

$$\widetilde{c^k_{ij}} = c^k_{ij} \tag{8.11}$$

$$\widetilde{c^{\alpha}_{ij}} = \frac{1}{f} c^{\alpha}_{ij}. \tag{8.12}$$

A similar argument as above applied to (8.5) and (8.6) yields

$$\widetilde{c^k_{i\alpha}} = f c^k_{i\alpha}. \tag{8.13}$$

and another pair of equations. Notice that if $\alpha = \beta$, then (8.6) will yield $f\widetilde{c^{\beta}_{i\beta}} = e_i(f) + f c^{\beta}_{i\beta}$, while if $\beta \neq \alpha$, then the contribution of e_α is negligible, and so $f\widetilde{c^{\beta}_{i\alpha}} = f c^{\beta}_{i\alpha}$. Hence, combining the cases, we have

$$\widetilde{c^{\beta}_{i\beta}} = \frac{1}{f} e_i(f) + c^{\beta}_{i\beta}, \tag{8.14}$$

$$\widetilde{c^{\beta}_{i\alpha}} = c^{\beta}_{i\alpha} (\text{when } \beta \neq \alpha). \tag{8.15}$$

A similar argument to (8.7) and (8.8) yields

$$\widetilde{c^k_{\alpha i}} = f c^k_{\alpha i}, \tag{8.16}$$

$$\widetilde{c^{\beta}_{\beta i}} = c^{\beta}_{\beta i} - \frac{1}{f} e_i(f), \tag{8.17}$$

$$\widetilde{c^{\beta}_{i\alpha}} = c^{\beta}_{i\alpha} (\text{when } \beta \neq \alpha), \tag{8.18}$$

Finally, in Case (8.9) and (8.10), one equation we get is

$$\widetilde{c_{\alpha\beta}^{k}} = f^2 c_{\alpha\beta}^{k}, \tag{8.19}$$

and similar to the other cases, we get two following equations

$$\begin{aligned} \widetilde{c_{\gamma\beta}^{\gamma}} &= f c_{\gamma\beta}^{\gamma} - e_\beta(f)(\text{when } \gamma \neq \beta), & (8.20)\\ \widetilde{c_{\alpha\gamma}^{\gamma}} &= f c_{\alpha\gamma}^{\gamma} + e_\alpha(f)(\text{when } \gamma \neq \alpha). & (8.21) \end{aligned}$$

Summarizing even more, we will have 8 relations as below as the relations desired.

$$\begin{aligned} \widetilde{c_{ij}^{k}} &= c_{ij}^{k}, \\ \widetilde{c_{ij}^{\alpha}} &= \frac{1}{f} c_{ij}^{\alpha}, \\ \widetilde{c_{i\alpha}^{k}} &= f c_{i\alpha}^{k}, \\ \widetilde{c_{i\beta}^{\beta}} &= \frac{1}{f} e_i(f) + c_{i\beta}^{\beta}, \\ \widetilde{c_{i\alpha}^{\beta}} &= c_{i\alpha}^{\beta}(\text{when } \beta \neq \alpha), \\ \widetilde{c_{\alpha\beta}^{k}} &= f^2 c_{\alpha\beta}^{k} \\ \widetilde{c_{\gamma\beta}^{\gamma}} &= f c_{\gamma\beta}^{\gamma} - e_\beta(f)(\text{when } \gamma \neq \beta), \\ \widetilde{c_{\alpha\beta}^{\gamma}} &= f c_{\alpha\beta}^{\gamma}. \end{aligned}$$

Hence, Theorem 13 is proven. □

Since the derivative of a constant function is 0, an immediate consequence of Theorem 13 is

Corollary 2. *If $f = a$ is a constant, then the structure functions will undergo the following* *changes.*

$$\begin{aligned}
\widetilde{c_{ij}^{k}} &= c_{ij}^{k} \\
\widetilde{c_{ij}^{\alpha}} &= \frac{1}{a} c_{ij}^{\alpha} \\
\widetilde{c_{i\alpha}^{k}} &= a c_{i\alpha}^{k} \\
\widetilde{c_{i\beta}^{\beta}} &= c_{i\beta}^{\beta} \\
\widetilde{c_{i\alpha}^{\beta}} &= c_{i\alpha}^{\beta}\,, (\textit{when } \beta \neq \alpha) \\
\widetilde{c_{\alpha\beta}^{k}} &= a^2 c_{\alpha\beta}^{k} \\
\widetilde{c_{\gamma\beta}^{\gamma}} &= a c_{\gamma\beta}^{\gamma}\,, (\textit{when } \gamma \neq \beta) \\
\widetilde{c_{\alpha\beta}^{\gamma}} &= a c_{\alpha\beta}^{\gamma}
\end{aligned}$$

8.3 The Relationship between the Scalar Curvatures S , S_1, and S_2:

In this section, we will prove

Theorem 14. *In the set up as in the introductory paragraph to this chapter, we have the following relationship, where the repeated indices are summed over their appropriate ranges:*

$$\begin{aligned}
S &= \frac{1}{2}S_1 + \frac{1}{2}a^2 S_2 - \frac{1}{4}a^4 c_{\gamma\beta}^{i} c_{\gamma\beta}^{i} \\
&+ \left(2e_\gamma(c_{\gamma j}^{j}) - c_{\alpha k}^{k} c_{\alpha j}^{j} - 2c_{\alpha k}^{k} c_{\alpha\beta}^{\beta} - \frac{1}{2} c_{i\beta}^{k} c_{k\beta}^{i} - c_{\gamma\beta}^{i} c_{i\beta}^{\gamma} - \frac{1}{2} c_{\gamma j}^{i} c_{\gamma j}^{i}\right) a^2 \\
&+ \left(2e_k(c_{k\beta}^{\beta}) - c_{i\gamma}^{\gamma} c_{i\beta}^{\beta} - 2c_{i\gamma}^{\gamma} c_{ij}^{j} - \frac{1}{2} c_{\gamma j}^{\alpha} c_{\alpha j}^{\gamma} - c_{\gamma j}^{i} c_{ij}^{\gamma} - \frac{1}{2} c_{k\beta}^{\alpha} c_{k\beta}^{\alpha}\right) \\
&- \frac{1}{4}\left(\frac{1}{a^2}\right)(c_{kj}^{\alpha} c_{kj}^{\alpha}).
\end{aligned}$$

8.3.1 Decomposing the Scalar Curvature in terms of the Indices of the Structure Functions

First, we need to express S explicitly in terms of the structure functions of both Greek and Roman indices as indicated in the introductory paragraphs. However, we saw at the end of Chapter

7 that in terms of the structure functions c_{IJ}^{K}, we have

$$S = \sum_{J,K=1}^{n} 2e_K(c_{KJ}^{J}) + \sum_{I,J,K=1}^{n} -c_{IK}^{K}c_{IJ}^{J} - \frac{1}{2}c_{KJ}^{I}c_{IJ}^{K} - \frac{1}{4}(c_{KJ}^{I})^2.$$

Now, using the convention (†), we can decompose the above formula as followed:

$$\begin{aligned} S &= \sum_{j,k=1}^{r} 2e_k(c_{kj}^{j}) - \sum_{j,k=1}^{r} c_{ik}^{k}c_{ij}^{j} - \frac{1}{2}c_{kj}^{i}c_{ij}^{k} - \frac{1}{4}(c_{kj}^{i})^2 \\ &+ \sum_{\beta,\gamma=r+1}^{r+s} 2e_\gamma(c_{\gamma\beta}^{\beta}) - \sum_{\alpha,\beta,\gamma=r+1}^{r+s} c_{\alpha\gamma}^{\gamma}c_{\alpha\beta}^{\beta} - \frac{1}{2}c_{\gamma\beta}^{\alpha}c_{\alpha\beta}^{\gamma} - \frac{1}{4}(c_{\gamma\beta}^{\alpha})^2 \\ &+ \sum_{i=1}^{r}\sum_{\gamma,\beta=r+1}^{r+s} -c_{i\gamma}^{\gamma}c_{i\beta}^{\beta} - \frac{1}{2}c_{\gamma\beta}^{i}c_{i\beta}^{\gamma} - \frac{1}{4}(c_{\gamma\beta}^{i})^2 \\ &+ \sum_{j=1}^{r}\sum_{\gamma=r+1}^{r+s} 2e_\gamma(c_{\gamma j}^{j}) - \sum_{i,j=1}^{r}\sum_{\gamma=r+1}^{r+s} c_{i\gamma}^{\gamma}c_{ij}^{j} - \frac{1}{2}c_{\gamma j}^{i}c_{ij}^{\gamma} - \frac{1}{4}(c_{\gamma j}^{i})^2 \\ &+ \sum_{k=1}^{r}\sum_{\beta=r+1}^{r+s} 2e_k(c_{k\beta}^{\beta}) - \sum_{i,k=1}^{r}\sum_{\beta=r+1}^{r+s} c_{ik}^{k}c_{i\beta}^{\beta} - \frac{1}{2}c_{k\beta}^{i}c_{i\beta}^{k} - \frac{1}{4}(c_{k\beta}^{i})^2 \\ &+ \sum_{j=1}^{r}\sum_{\alpha,\gamma=r+1}^{r+s} -c_{\alpha\gamma}^{\gamma}c_{\alpha j}^{j} - \frac{1}{2}c_{\gamma j}^{\alpha}c_{\alpha j}^{\gamma} - \frac{1}{4}(c_{\gamma j}^{\alpha})^2 \\ &+ \sum_{j,k=1}^{r}\sum_{\alpha=r+1}^{r+s} -c_{\alpha k}^{k}c_{\alpha j}^{j} - \frac{1}{2}c_{kj}^{\alpha}c_{\alpha j}^{k} - \frac{1}{4}(c_{kj}^{\alpha})^2 \\ &+ \sum_{k=1}^{r}\sum_{\alpha,\beta=r+1}^{r+s} -c_{\alpha k}^{k}c_{\alpha\beta}^{\beta} - \frac{1}{2}c_{k\beta}^{\alpha}c_{\alpha\beta}^{k} - \frac{1}{4}(c_{k\beta}^{\alpha})^2. \end{aligned}$$

Let us investigate the terms Q containing the structure functions with both Greek and Roman indices above. From now on, all the summation will be suppressed, with the implication that the indices are summed over their appropriate range. When we rearrange the cross terms, they will become

$$
\begin{aligned}
Q &= 2e_\gamma(c^j_{\gamma j}) + 2e_k(c^\beta_{k\beta}) \\
&- c^\gamma_{i\gamma}c^\beta_{i\beta} - c^k_{\alpha k}c^j_{\alpha j} - (c^\gamma_{i\gamma}c^j_{ij} + c^k_{ik}c^\beta_{i\beta}) - (c^k_{\alpha k}c^\beta_{\alpha\beta} + c^\gamma_{\alpha\gamma}c^j_{\alpha j}) \\
&- \frac{1}{2}c^i_{k\beta}c^k_{i\beta} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - \left(\frac{1}{2}c^i_{\gamma\beta}c^\gamma_{i\beta} + \frac{1}{2}c^\alpha_{k\beta}c^k_{\alpha\beta}\right) - \left(\frac{1}{2}c^i_{\gamma j}c^\gamma_{ij} + \frac{1}{2}c^\alpha_{kj}c^k_{\alpha j}\right) \\
&- \frac{1}{4}(c^i_{\gamma\beta})^2 - \frac{1}{4}(c^\alpha_{kj})^2 - \left(\frac{1}{4}(c^i_{\gamma j})^2 + \frac{1}{4}(c^i_{k\beta})^2\right) - \left(\frac{1}{4}(c^\alpha_{k\beta})^2 + \frac{1}{4}(c^\alpha_{\gamma j})^2\right).
\end{aligned}
$$

Since the indices are summed over, they are dummy indices, and so we can rename one index with another, provided that we only replace Greek indices with Greek indices and Latin with Latin. Doing that, all the parenthetical terms above will simplify, and we get

$$
\begin{aligned}
Q &= 2e_\gamma(c^j_{\gamma j}) + 2e_k(c^\beta_{k\beta}) \\
&- c^\gamma_{i\gamma}c^\beta_{i\beta} - c^k_{\alpha k}c^j_{\alpha j} - 2c^\gamma_{i\gamma}c^j_{ij} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} \\
&- \frac{1}{2}c^i_{k\beta}c^k_{i\beta} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - c^i_{\gamma\beta}c^\gamma_{i\beta} - c^i_{\gamma j}c^\gamma_{ij} \\
&- \frac{1}{4}(c^i_{\gamma\beta})^2 - \frac{1}{4}(c^\alpha_{kj})^2 - \frac{1}{2}(c^i_{\gamma j})^2 - \frac{1}{2}(c^\alpha_{k\beta})^2.
\end{aligned}
$$

All in all, we will get the formula for the scalar curvature of M as

$$
\begin{aligned}
S &= 2e_k(c_{kj}^j) - c_{ik}^k c_{ij}^j - \frac{1}{2}c_{kj}^i c_{ij}^k - \frac{1}{4}(c_{kj}^i)^2 \\
&+ 2e_\gamma(c_{\gamma\beta}^\beta) - c_{\alpha\gamma}^\gamma c_{\alpha\beta}^\beta - \frac{1}{2}c_{\gamma\beta}^\alpha c_{\alpha\beta}^\gamma - \frac{1}{4}(c_{\gamma\beta}^\alpha)^2 \\
&+ 2e_\gamma(c_{\gamma j}^j) + 2e_k(c_{k\beta}^\beta) \\
&- c_{i\gamma}^\gamma c_{i\beta}^\beta - c_{\alpha k}^k c_{\alpha j}^j - 2c_{i\gamma}^\gamma c_{ij}^j - 2c_{\alpha k}^k c_{\alpha\beta}^\beta \\
&- \frac{1}{2}c_{k\beta}^i c_{i\beta}^k - \frac{1}{2}c_{\gamma j}^\alpha c_{\alpha j}^\gamma - c_{\gamma\beta}^i c_{i\beta}^\gamma - c_{\gamma j}^i c_{ij}^\gamma \\
&- \frac{1}{4}(c_{\gamma\beta}^i)^2 - \frac{1}{4}(c_{kj}^\alpha)^2 - \frac{1}{2}(c_{\gamma j}^i)^2 - \frac{1}{2}(c_{k\beta}^\alpha)^2 \\
&= 2e_k(c_{kj}^j) - c_{ik}^k c_{ij}^j - \frac{1}{2}c_{kj}^i c_{ij}^k - \frac{1}{4}(c_{kj}^i)^2 \\
&+ 2e_\gamma(c_{\gamma\beta}^\beta) - c_{\alpha\gamma}^\gamma c_{\alpha\beta}^\beta - \frac{1}{2}c_{\gamma\beta}^\alpha c_{\alpha\beta}^\gamma - \frac{1}{4}(c_{\gamma\beta}^\alpha)^2 \\
&+ 2e_\gamma(c_{\gamma j}^j) + 2e_k(c_{k\beta}^\beta) \\
&- c_{i\gamma}^\gamma c_{i\beta}^\beta - c_{\alpha k}^k c_{\alpha j}^j - 2c_{i\gamma}^\gamma c_{ij}^j - 2c_{\alpha k}^k c_{\alpha\beta}^\beta \\
&- \frac{1}{2}c_{k\beta}^i c_{i\beta}^k - \frac{1}{2}c_{\gamma j}^\alpha c_{\alpha j}^\gamma - c_{\gamma\beta}^i c_{i\beta}^\gamma - c_{\gamma j}^i c_{ij}^\gamma \\
&- \frac{1}{4}(c_{\gamma\beta}^i)^2 - \frac{1}{4}(c_{kj}^\alpha)^2 - \frac{1}{2}(c_{\gamma j}^i)^2 - \frac{1}{2}(c_{k\beta}^\alpha)^2.
\end{aligned}
$$

Rearranging the formula once more, we will get

$$
\begin{aligned}
S &= 2e_k(c_{kj}^j) + 2e_\gamma(c_{\gamma\beta}^\beta) + 2e_\gamma(c_{\gamma j}^j) + 2e_k(c_{k\beta}^\beta) \\
&- c_{ik}^k c_{ij}^j - \frac{1}{2}c_{kj}^i c_{ij}^k - \frac{1}{4}(c_{kj}^i)^2 \\
&- c_{\alpha\gamma}^\gamma c_{\alpha\beta}^\beta - \frac{1}{2}c_{\gamma\beta}^\alpha c_{\alpha\beta}^\gamma - \frac{1}{4}(c_{\gamma\beta}^\alpha)^2 \\
&- c_{i\gamma}^\gamma c_{i\beta}^\beta - c_{\alpha k}^k c_{\alpha j}^j - 2c_{i\gamma}^\gamma c_{ij}^j - 2c_{\alpha k}^k c_{\alpha\beta}^\beta \\
&- \frac{1}{2}c_{k\beta}^i c_{i\beta}^k - \frac{1}{2}c_{\gamma j}^\alpha c_{\alpha j}^\gamma - c_{\gamma\beta}^i c_{i\beta}^\gamma - c_{\gamma j}^i c_{ij}^\gamma \\
&- \frac{1}{4}(c_{\gamma\beta}^i)^2 - \frac{1}{4}(c_{kj}^\alpha)^2 - \frac{1}{2}(c_{\gamma j}^i)^2 - \frac{1}{2}(c_{k\beta}^\alpha)^2.
\end{aligned}
$$

8.3.2 The Proof of Theorem 14

With the decomposition of the scalar curvature according to the indices of the structure functions established, we proceed to the proof of Theorem 14. In the computations below, we should keep in mind the notations and conventions used in the previous theorems of this chapter.

Proof of Theorem 14. By the calculations in the previous subsection, we have

$$
\begin{aligned}
S' &= 2\widetilde{e_k}(\widetilde{c^j_{kj}}) + 2\widetilde{e_\gamma}(\widetilde{c^\beta_{\gamma\beta}}) + 2\widetilde{e_\gamma}(\widetilde{c^j_{\gamma j}}) + 2\widetilde{e_k}(\widetilde{c^\beta_{k\beta}}) \\
&- \widetilde{c^k_{ik}}\widetilde{c^j_{ij}} - \frac{1}{2}\widetilde{c^i_{kj}}\widetilde{c^k_{ij}} - \frac{1}{2}(\widetilde{c^i_{kj}})^2 \\
&- \widetilde{c^\gamma_{\alpha\gamma}}\widetilde{c^\beta_{\alpha\beta}} - \frac{1}{2}\widetilde{c^\alpha_{\gamma\beta}}\widetilde{c^\gamma_{\alpha\beta}} - \frac{1}{4}(\widetilde{c^\alpha_{\gamma\beta}})^2 \\
&- \widetilde{c^\gamma_{i\gamma}}\widetilde{c^\beta_{i\beta}} - \widetilde{c^k_{\alpha k}}\widetilde{c^j_{\alpha j}} - 2\widetilde{c^\gamma_{i\gamma}}\widetilde{c^j_{ij}} - 2\widetilde{c^k_{\alpha k}}\widetilde{c^\beta_{\alpha\beta}} \\
&- \frac{1}{2}\widetilde{c^i_{k\beta}}\widetilde{c^k_{i\beta}} - \frac{1}{2}\widetilde{c^\alpha_{\gamma j}}\widetilde{c^\gamma_{\alpha j}} - \widetilde{c^i_{\gamma\beta}}\widetilde{c^\gamma_{i\beta}} - \widetilde{c^i_{\gamma j}}\widetilde{c^\gamma_{ij}} \\
&- \frac{1}{4}(\widetilde{c^i_{\gamma\beta}})^2 - \frac{1}{4}(\widetilde{c^\alpha_{kj}})^2 - \frac{1}{2}(\widetilde{c^i_{\gamma j}})^2 - \frac{1}{2}(\widetilde{c^\alpha_{k\beta}})^2.
\end{aligned}
$$

Now, using the fact that $f = a$ is constant by assumption and Corollary 2 to Theorem 13 above, we can relate the new structure functions back to the old structure functions in the formula above.

Expressing the formula above in terms of the old structure functions, we get

$$
\begin{aligned}
S &= 2e_k(c^j_{kj}) + 2ae_\gamma(kc^\beta_{\gamma\beta}) + 2ae_\gamma(kc^j_{\gamma j}) + 2e_k(c^\beta_{k\beta}) \\
&- c^k_{ik}c^j_{ij} - c^i_{kj}c^k_{ij} - \frac{1}{4}(c^i_{kj})^2 \\
&- (ac^\gamma_{\alpha\gamma})(ac^\beta_{\alpha\beta}) - \frac{1}{2}(ac^\alpha_{\gamma\beta})(ac^\gamma_{\alpha\beta}) - \frac{1}{4}(ac^\alpha_{\gamma\beta})^2 \\
&- c^\gamma_{i\gamma}c^\beta_{i\beta} - (ac^k_{\alpha k})(ac^j_{\alpha j}) - 2c^\gamma_{i\gamma}c^j_{ij} - 2(ac^k_{\alpha k})(ac^\beta_{\alpha\beta}) \\
&- \frac{1}{2}(ac^i_{k\beta})(ac^k_{i\beta}) - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - (a^2c^i_{\gamma\beta})c^\gamma_{i\beta} - (ac^i_{\gamma j})\left(\frac{1}{a}c^\gamma_{ij}\right) \\
&- \frac{1}{4}(a^2c^i_{\gamma\beta})^2 - \frac{1}{4}\left(\frac{1}{a}c^\alpha_{kj}\right)^2 - \frac{1}{2}(ac^i_{\gamma j})^2 - \frac{1}{2}(c^\alpha_{k\beta})^2 \\
&= 2e_k(c^j_{kj}) + 2a^2e_\gamma(c^\beta_{\gamma\beta}) + 2a^2e_\gamma(c^j_{\gamma j}) + 2e_k(c^\beta_{k\beta}) \\
&- c^k_{ik}c^j_{ij} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{4}(c^i_{kj})^2 \\
&- a^2c^\gamma_{\alpha\gamma}c^\beta_{\alpha\beta} - \frac{1}{2}a^2c^\alpha_{\gamma\beta}c^\gamma_{\alpha\beta} - \frac{1}{4}a^2(c^\alpha_{\gamma\beta})^2 \\
&- c^\gamma_{i\gamma}c^\beta_{i\beta} - a^2c^k_{\alpha k}c^j_{\alpha j} - 2c^\gamma_{i\gamma}c^j_{ij} - 2a^2\ c^k_{\alpha k}c^\beta_{\alpha\beta} \\
&- \frac{1}{2}a^2c^i_{k\beta}c^k_{i\beta} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - a^2c^i_{\gamma\beta}c^\gamma_{i\beta} - c^i_{\gamma j}c^\gamma_{ij} \\
&- \frac{1}{4}a^4(c^i_{\gamma\beta})^2 - \frac{1}{4}\left(\frac{1}{a^2}\right)(c^\alpha_{kj})^2 - \frac{1}{2}a^2(c^i_{\gamma j})^2 - \frac{1}{2}(c^\alpha_{k\beta})^2.
\end{aligned}
$$

Rearranging the terms above, we get

$$
\begin{aligned}
S &= 2e_k(c^k_{kj}) - c^k_{ik}c^j_{ij} - \frac{1}{2}c^i_{kj}c^k_{ij} - \frac{1}{4}(c^i_{kj})^2 \\
&+ 2a^2e_\gamma(c^\beta_{\gamma\beta}) - a^2c^\gamma_{\alpha\gamma}c^\beta_{\alpha\beta} - \frac{1}{2}a^2c^\alpha_{\gamma\beta}c^\gamma_{\alpha\beta} - \frac{1}{4}a^2(c^\alpha_{\gamma\beta})^2 \\
&- \frac{1}{4}a^4(c^i_{\gamma\beta})^2 \\
&+ \left(2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}(c^i_{\gamma j})^2\right)a^2 \\
&+ \left(2e_k(c^\beta_{k\beta}) - c^\gamma_{i\gamma}c^\beta_{i\beta} - 2c^\gamma_{i\gamma}c^j_{ij} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - c^i_{\gamma j}c^\gamma_{ij} - \frac{1}{2}(c^\alpha_{k\beta})^2\right) \\
&- \frac{1}{4}\left(\frac{1}{a^2}\right)(c^\alpha_{kj})^2 \\
&= S_1 + a^2S_2 - \frac{1}{4}a^4(c^i_{\gamma\beta})^2 \\
&+ \left(2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}(c^i_{\gamma j})^2\right)a^2 \\
&+ \left(2e_k(c^\beta_{k\beta}) - c^\gamma_{i\gamma}c^\beta_{i\beta} - 2c^\gamma_{i\gamma}c^j_{ij} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - c^i_{\gamma j}c^\gamma_{ij} - \frac{1}{2}(c^\alpha_{k\beta})^2\right) \\
&- \frac{1}{4}\left(\frac{1}{a^2}\right)(c^\alpha_{kj})^2.
\end{aligned}
$$

Using the convention that repeated indices get summed over their appropriate ranges, we can write S above as follows:

$$
\begin{aligned}
S &= S_1 + a^2S_2 - \frac{1}{4}a^4c^i_{\gamma\beta}c^i_{\gamma\beta} \\
&+ \left(2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}c^i_{\gamma j}c^i_{\gamma j}\right)a^2 \\
&+ \left(2e_k(c^\beta_{k\beta}) - c^\gamma_{i\gamma}c^\beta_{i\beta} - 2c^\gamma_{i\gamma}c^j_{ij} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - c^i_{\gamma j}c^\gamma_{ij} - \frac{1}{2}c^\alpha_{k\beta}c^\alpha_{k\beta}\right) \\
&- \frac{1}{4}\left(\frac{1}{a^2}\right)(c^\alpha_{kj}c^\alpha_{kj}).
\end{aligned}
$$

Therefore, Theorem 14 is proven. □

8.4 Corollaries of Theorem 14:

In this section, we will discuss the consequences of Theorem 14 when the metric is collapsed in the sub-bundle Y (in other words, when a increases without bounds). However, first, we will discuss foliation theory and some important geometric conditions on a manifold M.

8.4.1 Foliation Theory

In this section, we will define some extra conditions on a manifold M that are helpful for our later discussions. We have

Definition 30. *Let Y be a sub-bundle of TM of constant rank of a smooth manifold M. Y is called **involutive** if for all vector fields $U, V \in \Gamma(M, Y)$, we also have $[U, V] \in \Gamma(M, Y)$, where Γ denotes smooth sections of the bundle Y.*

A famous theorem concerning involutive sub-bundles of a manifold is

Theorem 15 (Frobenius' Theorem)**.** *If Y is a sub-bundle of TM of constant rank on a smooth manifold M, then Y is involutive if and only if Y is the tangent space of a foliation of M.*

For an exposition on foliations on Riemannian manifolds, see [7].

Corollary 3. *Let (M, g) be a Riemannian manifold, and $B = \{e_1, \cdots, e_r, e_{r+1}, \cdots e_n\}$ be a local adapted orthonormal frame. Then, using the notation below, a sub-bundle Y of M is involutive if and only if $c^i_{\alpha\beta} = 0$ for all α , β , i.*

The definition of an adapted frame is technical, but one can find it in [2].

Definition 31. *Let (M, g) be a Riemannian manifold and Y be an involutive sub-bundle of M. Let $X = Y^{\perp}$. Then we say the metric g is **bundle-like** with respect to Y if for all $x \in M$, there exists a local adapted orthonormal frame $B = \{e_1, \cdots, e_r, e_{r+1}, \cdots, e_n\}$ defined on an open neighborhood U of x so that $X|_U = span\{e_1, \cdots, e_r\}_U$, $Y|_U = span\{e_{r+1}, \cdots, e_n\}_U$, and for all α, j, $[e_\alpha, e_j] \in \Gamma(U, Y)$.*

Corollary 4. *Let (M, g) be a Riemannian manifold and Y be an involutive sub-bundle of M. Let $X = Y^{\perp}$. Then g is bundle-like with respect to Y if and only if for all $x \in M$ there exists a local adapted orthonormal frame $B = \{e_1, \cdots, e_r, e_{r+1}, \cdots, e_n\}$ in an open neighborhood of x so that for all α, j, k, $c^k_{\alpha j} = 0$.*

Definition 32. *Let (M, g) be a Riemannian manifold. We say that g is* ***nearly bundle-like*** *with respect to a vector sub-bundle Y of M if there exists a local orthonormal frame in an open neighborhood around every point $x \in M$ so that the inequality $S_2 > 2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}c^i_{\gamma j}c^i_{\gamma j}$, where S_2 is the scalar curvature of M restricted to Y, holds true.*

This definition is what we have come up with to aid the statement of our theorem, and in the future, we are working towards a clearer geometric interpretation of the condition above to better understand the structure of such a metric and a sub-bundle.

Corollary 5. *Let (M, g) be a Riemannian manifold and Y be a sub-bundle of M. Then if S_2, the scalar curvature of M restricted to Y, is positive and g is bundle-like with respect to Y, then g is also nearly bundle-like with respect to Y.*

Proof. Notice that if g is bundle-like with respect to Y, then by Corollary 4, for all $x \in M$, there exists a local orthonormal frame in an open neighborhood around x so that for all α, j, k, we have $c^k_{\alpha j} = 0$. Hence, along with the condition that Y is involutive, which, by Corollary 1, implies that $c^i_{\gamma\beta} = 0$, the right-hand side of the inequality specified in the definition of a nearly bundle-like metric vanishes to 0.

Hence, since $S_2 > 0$ by hypothesis, we have

$$S_2 > 0 = 2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}c^i_{\gamma j}c^i_{\gamma j},$$

which implies, by Definition 3, that g is nearly bundle-like with respect to Y. Hence, the corollary is proven. □

We have

Definition 33. *A distribution Y of M is called* ***everywhere non-involutive*** *if there exists two vector fields $U, V \in \Gamma(Y)$ so that $[U,V]_x \notin Y_x$ for all $x \in M$.*

8.4.2 Corollaries of Theorem 14

In this section, we will conclude our thesis with a few corollaries that tells us about the topological and geometric conditions on a Riemannian manifold and the constant a as in Theorem 14 that will make S positive provided that we know $S_2 > 0$. This is the goal that we set up in the beginning of the project.

We will begin with

Corollary 6. *If the sub-bundle Y is involutive, then the formula in Theorem 14 becomes*

$$\begin{aligned} S &= \frac{1}{2}S_1 + \frac{1}{2}a^2 S_2 \\ &+ \left(2e_\gamma(c^j_{\gamma j}) - c^k_{\alpha k}c^j_{\alpha j} - 2c^k_{\alpha k}c^\beta_{\alpha\beta} - \frac{1}{2}c^k_{i\beta}c^i_{k\beta} - c^i_{\gamma\beta}c^\gamma_{i\beta} - \frac{1}{2}c^i_{\gamma j}c^i_{\gamma j}\right)a^2 \\ &+ \left(2e_k(c^\beta_{k\beta}) - c^\gamma_{i\gamma}c^\beta_{i\beta} - 2c^\gamma_{i\gamma}c^j_{ij} - \frac{1}{2}c^\alpha_{\gamma j}c^\gamma_{\alpha j} - c^i_{\gamma j}c^\gamma_{ij} - \frac{1}{2}c^\alpha_{k\beta}c^\alpha_{k\beta}\right) \\ &- \frac{1}{4}\left(\frac{1}{a^2}\right)(c^\alpha_{kj}c^\alpha_{kj}). \end{aligned}$$

A consequence of this corollary is

Corollary 7. *If, in addition to the hypothesis in Corollary 6, we also have that Y is nearly bundle-like, then there exists a constant $A > 0$ so that for all $a \geq A$, $S > 0$.*

Another corollary of Theorem 14 is

Theorem 16. *If the metric g is bundle-like with respect to Y then the formula in Theorem 14* *becomes*

$$
\begin{aligned}
S &= \frac{1}{2}S_1 + \frac{1}{2}a^2 S_2 \\
&+ \left(2e_k(c^{\beta}_{k\beta}) - c^{\gamma}_{i\gamma}c^{\beta}_{i\beta} - 2c^{\gamma}_{i\gamma}c^{j}_{ij} - \frac{1}{2}c^{\alpha}_{\gamma j}c^{\gamma}_{\alpha j} - c^{i}_{\gamma j}c^{\gamma}_{ij} - \frac{1}{2}c^{\alpha}_{k\beta}c^{\alpha}_{k\beta}\right) \\
&- \frac{1}{4}\left(\frac{1}{a^2}\right)(c^{\alpha}_{kj}c^{\alpha}_{kj}).
\end{aligned}
$$

In particular, there exists a constant $A > 0$ so that for all $a \geq A$, $S_2 > 0$ implies that $S > 0$.

Finally, we have a nice corollary to determine in which case, there is a metric of negative scalar curvature on M.

Theorem 17. *If Y is everywhere non-involutive, then there exists a constant A so that for all $a \geq A$, $S < 0$.*

Appendices

APPENDIX A. THE VOLUME OF THE N-DIMENSIONAL SPHERE

In this appendix, we will calculate the volume of a sphere of radius R in n-dimensional space. The purpose of this calculation is that, by the result in Chapter 3, we have that the volume of a ball in an n-dimensional manifold M is related to the volume of the ball of the same radius in flat space by a formula involving the scalar curvature S. Hence, since the volume of the ball in flat space is known, by taking the derivative of the volume function in r, we can interpret S as the volume rate of change of a certain ball in M.

We will begin by examining the Gaussian integral, which has the value

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Now, taking both sides to the power of n, we get

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^n = \pi^{\frac{n}{2}}.$$

However, expanding the left hand side and collapsing the product of integrals into a single integral (just like for the 2-dimensional case of the Gaussian integral), we get:

$$\pi^{\frac{n}{2}} = \int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)} dx_1 \cdots dx_n.$$

We will evaluate the right hand side by changing to n-dimensional spherical coordinates. First of all, note that the volume element in n-dimensional spherical coordinate is calculated by $dV = dx_1 \cdots dx_n = r^{n-1}\, dr\, dA$, where dA is the volume element in the (n-1)-spherical coordinate.

Hence, we have the following calculations:

$$
\begin{aligned}
\pi^{\frac{n}{2}} &= \int\limits_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)} dx_1 \cdots dx_n \\
&= \int\limits_0^\infty \int\limits_{S^{(n-1)}} r^{n-1} e^{-r^2} dr dA (\text{where } S^{(n-1)} \text{ is the (n-1)-sphere with radius } 1) \\
&= \int\limits_0^\infty r^{n-1} e^{-r^2} dr \int\limits_{S^{(n-1)}} dA \\
&= A(S^{n-1}(1)) \int\limits_0^\infty r^{n-1} e^{-r^2} dr,
\end{aligned}
$$

where the coefficient in front of the integral is the area of the $(n-1)$-sphere with radius 1.

For the integral, let $u = r^2$. Hence, $du = 2rdr$ and $r^{n-2} = u^{\frac{n}{2}-1}$. Hence, the integral becomes

$$
\frac{1}{2} \int\limits_0^\infty u^{\frac{n}{2}-1} e^{-u} du
$$

which is half of the gamma function $\Gamma\left(\frac{n}{2}\right)$.

Hence, we combine the equation above and the gamma function to get

$$
\frac{1}{2} A(S^{(n-1)}(1)) \Gamma\left(\frac{n}{2}\right) = \pi^{\frac{n}{2}} \Rightarrow A(S^{(n-1)}(1)) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.
$$

Hence, we have calculated the $(n-1)$-dimensional volume of the unit $(n-1)$-sphere. Now, consider the transformation $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $T(x) = rx$. Clearly, T is a linear transformation, and furthermore, T sends the unit $(n-1)$-sphere to a sphere of radius r of the same dimension. Hence, from linear algebra, we have that $A(S^{(n-1)}(r)) = |\text{det}T| \cdot A(S^{(n-1)}(1)) = r^{n-1} \cdot \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ (the geometric meaning of the determinant of a transformation).

Now, the volume of the n-dimensional ball of radius R can be calculated by integrating the areas of the cross sectional $(n-1)$-spheres from 0 to R. Hence, the desired formula is

$$\begin{aligned} V &= \int_0^R A(S^{(n-1)}(r))dr \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^R r^{n-1}dr \\ &= \frac{2\pi^{\frac{n}{2}} R^n}{n\Gamma(\frac{n}{2})} \\ &= \frac{2\pi^{\frac{n}{2}} R^n}{2\frac{n}{2}\Gamma(\frac{n}{2})} \\ &= \frac{\pi^{\frac{n}{2}} R^n}{\Gamma(\frac{n}{2}+1)}. \end{aligned}$$

where we used the identity $x\Gamma(x) = \Gamma(x+1)$.

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