TOEPLITZ OPERATORS WITH SYMBOLS FROM CERTAIN ROTATION ALGEBRAS AND THEIR INDEX FORMULAS

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Submitted to the Graduate Faculty of the
College of Science and Engineering
Texas Christian University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy in Mathematics

December 2018

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ACKNOWLEDGMENTS

I would like to thank my advisor, Efton Park, for the time, effort, guidance, and encouragement he has provided. His patience and wit have been indefatigable. I would also like to thank José Carrión, Greg Friedman, and Robert Doran for their help and instruction over the past several years. Further, I would like to thank my family for their constant support. In particular, I would like to thank my father, Saul Hellerman, for introducing me to Zeno's paradox at an early age — the rest of my academic path has followed logically.

Contents

1 Introduction

The kernel is a particularly desirable detail to know about an operator because it provides the solution set of a linear equation. In many cases, however, the kernel is not easy to find. A general method for calculating even the dimension of the kernel of an operator is problematic because this number is not invariant under homotopy or compact perturbation. Instead, index theory uses Fredholm index as a workable alternative.

The Fredholm index of an operator can give a sense of the size of solution sets to equations involving the operator. An operator W, on a Hilbert space, is a Fredholm operator if W and its adjoint, W^*, have finite dimensional kernels and if the range of W is closed. Equivalently, W is Fredholm if it is invertible modulo compact operators. Fredholm index is defined by the formula $ind(W) = dim(ker(W)) - dim(ker(W^*))$. Unlike the dimension of the kernel of an operator, Fredholm index is invariant under homotopy and compact perturbation. It can still be difficult to calculate an operator's index directly, so it is a goal of index theory to relate the index of an operator to its geometric or topological properties. This can result in a formula for the index requiring only straightforward calculations.

Our work will give explicit index formulas for Toeplitz operators with symbols from the non-abelian algebras $C(S^1) \rtimes_\alpha \mathbb{Z}_n$, where $\alpha(z) = e^{2\pi i/n}z$, and $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, where $\alpha(w, z) = (w, e^{2\pi i/n}z)$. In the latter case, it turns out that the different parts of the index formula will also allow us to distinguish between elements which have the same index but are not homotopic. We begin with an overview of some facts about Toeplitz operators, crossed product algebras, and K-theory that we will need.

1.1 Toeplitz Operators on the Circle

The Hardy space, $H^2(S^1)$, is the subspace of $L^2(S^1)$ which has an orthonormal basis of non-negative powers of z; that is, $H^2(S^1)$ is the closed linear span of the basis $\{e^{in\theta}\}_{n\geq 0}$. (Convenience outweighing consistency in this case, we will, without further apology, use both expressions z^n and $e^{in\theta}$ in this paper.) The Hardy space can also be understood as the subspace of functions in $L^2(S^1)$ which have analytic extensions to the unit disk [2, Introduction]. Toeplitz operators, in the classical understanding, are operators on the Hardy space, $H^2(S^1)$, usually with symbols from either $L^\infty(S^1)$ or $C(S^1)$. For our purposes we will only consider Toeplitz operators with continuous symbols. Let P be the projection from $L^2(S^1)$ to $H^2(S^1)$. We can consider P as cutting off any negative powers of z from an element in $L^2(S^1)$. With this projection we define a Toeplitz operator.

Definition 1.1.1. A Toeplitz operator with continuous symbol f, notated T_f, is an operator on the Hardy space such that for any $g \in H^2(S^1)$, $T_f(g) = P(fg)$. Further, let $\mathcal{T}(C(S^1))$ be the C^*-subalgebra of $B(H^2(S^1))$ generated by the Toeplitz operators with symbols from $C(S^1)$. The $*$-map is $(T_f)^* = T_{\bar{f}}$.

Operators with the form $T_fT_g - T_{fg}$, where f and g are continuous functions, are called semi-commutators. ♢

Example 1.1.2. We consider the Toeplitz operator T_z which acts as a unilateral shift on the Hardy space. Since $T_z(z^n) = z^{n+1}$ for $n \geq 0$, the range of T_z is the closed span of $\{z^n\}_{n>0}$ and the kernel is $\{0\}$. The adjoint is $T_{\bar{z}}$. Since $T_{\bar{z}}(1) = P(z^{-1}) = 0$, and, for $n > 0$, $T_{\bar{z}}(z^n) = z^{n-1}$, we conclude that the kernel is the linear span of $\{1\}$. Thus T_z is Fredholm and has index -1. We further note that $T_{\bar{z}}T_z = 1$ whereas $1 - T_z T_{\bar{z}}$ is the one-dimensional projection onto the subspace spanned by $\{1\}$. Similarly, it can be shown that, for any integer m, T_{z^m} has an index of $-m$. $\diamondsuit$

Let $\mathcal{K}$ be the algebra of compact operators on the Hardy space. Using the fact that semi-commutators of Toeplitz operators are compact, it can be shown that $\mathcal{T}(C(S^1))/\mathcal{K}$ is isomorphic to $C(S^1)$ by the symbol map σ which sends T_f to f [4, Theorem 7.23]. This further implies that a Toeplitz operator with continuous symbol is Fredholm if and only if its symbol is invertible [4, Theorem 7.26]. This isomorphism also gives rise to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^1)) \xrightarrow{\sigma} C(S^1) \to 0.$$

It is a standard example in K-theory to show that the connecting map for this short exact sequence, $\delta : K_1(C(S^1)) \to K_0(\mathcal{K})$, is an isomorphism. In fact, [8, Example 9.4.4] shows that δ sends the generator $[z]_1$ to $-[1 - T_z T_{\bar{z}}]_0$. From Example 1.1.2 we know that this is the class of a rank one projection and thus a generator for $K_0(\mathcal{K})$.

A further classical argument, using the trace on $\mathcal{K}$, establishes that the connecting map, δ, can be used to find the index of a Toeplitz operator. Since, with $f \in C(S^1)$, T_f

is Fredholm only if f is invertible, polar decomposition of f yields a unitary $g \in C(S^1)$ such that f is homotopic to g. Through index invariance under homotopy and compact perturbation, $ind(T_f) = ind(T_g)$. Using index properties and K-theory, an application of [8, Proposition 9.4.2] establishes that $ind(T_g) = -K_0(tr)(\delta([g]_1))$.

Of course, for any particular invertible function f in $C(S^1)$, it may not be immediately obvious which unitary g is homotopic to f, or what the trace of $\delta([g]_1)$ will be. What we actually desire is a formula that uses topological properties of f to calculate the index of T_f. This general goal of using topology-based formulas to yield operator algebra results will be a recurring theme.

For continuous functions on the circle, the topological property we need is winding number. It is invariant under homotopy, obeys the index-like rule that $wn(fg) = wn(f) + wn(g)$, and $wn(z) = 1$. An argument from these well-known properties in [4, Theorem 7.26, pg. 185] concludes that, when $f \in C^1(S^1)$, the index of T_f is equal to the negative of the winding number of f, or, using the formula for winding number,

$$ind(T_f) = -\frac{1}{2\pi i} \int_{S^1} f^{-1} df.$$

We can slightly modify this formula so that it applies to matrices as well. For X in $M_n(C^1(S^1))$, we define T_X as the $n \times n$ matrix where the (i, j)th entry is the Toeplitz operator whose symbol is the (i, j)th entry of X. Considering T_X as an operator on n copies of $H^2(S^1)$, the previous argument for the one-dimensional case can be extended, as in [9, Exercise 4.4.30(2)], yielding the following index formula:

$$ind(T_X) = -\frac{1}{2\pi i}\int_{S^1} tr(X^{-1}dX).$$

This example of Toeplitz operators with continuous symbol serves as both a set of basic facts to be used and a paradigm for the rest of this dissertation. We will look at Toeplitz operators with symbols from algebras other than $C(S^1)$, but our goals will still be to establish a short exact sequence, show that the Toeplitz algebra modulo compacts is isomorphic to the symbol algebra, and find an integral formula which will give us the index of a Toeplitz operator.

1.2 Toeplitz Operators on the 3-Sphere

Toeplitz operators can also be defined on manifolds other than the circle. Much of this dissertation will concern extensions of the Toeplitz algebra on the 3-sphere. The definitions and basic results for Toeplitz operators with symbols from $C(S^3)$ were detailed by Coburn [2].

Definition 1.2.1. Consider S^3 as $\{(w,z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\}$. The Hardy space on S^3, a subspace of $L^2(S^3)$ which we will notate as $H^2(S^3)$, is defined as the closed linear span of the orthonormal basis $\{h_{m_1,m_2}\}$ where m_1 and m_2 are non-negative integers and

$$h_{m_1,m_2} = c_{m_1,m_2} w^{m_1} z^{m_2}, \text{ where } c_{m_1,m_2} = \sqrt{\frac{(1+m_1+m_2)!}{m_1! m_2!}}.$$

Define $P : L^2(S^3) \to H^2(S^3)$ as the projection onto this Hardy space. For any positive integer n, extend P to the projection of $M_n(L^2(S^3))$ onto $M_n(H^2(S^3))$ which applies P to each entry. We then define a Toeplitz operator T_X, where $X \in M_n(C(S^3))$, as the operator on $(H^2(S^3))^n$ where $T_X(Y) = P(XY)$ for each $Y \in (H^2(S^3))^n$.

Let $\mathcal{T}(M_n(C(S^3)))$ be the algebra generated by Toeplitz operators with symbols from $M_n(C(S^3))$. An argument nearly identical to the argument in [4, Proposition 7.4] establishes that the $*$-map for this algebra can be defined by $(T_X)^* = T_{X^*}$. $\diamondsuit$

It will be useful for differential and integral calculations to use hyperspherical coordinates for S^3: $(e^{i\theta_1}, e^{i\theta_2}, \eta)$ where θ_1 and θ_2 are between 0 and 2π and η is between 0 and $\pi/2$. It is well known that the volume of S^3 is $2\pi^2$. In [1] it is shown that the hyperspherical coordinates are related to the coordinates (w, z) by

$$w = e^{i\theta_1}\sin(\eta) \text{ and } z = e^{i\theta_2}\cos(\eta)$$

and that the volume form is $dV = \sin(\eta)\cos(\eta)d\eta d\theta_1 d\theta_2$.

Example 1.2.2. It is clear that $P(1) = 1$ and that $P(\bar{z}) = 0$. However, it is less clear how elements involving $|z|^2$ relate to the Hardy space. We begin by using the L^2 norm and the basis of H^2 presented in Definition 1.2.1 to find the coefficients of the expansion of $P(|z|^2)$. We use the fact that $|z|^2 = z\bar{z} = \cos^2(\eta)$. We start with the basis element of 1.

$$\frac{1}{2\pi^2}\int_{S^3} 1 \cdot |z|^2 dV = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} \cos^2(\eta)\sin(\eta)\cos(\eta)d\eta d\theta_1 d\theta_2 = \frac{1}{2}.$$

However, for any $w^{m_1}z^{m_2}$ where either m_1 or m_2 is not 0,

$$\frac{1}{2\pi^2}\int_{S^3} w^{m_1}z^{m_2}\cdot|z|^2dV = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} e^{im_1\theta_1}e^{im_2\theta_2}\sin^{m_1+1}(\eta)\cos^{m_2+3}(\eta)d\eta d\theta_1 d\theta_2.$$

But this will equal 0 as $\int_0^{2\pi} e^{im\theta}d\theta = 0$ for any non-zero integer m. Thus $P(|z|^2) = 1/2$.

$\diamondsuit$

There are similar results for $P(|z|^2w^{m_1}z^{m_2})$ and $P(|w|^2w^{m_1}z^{m_2})$. The precise constants are not essential, but we will require the following lemma:

Lemma 1.2.3. *If $X = |w|^{2l_1}|z|^{2l_2}w^{n_1}z^{n_2}$ where l_1, l_2, n_1, and n_2 are non-negative integers, then $P(X) = aw^{n_1}z^{n_2}$, where a is some positive number.*

Proof. As in the previous example, we calculate the projection onto each basis element $k_{m_1,m_2}w^{m_1}z^{m_2}$ of $H^2(S^3)$.

$$\frac{1}{2\pi^2}\int_{S^3} X\cdot\overline{k_{m_1,m_2}w^{m_1}z^{m_2}}dV = \frac{k_{m_1,m_2}}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} e^{i(n_1-m_1)\theta_1}e^{i(n_2-m_2)\theta_2}$$

$$\cdot\sin^{n_1+m_1+2l_1+1}(\eta)\cos^{n_2+m_2+2l_2+1}(\eta)d\eta d\theta_1 d\theta_2.$$

If $n_1 \neq m_1$ or $n_2 \neq m_2$ then this integral will be 0. If $n_1 = m_1$ and $n_2 = m_2$, then the integral equals $2k_{m_1,m_2}\int_0^{\pi/2} \sin^{2m_1+l_1+1}(\eta)\cos^{2m_2+2l_2+1}(\eta)d\eta$. But from Definition 1.2.1 $k_{m_1,m_2} > 0$, and the integral will also be positive as sine and cosine are non-negative between 0 and $\pi/2$. $\square$

Since the z coordinate can be 0 on the 3-sphere, the function $f(w,z) = z$ is not invertible. This will be a key factor later. For now we observe that operators such as $T_{\bar{z}}$, $T_{\bar{w}}$, and T_{wz} are not Fredholm: each has an infinite dimensional kernel or cokernel. (The kernel of $T_{\bar{z}}$ contains all non-negative integer powers of w. The others have similar issues.)

Coburn [2, Lemma 2] shows that, as in the case of Toeplitz operators on the circle, T_X is compact if and only if $X = 0$ and that the symbol map σ induces an isomorphism between $\mathcal{T}(C(S^3))/\mathcal{K}$ and $C(S^3)$. Further, [2] concludes with an example of an index one operator in $\mathcal{T}(M_2(C(S^3)))$. As we will see shortly, there are no one-dimensional Toeplitz operators on the 3-sphere with non-zero index. First, to provide a basic example of the index arguments we will be making, we demonstrate that the operator from Coburn has an index of one.

Example 1.2.4. Let $X = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}$. We consider $T_X \in \mathcal{T}(M_2(C(S^3)))$.

Suppose $\begin{bmatrix} f \\ g \end{bmatrix}$, with $f, g \in H^2(S^3)$, is in the kernel of T_X. This implies that $P(wf) - P(\bar{z}g) = 0$ and that $P(zf) + P(\bar{w}g) = 0$. Rearranging, we get $wf = P(\bar{z}g)$ and $zf = -P(\bar{w}g)$. Multiplying the first equation by z and the second by w, we get

$$zP(\bar{z}g) = wzf = -wP(\bar{w}g).$$

Replace g with its expansion $\sum_{i,j=0}^{\infty} g_{i,j} w^i z^j$. Then

$$\bar{z}g = \sum_{i=0}^{\infty} g_{i,0} \bar{z} w^i + \sum_{i=0,j=1}^{\infty} g_{i,j} w^i |z|^2 z^{j-1}.$$

Applying Lemma 1.2.3, we see that

$$zP(\bar{z}g) = z \cdot \sum_{i=0,j=1}^{\infty} a_{i,j} g_{i,j} w^i z^{j-1} = \sum_{i=0,j=1}^{\infty} a_{i,j} g_{i,j} w^i z^j,$$

where every $a_{i,j} > 0$.

Similarly $-wP(\bar{w}g) = -w \cdot \sum_{i=1,j=0}^{\infty} b_{i,j} g_{i,j} w^{i-1} z^j = -\sum_{i=1,j=0}^{\infty} b_{i,j} g_{i,j} w^i z^j$, where every $b_{i,j} > 0$. Our equation then becomes $\sum_{i=0,j=1}^{\infty} a_{i,j} g_{i,j} w^i z^j = -\sum_{i=1,j=0}^{\infty} b_{i,j} g_{i,j} w^i z^j$.

Thus $g_{0,0}$ is unrestricted, and we can see that $f(w,z) = 0$, $g(w,z) = 1$ is one solution to the system. However, all $g_{i,0}$ with $i > 0$ and $g_{0,j}$ with $j > 0$ must be 0, since they only appear on one half of the equation. Further, for $i, j > 0$, $a_{i,j} g_{i,j} = -b_{i,j} g_{i,j}$. But each $a_{i,j}$ and $b_{i,j}$ was positive, so, except for $i = j = 0$ every $g_{i,j}$ must equal 0.

Finally, since g must be a constant, $P(\bar{w}g) = 0$, which implies that $wzf = 0$ everywhere on S^3. Thus $f(w,z) = 0$ off of the measure 0 subsets $\{(w,z) \,|\, w = 0\}$ and $\{(w,z) \,|\, z = 0\}$. Since f is an L^2 function, we conclude that $f = 0$ and that the kernel of T_X is of dimension 1.

We now check the adjoint $(T_X)^*$, which we know from Definition 1.2.1 is equal to T_{X^*}. But $X^* = \begin{bmatrix} \bar{w} & \bar{z} \\ -z & w \end{bmatrix}$.

Suppose $P(\bar{w}f) + P(\bar{z}g) = 0$ and $P(-zf) + P(wg) = 0$. Then $P(\bar{w}f) = -P(\bar{z}g)$ and $zf = wg$. Replace f, g with their expansions $\sum_{i=0,j=0}^{\infty} f_{i,j} w^i z^j$, $\sum_{i=0,j=0}^{\infty} g_{i,j} w^i z^j$, respectively. The first equation then becomes

$$P\left(\sum_{i=0,j=0}^{\infty} f_{i,j} |w|^2 w^{i-1} z^j\right) = -P\left(\sum_{i=0,j=0}^{\infty} |z|^2 g_{i,j} w^i z^{j-1}\right),$$

or, applying Lemma 1.2.3,

$$\sum_{i=1,j=0}^{\infty} a_{i,j} f_{i,j} |w|^2 w^{i-1} z^j = -\sum_{i=0,j=1}^{\infty} b_{i,j} g_{i,j} |z|^2 w^i z^{j-1},$$

where $a_{i,j}$, $b_{i,j}$ are positive values. We conclude from this that, for i, $j > 0$, $a_{i,j} f_{i,j} = -b_{i-1,j+1} g_{i-1,j+1}$, so $f_{i,j}$ and $g_{i-1,j+1}$ have opposite signs.

Similarly, with the second equation, we get

$$\sum_{i=0,j=0}^{\infty} f_{i,j} w^i z^{j+1} = \sum_{i=0,j=0}^{\infty} g_{i,j} w^{i+1} z^j.$$

We conclude from this that when $i = 0$, $f_{i,j} = 0$, and when $j = 0$, $g_{i,j} = 0$. Further, when $i > 0$, $f_{i,j} = g_{i-1,j+1}$. But these were supposed to have different signs. So $f_{i,j} = 0$. And when $j > 0$, $g_{i,j} = f_{i+1,j-1}$. So $g_{i,j} = 0$.

Thus the kernel of T_X^* is trivial, and the index of T_X is 1. $\diamondsuit$

We will now use these facts from Coburn to find an index formula for Toeplitz operators on $C(S^3)$.

Theorem 1.2.5. *For any positive integer n, if X is in $GL_n(C^\infty(S^3))$, then the Fredholm index of the Toeplitz operator T_X is $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} tr((X^{-1}dX)^3)$.*

Proof. As a specific instance of the Chern character from [5, Proposition 1.2], the formula $tr((X^{-1}dX))^3$ is a map from $K_1(C(S^3))$ to the de Rham cohomology of S^3. Since K_1 classes are closed under homotopy, the map is homotopy invariant. It also follows from the group operations of K_1 and de Rham cohomology that for any invertible $X, Y \in C^\infty(S^3)$,

$$\int_{S^3} tr(((XY)^{-1}d(XY))^3) = \int_{S^3} tr((X^{-1}dX)^3) + \int_{S^3} tr((Y^{-1}dY)^3).$$

This leaves us to show that, for $X = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}$, $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} tr((X^{-1}dX)^3) = 1.$

We switch X to hyperspherical coordinates and calculate:

$$X = \begin{bmatrix} e^{i\theta_1}\sin(\eta) & -e^{-i\theta_2}\cos(\eta) \\ e^{i\theta_2}\cos(\eta) & e^{-i\theta_1}\sin(\eta) \end{bmatrix}.$$

So $dX = \begin{bmatrix} ie^{i\theta_1}\sin(\eta)d\theta_1 + e^{i\theta_1}\cos(\eta)d\eta & ie^{-i\theta_2}\cos(\eta)d\theta_2 + e^{-i\theta_2}\sin(\eta)d\eta \\ ie^{i\theta_2}\cos(\eta)d\theta_2 - e^{i\theta_2}\sin(\eta)d\eta & -ie^{-i\theta_1}\sin(\eta)d\theta_1 + e^{-i\theta_1}\cos(\eta)d\eta \end{bmatrix},$

$$\text{while } X^{-1} = \begin{bmatrix} e^{-i\theta_1}\sin(\eta) & e^{-i\theta_2}\cos(\eta) \\ -e^{i\theta_2}\cos(\eta) & e^{i\theta_1}\sin(\eta) \end{bmatrix}.$$

Thus $X^{-1}dX =$

$$\begin{bmatrix} i\sin^2(\eta)d\theta_1 + i\cos^2(\eta)d\theta_2 & e^{-i(\theta_1+\theta_2)}(d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1)) \\ e^{i(\theta_1+\theta_2)}(-d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1)) & -i\sin^2(\eta)d\theta_1 - i\cos^2(\eta)d\theta_2 \end{bmatrix}.$$

Next, $tr((X^{-1}dX)^3) =$

$$3(i\sin^2(\eta)d\theta_1 + i\cos^2(\eta)d\theta_2)(d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1))$$

$$(-d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1))$$

$$-3(d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1))(-d\eta + i\sin(\eta)\cos(\eta)(d\theta_2 - d\theta_1))$$

$$(-i\sin^2(\eta)d\theta_1 - i\cos^2(\eta)d\theta_2)$$

$$= 3(i\sin^2(\eta)d\theta_1 + i\cos^2(\eta)d\theta_2)(2i\sin(\eta)\cos(\eta)d\eta \wedge (d\theta_2 - d\theta_1))$$

$$-3(2i\sin(\eta)\cos(\eta)d\eta \wedge (d\theta_2 - d\theta_1))(-i\sin^2(\eta)d\theta_1 - i\cos^2(\eta)d\theta_2)$$

$$= 12\sin(\eta)\cos(\eta)d\eta \wedge d\theta_1 \wedge d\theta_2.$$

In conclusion,

$$\frac{1}{2\pi^2}\int_{S^3}\frac{1}{12}tr((X^{-1}dX)^3) = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\sin(\eta)\cos(\eta)d\eta \wedge d\theta_1 \wedge d\theta_2 = 1.$$

□

Note that for any invertible Y in $C^\infty(S^3)$, $(Y^{-1}dY)^3 = 0$, since $dY \wedge dY = 0$, so T_Y will have an index of 0. Thus Toeplitz operators on the 3-sphere with non-zero index require matrix symbols.

We will use Theorem 1.2.5 extensively in Chapter 3.

1.3 Crossed Product Algebras

We will be concerned with the algebras of continuous functions on the circle crossed by $\mathbb{Z}_n$ and on the 3-sphere crossed by $\mathbb{Z}_n$, for any positive integer n. In both cases the action will be the rotation of one complex coordinate by $2\pi/n$. We first review the definition of a crossed product algebra and some facts particular to algebras crossed by $\mathbb{Z}_n$.

Definition 1.3.1. Let A be a $*$-algebra and G a group which acts on A; that is, G is isomorphic to a subgroup of $*$-automorphisms on A. As in [7, 7.6.1], $A \rtimes G$ is defined as $C_c(G, A)$. When G is finite, elements in $A \rtimes G$ can be expressed in the form $\sum_{\gamma \in G} a_\gamma \gamma$, where each a_γ is an element of A [3, VIII, pages 216-217]. The $*$-algebra operations for these forms are as follows:

$$\sum_{\gamma \in G} a_\gamma \gamma + \sum_{\gamma \in G} b_\gamma \gamma = \sum_{\gamma \in G} (a_\gamma + b_\gamma)\gamma, \text{ where each } a_\gamma, b_\gamma \text{ is in } A,$$

and for $a, b \in A$ and $\gamma, \alpha \in G$

$$(a\gamma)(b\alpha) = a(\gamma \cdot b)(\gamma\alpha)$$

where $\gamma \cdot b$ is the element in A that results when γ acts on b. The $*$-map is defined by $(a\gamma)^* = (\gamma^{-1} \cdot a^*)\gamma^{-1}$.

For the crossed product algebra $A \rtimes \mathbb{Z}_n$, where n is a positive integer, we will write $A \rtimes_\alpha \mathbb{Z}_n$ instead where α is the isomorphism from $\mathbb{Z}_n$ to a subgroup of automorphisms on A. Since this subgroup is generated by $\alpha(1)$, and $\alpha(m) = \alpha^m(1)$, we will use just α to stand for $\alpha(1)$ as well.

In particular, we will be concerned with $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ where $\alpha \cdot f(z) = f(e^{2\pi i/n} z)$. We will work with the representation of this algebra as the subalgebra of bounded operators on $L^2(S^1)$ where each continuous function operates by multiplication and α is represented by the operator V such that, for every $h \in L^2(S^1)$, $V(h)(z) = h(e^{2\pi i/n} z)$.

Similarly, we will consider $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, where $\alpha \cdot f(w, z) = f(w, e^{2\pi i/n} z)$, and work with the representation of this algebra as the subalgebra of bounded operators on $L^2(S^3)$ where each continuous function operates by multiplication and α is represented by the operator V such that, for every $h \in L^2(S^3)$, $V(h)(w, z) = h(w, e^{2\pi i/n} z)$.

In either case it is clear that when V operates on a continuous function f from L^2, $V(f) = \alpha \cdot f$. $\diamondsuit$

Example 1.3.2. Consider $C(S^1) \rtimes_\alpha \mathbb{Z}_3$, where α is the rotation action such that for any f in $C(S^1)$, $\alpha \cdot f(z) = f(e^{2\pi i/3} z)$. A typical element in this algebra will have the form

$f + gV + hV^2$ where f, g, and h are in $C(S^1)$ and V is the generator of $\mathbb{Z}_3$ which acts on $C(S^1)$ by α. Note that $(zV)^* = (\alpha^2 \cdot \bar{z})V^{-1} = \overline{e^{4\pi i/3} z} V^2 = e^{2\pi i/3}\bar{z}V^2$. Further, note that zV is a unitary as $(zV)^*(zV) = (e^{2\pi i/3}\bar{z}V^2)(zV) = e^{2\pi i/3}\bar{z}e^{4\pi i/3}z = 1$, and similarly $(zV)(zV)^* = 1$. ♢

We will use the K-theory of such algebras extensively. Paschke [6] outlines a procedure to determine the K-theory of $A \rtimes_\alpha \mathbb{Z}_2$. The more general case of $A \rtimes_\alpha \mathbb{Z}_n$ was not completely addressed due to the more extensive "bookkeeping" required. Luckily for us, a more recent paper by Schwietzer [10] addresses this bookkeeping and finds the K-theory of $A \rtimes_\alpha \mathbb{Z}_n$. We outline the notation and an important isomorphism from these papers. We will consider the K-theory results in the next section.

Definition 1.3.3. With n a positive integer, let A be a C^*-algebra and α a $\mathbb{Z}_n$ action on A. Following the notation in [10], we will construct a matrix representation of $A \rtimes_\alpha \mathbb{Z}_n$.

Let $A_m = \{f \in A \,|\, \alpha \cdot f = e^{2\pi m i/n} f\}$. Note that A_0 is the set of elements f such that $\alpha \cdot f = f$. Further, note that for each $m < n$, $A_m A_{n-m}$ (the closed linear span of $\{ab \,|\, a \in A_m, b \in A_{n-m}\}$) is a two-sided ideal in A_0. Let $I_m = \displaystyle\sum_{j=1}^{n-1-m} A_j A_{n-j}$.

Define the projection $E_m : A \to A_m$ by $E_m(g) = \displaystyle\frac{1}{n}\sum_{j=0}^{n-1} e^{-m\pi i/n}\alpha^j \cdot g$ for $g \in A$.

A useful result from [10, Lemma 1] is an isomorphism

$$\phi : A \rtimes \mathbb{Z}_n \to \begin{bmatrix} A_0 & A_1 & \dots & A_{n-1} \\ A_{n-1} & A_0 & \dots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}.$$

Note that the image of ϕ is a subalgebra of $M_n(A)$ — a fact we will use later to shorten some calculations.

Letting V be a generator of $\mathbb{Z}_n$, the isomorphism is defined by

$$\phi(fV^m) = \begin{bmatrix} E_0(f) & e^{-2m\pi i/n}E_1(f) & \dots & e^{-m(n-1)2\pi i/n}E_{n-1}(f) \\ E_{n-1}(f) & e^{-2m\pi i/n}E_0(f) & \dots & e^{-m(n-1)2\pi i/n}E_{n-2}(f) \\ \vdots & \vdots & \ddots & \vdots \\ E_1(f) & e^{-2m\pi i/n}E_2(f) & \dots & e^{-m(n-1)2\pi i/n}E_0(f) \end{bmatrix}.$$

Schwietzer [10, Lemma 1] also provides an inverse map for any matrix X in

$$\begin{bmatrix} A_0 & A_1 & \dots & A_{n-1} \\ A_{n-1} & A_0 & \dots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}$$

defined by

$$\phi^{-1}(X) = \frac{1}{n} \sum_{m=0}^{n-1} \left(\sum_{k=1}^{n} \sum_{l=1}^{n} e^{2(k-1)m\pi i/n} X_{l,k} \right) V^m.$$

Since all the sums are finite, we can rewrite the inverse as

$$\phi^{-1}(X) = \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} X_{l,k} \left(\sum_{m=0}^{n-1} e^{2(k-1)m\pi i/n} V^m \right).$$

This last form is easier to calculate, as $\sum_{l=1}^{n} X_{l,k}$ is the sum of the kth column. It will also be useful for determining the index of certain operators in Chapter 3. $\diamondsuit$

The isomorphism ϕ will be a great convenience for us, as the next example demonstrates.

Example 1.3.4. Consider $f + gV$ in $C(S^1) \rtimes_\alpha \mathbb{Z}_2$ where $\alpha \cdot h(z) = h(-z)$ for $h \in C(S^1)$.

Now $\phi(f+gV) = \begin{bmatrix} E_0(f+g) & E_1(f-g) \\ E_1(f+g) & E_0(f-g) \end{bmatrix}$, and the determinant of this matrix is

$$\begin{aligned} & E_0(f+g)E_0(f-g) - E_1(f-g)E_1(f+g) \\ =& \frac{1}{2}(f+g+\alpha\cdot f+\alpha\cdot g)\frac{1}{2}(f-g+\alpha\cdot f-\alpha\cdot g) \\ & -\frac{1}{2}(f-g-\alpha\cdot f+\alpha\cdot g)\frac{1}{2}(f+g-\alpha\cdot f-\alpha\cdot g) \\ =& f(\alpha\cdot f) - g(\alpha\cdot g). \end{aligned}$$

We conclude that $f+gV$ is invertible if and only if $f(\alpha\cdot f)-g(\alpha\cdot g)$ is never 0.

It is worth noting that, for the algebra in this example, A_0 is the set of even functions and A_1 the set of odd functions. Further, even if S^1 is replaced with S^3 or another manifold, the same conclusion about invertibility will hold. $\diamondsuit$

Example 1.3.5. Let $A = C(S^3) \rtimes_\alpha \mathbb{Z}_2$. We observe that

$$X = (w+\bar{w}+z-\bar{z}) + (w-\bar{w}+z+\bar{z})V$$

is a unitary because $\phi(X) = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}$, which is a unitary in $\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$.

Similarly, for any integer $n > 1$, the element Y in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ such that

$$\phi(Y) = \begin{bmatrix} 1 & 0 & & \dots & & 0 \\ 0 & \ddots & & & & \\ & & w & -\bar{z} & & \vdots \\ \vdots & & z & \bar{w} & & \vdots \\ & & & & \ddots & \\ 0 & & \dots & \dots & & 1 \end{bmatrix},$$

will also be a unitary. ♢

Paschke [6, Section 2] further argues that when X is a locally compact T_2 space, then, for the algebra $C_0(X) \rtimes_\alpha \mathbb{Z}_2$, $I_0 = \sum_{j=1}^{n-1} A_j A_{n-j}$ will be the ideal in A_0 of functions that vanish on the space of points fixed by α. We label the fixed point space Y. Paschke [6, Section 2] concludes that $A_0/I_0 \cong C_0(Y)$. We need a similar result for $A \rtimes_\alpha \mathbb{Z}_n$. The argument parallels Paschke's, but we include it here for completeness.

Lemma 1.3.6. *Let X be a locally compact T_2 space and α a $\mathbb{Z}_n$ action on X. Let $A_m = \{f \in A \,|\, \alpha \cdot f = e^{2\pi m i/n} f\}$ and $I_l = \sum_{j=1}^{n-1-l} A_j A_{n-j}$. Further, let Y be the subspace of points in X which are fixed under α. For any non-negative integer l less than n, A_0/I_l is isomorphic to $C_0(Y)$.*

Proof. Note that if $f \in I_l$, then $f = f_1 f_2$ where $f_1 \in A_k$ and $f_2 \in A_j$ such that $k, j > 0$ and $k + j = n$. But this means that $f_1(\alpha(x)) = e^{2\pi i k/n} f_1(x)$ for every $x \in X$. But Y is the set of points fixed by α, so for $y \in Y$, $f_1(y) = e^{2\pi i k/n} f_1(y)$. Thus f_1 and f will vanish on Y. Further, for every other point x in the complement of Y, there will exist functions $g_1 \in A_1$ and $g_2 \in A_{n-1}$ such that $g_1 g_2(x) \neq 0$.

Let Z be the space formed from X by identifying each x with $\alpha^k(x)$ for each positive integer k less than n. Note that $A_0 \cong C(Z)$. Thus $I_l = \{f \in A_0 \mid f(Y) = \{0\}\}$ as ideals in A_0 are defined by the sets of points on which all the functions in the ideal vanish. We conclude that $A_0/I_l \cong C(Y)$ through the restriction map. $\square$

1.4 The K-Theory of $A \rtimes_\alpha \mathbb{Z}_n$

We now focus on the K-theory results of Paschke and Schwietzer relevant to this paper.

Definition 1.4.1. Using the 2×2 matrix form of $A \rtimes_\alpha \mathbb{Z}_2$ (see Definition 1.3.3), define $\psi : A \rtimes_\alpha \mathbb{Z}_2 \to A_0/I_0$ by

$$\psi\left(\begin{bmatrix} E_0(x_1) & E_1(x_2) \\ E_1(x_1) & E_0(x_2) \end{bmatrix}\right) = E_0(x_1) + I_0.$$

Note that ψ is a homomorphism as

$$\psi\left(\begin{bmatrix} E_0(x_1) & E_1(x_2) \\ E_1(x_1) & E_0(x_2) \end{bmatrix}\begin{bmatrix} E_0(y_1) & E_1(y_2) \\ E_1(y_1) & E_0(y_2) \end{bmatrix}\right)$$

$$=\psi\left(\begin{bmatrix} E_0(x_1)E_0(y_1)+E_1(x_2)E_1(y_1) & E_0(x_1)E_1(y_1)+E_1(x_2)E_0(y_2) \\ E_1(x_1)E_0(y_1)+E_0(x_2)E_1(y_1) & E_1(x_1)E_1(y_2)+E_0(x_2)E_0(y_2) \end{bmatrix}\right)$$

$$=E_0(x_1)E_0(y_1)+I_0=\psi\left(\begin{bmatrix} E_0(x_1) & E_1(x_2) \\ E_1(x_1) & E_0(x_2) \end{bmatrix}\right)\psi\left(\begin{bmatrix} E_0(y_1) & E_1(y_2) \\ E_1(y_1) & E_0(y_2) \end{bmatrix}\right)$$

as $E_1(x_2)E_1(y_1)$ is in I_0. ♢

Using the inclusion map $i : A_0 \to A \rtimes_\alpha \mathbb{Z}_2$, Paschke [6, Theorem 1] shows that there is a six-term exact sequence:

$$\begin{array}{ccccc} K_1(A_0) & \longrightarrow & K_1(A \rtimes_\alpha \mathbb{Z}_2) & \xrightarrow{K_1(\psi)} & K_1(A_0/I_0) \\ \uparrow & & & & \downarrow \\ K_0(A_0/I_0) & \xleftarrow[K_0(\psi)]{} & K_0(A \rtimes_\alpha \mathbb{Z}_2) & \longleftarrow & K_0(A_0). \end{array} \tag{1}$$

Although we will soon make more general conclusions using Schwietzer's results, we first examine two applications of this six-term sequence as they exemplify the broader results.

Lemma 1.4.2. *If α is the action on $C(S^1)$ such that $\alpha(f(z)) = f(-z)$, then both $K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$ and $K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$ are isomorphic to $\mathbb{Z}$.*

Proof. Using the proof of Lemma 1.3.6, A_0 will be isomorphic to $C(S^1)$, as identifying each point z with $\alpha(z) = -z$ quotients S^1 to itself. (The isomorphism sends z^{2m} to z^m for every integer m.) Further, since z and $\bar{z}$ are both in A_1, $1 = z\bar{z}$ is in the ideal I_0, which implies I_0 is isomorphic to A_0. Thus $A_0/I_0 \cong \{0\}$. The six-term exact sequence (1) then yields isomorphisms between $K_1(C(S^1))$ and $K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$ and between $K_0(C(S^1))$ and $K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$. A basic K-theory result informs us that $K_i(C(S^1)) \cong \mathbb{Z}$ for $i = 0, 1$. The desired result follows. □

We note in passing that 1 is not a generating projection for $K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$. Instead, $p = \frac{1}{2} + \frac{1}{2}V$ and $q = \frac{1}{2} - \frac{1}{2}V$ are. Note that $[p]_0 = [q]_0$ as, with $v = \frac{1}{2}z + \frac{1}{2}zV$, we have $p = v^*v$ and $q = vv^*$.

We will be concerned with unitaries and K_1 equivalence classes in the next chapter.

Lemma 1.4.3. *If α is the action on $C(S^3)$ such that $\alpha(f(w, z)) = f(w, -z)$, then both $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_2)$ and $K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_2)$ are isomorphic to $\mathbb{Z}^2$.*

Proof. Here, again using the proof of Lemma 1.3.6, A_0 is isomorphic to $C(S^3)$, as identifying each point (w, z) with $\alpha(w, z) = (w, -z)$ quotients S^3 to itself. The set of fixed points is $\{(e^{i\theta_1}, 0)\}$, so $A_0/I_0 \cong C(S^1)$. (Unlike the previous case, $z\bar{z} = |z|^2$ is not identically 1, so the function 1 need not be in I_0. In fact, 1 cannot be in I_0, as $I_0 \ncong A_0$.) Note that for every positive integer j, w^j and $\bar{w}^j$ are in A_0. Further, w^j and $\bar{w}^j$ are not in $I_0 = A_1^2$ as every function $f(w, z)$ in A_1 has the property that $f(w, -z) = -f(w, z)$, which requires that $f(w, 0) = 0$. Since $w^j = e^{ji\theta_1}$ and $\bar{w}^j = e^{-ji\theta_1}$ when restricted to $\{(e^{i\theta_1}, 0)\}$, and since these restrictions form a basis for $C(S^1)$, the positive powers of w and $\bar{w}$ will be a basis for A_0/I_0.

Basic K-theory informs us that $K_i(C(S^3)) \cong K_i(C(S^1)) \cong \mathbb{Z}$ so our six-term exact sequence (1) becomes

$$\begin{array}{ccccc} \mathbb{Z} \cong K_1(A_0) & \longrightarrow & K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_2) & \xrightarrow{K_1(\psi)} & K_1(A_0/I_0) \cong \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} \cong K_0(A_0/I_0) & \xleftarrow[K_0(\psi)]{} & K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_2) & \longleftarrow & K_0(A_0) \cong \mathbb{Z}. \end{array}$$

The elements in $K_1(A_0/I_0)$ can be represented by the positive powers of w and $\bar{w}$. Since the element X from Example 1.3.5 is a unitary in $C(S^3) \rtimes_\alpha \mathbb{Z}_2$, and X has the matrix representation of $\phi(X) = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}$, we conclude that $\psi(X) = w$. As w is a generator of $K_1(A_0/I_0)$, $K_1(\psi)$ will be surjective. Similarly, with Y such that $\phi(Y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\psi(Y) = 1$, and thus $K_0(\psi)$ is surjective. It follows that the connecting maps will be 0-maps. For $i = 0, 1$, we conclude from the short exact sequence

$$0 \to \mathbb{Z} \to K_i(C(S^3) \rtimes_\alpha \mathbb{Z}_2) \to \mathbb{Z} \to 0$$

that $K_i(C(S^3) \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}^2$. □

Here again $p = \frac{1}{2} + \frac{1}{2}V$ and $q = \frac{1}{2} - \frac{1}{2}V$ are generating projections. In this case they are not equivalent. We outline the argument. Suppose there exists $v \in C(S^3) \rtimes \mathbb{Z}_2$ such that $v^*v = p$ and $vv^* = q$. Using the isomorphism ϕ, $\phi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\phi(q) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. If $\phi(v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then a few matrix calculations show that $a = b = d = 0$ and $\bar{c}c = c\bar{c} = 1$. But $c \in A_1$, so $\bar{c}c \in A_1^2 = I_0$. This is a contradiction, as 1 is not in I_0.

Generalizing, Schwietzer [10] considers $\mathbb{Z}_n$ actions for any positive integer n. To calculate the K-theory of such algebras he uses the notation of the following definition.

Definition 1.4.4. For a $*$-algebra A, integers j and n such that $0 < j \leq n$, and a $\mathbb{Z}_n$ action α, we know that ϕ from Definition 1.3.3 is an isomorphism from $A \rtimes_\alpha \mathbb{Z}_n$ to

$$B = \begin{bmatrix} A_0 & A_1 & \dots & A_{n-1} \\ A_{n-1} & A_0 & \dots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}.$$

Following the notation of [10], we define jth lower right corner of an $n \times n$ matrix X in B as the projection of X onto the subalgebra

$$\begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & A_0 & A_1 & \dots & A_{n-j} \\ \vdots & & A_{n-1} & A_0 & \dots & A_{n-(j+1)} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & A_j & A_{j+1} & \dots & A_0 \end{bmatrix}.$$

We label this subalgebra B_j and note that the non-zero entries are contained in a $j \times j$ block.

Although we will only need to use the matrix form of this projection, the related projection in $A \rtimes_\alpha \mathbb{Z}_n$ would project $\phi^{-1}(X) = \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{k,l=1}^{n} e^{2(l-1)\pi i/n} X_{k,l}\right) V^m$ to $\frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{k,l=j}^{n} e^{2(l-1)\pi i/n} X_{k,l}\right) V^m$.

Let ψ_j be the quotient map of the (j,j)th entry of B_j (the upper left corner of the non-zero entries) into A_0/I_{j-1}. It can be shown that ψ_j is a homomorphism by an argument similar to the one for ψ in Definition 1.4.1. Note that B_1 is $A \rtimes_\alpha \mathbb{Z}_n$, and B_n is A_0. $\diamondsuit$

Schwietzer [10] then constructs the set of $n-1$ six-term exact sequences

$$\begin{array}{ccccc} K_1(B_i) & \longrightarrow & K_1(B_{i-1}) & \xrightarrow{K_1(\psi_{i-1})} & K_1(A_0/I_{i-2}) \\ \uparrow & & & & \downarrow \\ K_0(A_0/I_{i-2}) & \xleftarrow[K_0(\psi_{i-1})]{} & K_0(B_{i-1}) & \longleftarrow & K_0(B_i). \end{array} \tag{2}$$

Note that when $n = 2$, the one and only six-term exact sequence is the same as Paschke's.

We again find the relevant K-groups. Note that for an integer j such that $0 \le j < n$, $z^j \in A_j$ and $\bar{z}^j \in A_{n-j}$.

Lemma 1.4.5. *If α is the action on $C(S^1)$ such that $\alpha(f(z)) = f(e^{2\pi i/n} z)$, then both $K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ and $K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ are isomorphic to $\mathbb{Z}$.*

Proof. As before, applying the proof of Lemma 1.3.6, A_0 is isomorphic to $C(S^1)$ (this isomorphism is effected by sending z^{nm} to z^m for every integer m.) Further, for any integer j such that $0 \leq j < n$, $1 = z^j \bar{z}^j$ is in the ideal I_j, which implies I_j is isomorphic to A_0. Thus $A_0/I_j \cong \{0\}$. Each of the six-term exact sequences then yields isomorphisms between $K_*(B_j)$ and $K_*(B_{j-1})$. Thus $\mathbb{Z} \cong K_*(C(S^1)) \cong K_*(A_0) \cong K_*(B_n) \cong K_*(B_{n-1}) \cong ... \cong K_*(B_1) \cong K_*(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$. $\square$

Lemma 1.4.6. *If α is the action on $C(S^3)$ such that $\alpha(f(w,z)) = f(w, e^{2\pi i/n}z)$, then $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and $K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ are both isomorphic to $\mathbb{Z}^n$.*

Proof. As before, A_0 is isomorphic to $C(S^3)$. Our first sequence is then

$$\begin{array}{ccccc} K_1(B_n) \cong K_1(A_0) \cong \mathbb{Z} & \longrightarrow & K_1(B_{n-1}) & \xrightarrow{K_1(\psi_{n-1})} & K_1(A_0/I_{n-2}) \\ \uparrow & & & & \downarrow \\ K_0(A_0/I_{n-2}) & \xleftarrow[K_0(\psi_{n-1})]{} & K_0(B_{n-1}) & \longleftarrow & K_0(B_n) \cong K_0(A_0) \cong \mathbb{Z}. \end{array}$$

For any integer j such that $0 \leq j < n$, Lemma 1.3.6 informs us that $A_0/I_j \cong C(S^1)$. So our first sequence becomes

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_1(B_{n-1}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_0(B_{n-1}) & \longleftarrow & \mathbb{Z} \end{array}$$

and the successive sequences will be

$$\begin{array}{ccccc} K_1(B_j) & \longrightarrow & K_1(B_{j-1}) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_0(B_{j-1}) & \longleftarrow & K_0(B_j). \end{array}$$

Now, when $0 < j < n$, define X_j and Y_j as the elements in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ such that

$$\phi(X_j) = \begin{bmatrix} 1 & 0 & & \dots & & 0 \\ 0 & \ddots & & & & \\ & & w & -z & & \vdots \\ \vdots & & \bar{z} & \bar{w} & & \vdots \\ & & & & \ddots & \\ 0 & & \dots & \dots & & 1 \end{bmatrix}$$

with the w in the (j, j)th entry, and

$$\phi(Y_j) = \begin{bmatrix} 0 & 0 & & \dots & & 0 \\ 0 & \ddots & & & & \\ & & 1 & & & \vdots \\ \vdots & & & 0 & & \vdots \\ & & & & \ddots & \\ 0 & & \dots & \dots & & 0 \end{bmatrix},$$

with the 1 in the (j, j)th entry. Note that X_j is a unitary and Y_j a projection. Further, their projections into B_j are a unitary and a projection in B_j respectively. But ψ_j maps these elements to the generators of $K_*(C(S^1))$. Thus $K_*(\psi_j)$ will be surjective and the connecting maps will be 0-maps. It follows that each $K_*(B_{j-1})$ is isomorphic to $K_*(B_j) \oplus \mathbb{Z}$.

In the first sequence this results in $K_*(B_{n-1}) \cong \mathbb{Z}^2$. Each of the successive $n - 2$ sequences will result in another copy of $\mathbb{Z}$ being added on. We can conclude that $K_*(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}^n$. □

2 Toeplitz Operators with Symbols from $C(S^1) \rtimes_\alpha \mathbb{Z}_n$

In this chapter we will develop the index theory of Toeplitz operators with symbols from $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ where α is a rotation. We will start with examples and direct calculations of index. We will then find a short exact sequence, calculate the K-theory for the algebra, and construct an index formula.

2.1 Definitions and Examples

Definition 2.1.1. Having selected any positive integer n, let α be the homomorphism from the group $\mathbb{Z}_n$ to the automorphism group of $C(S^1)$ where the automorphism $\alpha(1)$ acts by $\alpha(1) \cdot f(z) = f(e^{2\pi i/n}z)$. As mentioned in Definition 1.3.1, we will use just α to stand for $\alpha(1)$ as well. The crossed product algebra that results, $C(S^1) \rtimes_\alpha \mathbb{Z}_n$, can be viewed as the subalgebra of $B(L^2(S^1))$ generated by multiplication operators with symbols from $C(S^1)$ and the operator, V, which sends $f(z)$ to $f(e^{2\pi i/n}z)$ for every f in $L^2(S^1)$.

We will need differential forms for our index formula, so we define a linear map d on the crossed product algebra by $d\left(\sum_{i=0}^{n-1} f_i V^i\right) = \sum_{i=0}^{n-1}(df_i)V^i$, where d on the right hand side is the usual exterior derivative. We decree that $d(V) = 0$. With $z = e^{i\theta}$, and for any continuous function f, set $V(fd\theta) = (\alpha \cdot f)d\theta V$. Because d will be part of the index formula, we will restrict the formula to elements of $C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n$. $\diamondsuit$

Example 2.1.2. Note that $dz = d(e^{i\theta}) = ie^{i\theta}d\theta = izd\theta$. Thus $V(dz) = V(izd\theta) = i(\alpha \cdot z)d\theta V = ie^{2\pi i/n}zd\theta V = e^{2\pi i/n}dzV$. We will frequently use the fact that V acts on dz similarly to how it acts on z.

More generally,

$$\begin{aligned} &d(V(f(z))) = d(f(e^{2\pi i/n}z)V) = e^{2\pi i/n}f'(e^{2\pi i/n}z)dzV \\ =&f'(e^{2\pi i/n}z)i(e^{2\pi i/n}z)d\theta V = V(f'(z)izd\theta) = V(f'(z)dz) = V(d(f(z))). \end{aligned} \qquad \diamondsuit$$

We next verify the product rule for d by direct calculation. There is nothing special about functions on the circle here — the lemma applies to any algebra $A \rtimes_\alpha \mathbb{Z}_n$ as long as an exterior derivative is defined on A.

Lemma 2.1.3. *Let A be an algebra with an exterior derivative defined on it, and let the derivative d be applied entrywise to the crossed product algebra $A \rtimes_\alpha \mathbb{Z}_n$. Let V be the element which implements the action α, and let $dV = 0$. If $X = \sum_{i=0}^{n-1} f_i V^i$ and $Y = \sum_{j=0}^{n-1} g_j V^j$ are in $A \rtimes_\alpha \mathbb{Z}_n$, then $d(XY) = (dX)Y + X(dY)$.*

Proof. $d(XY) = d\left(\left(\sum_{i=0}^{n-1} f_i V^i\right)\left(\sum_{j=0}^{n-1} g_j V^j\right)\right)$

$$\begin{aligned} &= d\left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_i(\alpha^i \cdot g_j)V^{i+j}\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\big((df_i)(\alpha^i \cdot g_j) + f_i d(\alpha^i \cdot g_j)\big)V^{i+j} \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(df_i)V^i g_j V^j + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f_i V^i (dg_j)V^j = (dX)Y + X(dY). \end{aligned} \qquad \square$$

Note that for any positive integer m, $M_m(A \rtimes_\alpha \mathbb{Z}_n) \cong M_m(A) \rtimes \mathbb{Z}_n$. Since, applying the derivative d entrywise, $M_m(A)$ is an algebra with an exterior derivative, Lemma 2.1.3 will apply to matrices of crossed products as well.

Definition 2.1.4. Let $\Omega^1(C(S^1))$ the space of 1-forms on the circle and $\Omega^1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ be the space of 1-forms on the crossed product algebra; that is, $\Omega^1(C(S^1) \rtimes_\alpha \mathbb{Z}_n) = \left\{ \sum_{i=0}^{n-1} \omega_i V^i \,|\, \text{each } \omega_i \in \Omega^1(C(S^1)) \right\}$.

In the following sections we will sometimes be interested in only part of a 1-form on $C(S^1) \rtimes_\alpha \mathbb{Z}_n$. To capture the part we are interested in, we define the map ν : $\Omega^1(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \to \Omega^1(C(S^1))$ by $\nu\left(\sum_{i=0}^{n-1} \omega_i V^i\right) = \omega_0$. Further, let π be the quotient map from $\Omega^1(C(S^1))$ to $\Omega^1(C(S^1))/Z$ where Z is the subspace of exact forms.

As in Section 1.1, let P be the projection onto the Hardy space and, for any $X \in C(S^1) \rtimes_\alpha \mathbb{Z}_n$, let $T_X = PX$. Define $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ as the C^*-algebra generated by the set of Toeplitz operators $\{T_X \,|\, X \in C(S^1) \rtimes_\alpha \mathbb{Z}_n\}$.

Note that the Hardy space is invariant under V, so $PV - VP = 0$. $\diamondsuit$

Although not essential for the arguments of this chapter, we provide a few examples that are of interest and may help illumine later results.

Example 2.1.5. We demonstrate that zV in $C(S^1) \rtimes_\alpha \mathbb{Z}_2$ is homotopic via invertible elements to z. Clearly zV is homotopic to izV, so we will present a homotopy through invertibles from izV to z.

For $t \in [0,1]$, let $X_t = tz + i(1-t)zV$. By our test for invertibility from Example 1.3.4, X_t will be invertible since $tz(\alpha\cdot(tz)) - i(1-t)z(\alpha\cdot(i(1-t)z)) = -t^2z^2 - (1-t)^2z^2 = (-2t^2+2t-1)z^2$ is nowhere vanishing for all $t \in [0,1]$. Since $X_0 = izV$ and $X_1 = z$, zV is homotopic to z. $\diamondsuit$

The next example gives an instance where T_X has index 0 but is not invertible — a phenomenon that does not happen when X is in $C(S^1)$ [4, Corollary 7.25]. It also shows the difficulty of calculating the index directly for even fairly basic examples of operators in $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ — something an index formula will save us from in the future.

Example 2.1.6. Consider $T_{z+(-z+\bar{z})V}$ in $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_2)$. Note that the adjoint is $T_{\bar{z}+(-z+\bar{z})V}$. We find the kernel for each.

Suppose $T_{z+(-z+\bar{z})V}(f) = 0$ for $f = \displaystyle\sum_{j=0}^{\infty} a_j z^j$ in the Hardy space. Then

$$\begin{aligned}
&P\left(z\sum_{j=0}^{\infty} a_j z^j\right) - P\left(z\sum_{j=0}^{\infty} a_j (-z)^j\right) + P\left(\bar{z}\sum_{j=0}^{\infty} a_j (-z)^j\right) = 0\\
\Rightarrow\ & 2z\sum_{j=0}^{\infty} a_{2j+1} z^{2j+1} = -P\left(\sum_{j=0}^{\infty} a_j (-1)^j z^{j-1}\right)\\
\Rightarrow\ & 2\sum_{j=0}^{\infty} a_{2j+1} z^{2j+2} = -\sum_{j=1}^{\infty} a_j (-1)^j z^{j-1}.
\end{aligned}$$

Since the left side of the last equation has only even powers of z, starting at z^2, matching with the odd indexed constants, every a_{2j}, except for a_0, must be 0. Our equation is then $2\sum_{j=0}^{\infty} a_{2j+1} z^{2j+2} = \sum_{j=0}^{\infty} a_{2j+1} z^{2j}$.

Matching powers of z from both sides, we get that $2a_{2j+1} = a_{2j+3}$ for $j \geq 0$. Clearly no such sequence of coefficients will converge unless each $a_{2j+1} = 0$. However, a_0 is still free, so the kernel of $T_{z+(-z+\bar{z})V}$ is the span of $\{1\}$.

We now check the adjoint.

Suppose $T_{\bar{z}+(-z+\bar{z})V}(f) = 0$ for $f = \sum_{j=0}^{\infty} a_j z^j$ in the Hardy space. Then

$$P\left(\bar{z}\sum_{j=0}^{\infty} a_j z^j\right) - P\left(z\sum_{j=0}^{\infty} a_j (-z)^j\right) + P\left(\bar{z}\sum_{j=0}^{\infty} a_j (-z)^j\right) = 0$$
$$\Rightarrow 2P\left(\sum_{j=0}^{\infty} a_{2j} z^{2j-1}\right) = z\sum_{j=0}^{\infty} a_j (-1)^j z^j$$
$$\Rightarrow 2\sum_{j=1}^{\infty} a_{2j} z^{2j-1} = \sum_{j=0}^{\infty} a_j (-1)^j z^{j+1}.$$

Since the left side of the last equation has only odd powers of z, starting at z^1, matching with the even indexed constants, every a_{2j+1} must be 0. Our equation is then $2\sum_{j=1}^{\infty} a_{2j} z^{2j-1} = \sum_{j=0}^{\infty} a_{2j} z^{2j+1}$.

Matching powers of z from both sides, we get that $2a_{2j} = a_{2j-2}$ for $j > 0$. Selecting any constant a_0, this will create a convergent sequence, so the kernel of $T_{\bar{z}+(-z+\bar{z})V}$ is the span of $\sum_{j=0}^{\infty} a_0 \frac{1}{2^j} z^{2j}$.

Since both of these kernels are one-dimensional, the index of $T_{z+(-z+\bar{z})V}$ is 0. But since the kernels are non-trivial, the operator is not invertible. $\diamondsuit$

Lastly, we give an example of some of the calculations that will be needed to use our promised index formula.

Example 2.1.7. Let $X = 3z^2 + z^{-4}V^2$ in $C(S^1) \rtimes_\alpha \mathbb{Z}_3$. Note that $dX = 6zdz - 4z^{-5}V^2dz$ and that $X^{-1} = \dfrac{9z^4 + e^{2\pi i/3}z^{-8}V - 3e^{4\pi i/3}z^{-2}V^2}{27z^6 + z^{-12}}$. (The inverse can be found by switching to the matrix form of Definition 1.3.3, or by inspection.)

We calculate $\nu(X^{-1}dX)$.

$$\nu(X^{-1}dX) = \frac{9z^4 \cdot 6zdz - e^{2\pi i/3}z^{-8}\alpha \cdot (4z^{-5}dz)}{27z^6 + z^{-12}} = \frac{54z^5 - 4z^{-13}}{27z^6 + z^{-12}}dz. \qquad \diamondsuit$$

2.2 Compact Operators and the Symbol Map

In Theorem 2.2.2 we will establish an isomorphism, which we label φ, between $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ and $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$. The isomorphism is defined by $\varphi(X) = T_X + \mathcal{K}$, where X is an element of $C(S^1) \rtimes_\alpha \mathbb{Z}_n$. With this isomorphism the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \to \mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K} \to 0.$$

can be rewritten as

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \xrightarrow{\sigma} C(S^1) \times_\alpha \mathbb{Z}_n \to 0.$$

where σ is the symbol map. This short exact sequence will later be used to find the K-theory of $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ and to help calculate the indices of operators.

As in Definition 1.1.1, for any symbols X and Y, we will call the Toeplitz operator $T_{XY} - T_X T_Y$ a semi-commutator. We start with a lemma about semi-commutators.

Lemma 2.2.1. *Let m and n be positive integers. For all X, Y in $M_m(C(S^1) \times_\alpha \mathbb{Z}_n)$, $T_{XY} - T_X T_Y$ is a compact operator.*

Proof. We begin with the case where $m = 1$.

Both [4, Proposition 7.22] and [2, Theorem 1] inform us that semi-commutators in $\mathcal{T}(C(S^1))$ are compact. For any $X = \sum_{l=0}^{n-1} f_l V^l, Y = \sum_{k=0}^{n-1} g_k V^k$ in $C(S^1) \rtimes_\alpha \mathbb{Z}_n$,

$$\begin{aligned}
T_{XY} - T_X T_Y &= PXY - PXPY \\
&= P\left(\sum_{l=0}^{n-1} f_l V^l\right)\left(\sum_{k=0}^{n-1} g_k V^k\right) - P\left(\sum_{l=0}^{n-1} f_l V^l\right) P\left(\sum_{k=0}^{n-1} g_k V^k\right) \\
&= P\left(\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} f_l V^l g_k V^k - f_l P V^l g_k V^k\right) \\
&= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1} (T_{f_l(\alpha^l \cdot g_k)} - T_{f_l} T_{(\alpha^l \cdot g_k)}) V^{l+k}.
\end{aligned}$$

But since each $T_{f_l(\alpha^l \cdot g_k)} - T_{f_l} T_{(\alpha^l \cdot g_k)}$ is compact, the sum is compact as well. It follows that $\varphi(XY) = T_{XY} + \mathcal{K} = T_X T_Y + \mathcal{K} = \varphi(X)\varphi(Y)$.

When $m > 1$, a calculation verifies that each entry of $T_{XY} - T_X T_Y$ is a sum of operators of the form $T_{X_{j_1 \cdot j_2} Y_{l_1,l_2}} - T_{X_{j_1,j_2}} T_{Y_{l_1,l_2}}$ where each X_{j_1,j_2} and Y_{l_1,l_2} is in $C(S^1) \rtimes_\alpha \mathbb{Z}_n$. Thus each entry, and therefore the entire semi-commutator, is compact. $\square$

Theorem 2.2.2. *The map $\varphi : C(S^1) \rtimes_\alpha \mathbb{Z}_n \to \mathcal{T}(C(S^1) \times_\alpha \mathbb{Z}_n)/\mathcal{K}$ defined by $\varphi(X) = T_X + \mathcal{K}$ for each X in $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ is an isomorphism.*

Proof. We begin by showing that for any X in $C(S^1) \times_\alpha \mathbb{Z}_n$, $\|X\| = \|T_X\|$. It will follow that φ is contractive and therefore continuous. Note that we are considering X as an operator, so the operator norm will be applied.

Let $X = \sum_{j=0}^{n-1} f_j V^j$ and $\epsilon > 0$. Clearly $\|X\| \geq \|T_X\|$. Next, pick h in $L^2(S^1)$ such that $\|h\| = 1$ and $\|Xh\| > \|X\| - \epsilon/2$. Note that

$$\|X(z^{nk}h)\| = \left\| \sum_{j=0}^{n-1} f_j(z) V^j(z^{nk}h(z)) \right\| = \|z^{nk}(X)h\| = \|Xh\|$$

for every integer k as $V(z^{nk}) = (e^{2\pi i/n}z)^{nk} = z^{nk}$. This fact allows us to shift Xh to $Xz^{kn}h$ without changing norm. However, as k increases, the expansion of $f_j z^{kn} h$ will be moved more and more into the Hardy space, so the impact of the Toeplitz projection P

will vanish. That is, for each f_j there exists an integer k_j such that for any integer $L \geq k_j$ $\|Pf_j(z)z^{nL}h(z) - f_j(z)z^{nL}h(z)\| < \epsilon/2n$. Letting $k = \max\{k_j\}_{0\leq j\leq n-1}$, it follows that $\|T_X(z^{nk}h(z)) - Xz^{nk}h(z)\| < \epsilon/2$ and that $\|T_X(z^{nk}h)\| > \|Xz^{nk}h\| - \epsilon/2 = \|Xh\| - \epsilon/2$. Thus $\|T_X\| \geq \|T_X(z^{nk}h)\| > \|Xh\| - \epsilon/2 > \|X\| - \epsilon$. So $\|X\| = \|T_X\|$.

Further, applying Lemma 2.2.1, we see that φ is an algebra homomorphism since for any X, Y in $C(S^1) \rtimes_\alpha \mathbb{Z}_n$, $\varphi(XY) = T_{XY} + \mathcal{K} = T_X T_Y + \mathcal{K} = \varphi(X)\varphi(Y)$.

Next, we show that φ is injective.

Suppose T_X is compact and $X = \sum_{j=0}^{n-1} f_j V^j$ with each f_j in $C(S^1)$. Because T_X is compact, for any $\epsilon > 0$ there exists N such that for every integer $m > N$, $\|T_X(z^m)\| < \epsilon/(2n)$. And, multiplying by z^m to shift each f_j more and more into the Hardy space, we can select $M > N$ such that for all integers $m > M$,

$$\|T_X(z^m)\| = \left\| P\left(\sum_{j=0}^{n-1} f_j(z)\cdot(e^{2\pi ij/n}z)^m\right)\right\| > \left\|\sum_{j=0}^{n-1} f_j(z)\cdot(e^{2\pi ij/n}z)^m\right\| - \epsilon/2.$$

Without loss of generality, let m be a multiple of n. So, for each integer l such that $0 \leq l < n$,

$$\|nf_l\| \leq \left\|\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} e^{2\pi i(j-l)k/n} f_j\right\| \leq \sum_{k=0}^{n-1}\left\|\sum_{j=0}^{n-1} e^{2\pi i(j-l)k/n} f_j\right\|$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \left\| \sum_{j=0}^{n-1} e^{2\pi ijk/n} f_j \right\| \\
&= \sum_{k=0}^{n-1} \left\| \left(\sum_{j=0}^{n-1} e^{2\pi ijk/n} f_j \right) z^{m+k} \right\| \\
&\leq \sum_{k=0}^{n-1} \left(\left\| P \left(\left(\sum_{j=0}^{n-1} e^{2\pi ijk/n} f_j \right) z^{m+k} \right) \right\| + \epsilon/(2n) \right) \\
&= \sum_{k=0}^{n-1} \| T_X(z^{m+k}) \| + \epsilon/2 < \epsilon.
\end{aligned}
$$

Thus each $f_l = 0$, so φ is injective.

Finally, as φ is an algebra homomorphism whose image includes all the generators of $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, it is surjective.

Thus φ is an isomorphism. The short exact sequence follows. □

Using Lemmas 2.2.1 and 2.2.2, it is straightforward to verify that, for any positive integer m, $M_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \cong M_m(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n))/\mathcal{K}$.

2.3 K-Theory

We now look at the K-theory of $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$. We will use the results from Lemma 1.4.5 and apply a similar argument to the classical one for the K-theory of $\mathcal{T}(C(S^1))$.

Theorem 2.3.1. *$K_1(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n))$ is isomorphic to 0 and $K_0(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n))$ is isomorphic to $\mathbb{Z}$.*

Proof. We turn our attention to the six-term exact sequence

$$\begin{array}{ccccc} K_1(\mathcal{K}) & \longrightarrow & K_1(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) & \xrightarrow{K_1(\sigma)} & K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \\ \uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_n) & \xleftarrow[K_0(\sigma)]{} & K_0(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) & \longleftarrow & K_0(\mathcal{K}) \end{array}$$

arising from the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \xrightarrow{\sigma} C(S^1) \rtimes_\alpha \mathbb{Z}_n \to 0.$$

From Lemma 1.4.5 we know that, for $i = 0, 1$, $K_i(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}$. Further, it is known from [8, Corollary 6.4.2, Example 8.2.9] that $K_0(\mathcal{K}) \cong \mathbb{Z}$ and $K_1(\mathcal{K}) \cong 0$. So the six-term sequence becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) & \xrightarrow{K_1(\sigma)} & \mathbb{Z} \cong K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \\ \uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ \mathbb{Z} & \xleftarrow[K_0(\sigma)]{} & K_0(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) & \longleftarrow & \mathbb{Z}. \end{array}$$

Clearly, $K_0(\sigma)$ is surjective and $K_1(\sigma)$ is injective. If δ_1 is an isomorphism, then the rest of the sequence will also be known. We now show this to be the case.

$K_0(\mathcal{K})$ is generated by the rank one projection, F_1, which projects onto the space spanned by the first basis element, 1, of the Hardy space. Using a standard definition of the index map [8, Proposition 9.2.4], we apply an argument similar to the one used in [8, Example 9.4.4, Exercise 12.4] to calculate $K_1(\mathcal{T}(C(S^1)))$.

Considering $[z]_1 \in K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, we compute $\delta_1([z]_1)$ by lifting the unitary $\begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$ in $M_2(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ to the unitary $\begin{bmatrix} T_z & 1 - T_z T_{\bar{z}} \\ 0 & T_{\bar{z}} \end{bmatrix}$ in $M_2(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n))$. We then evaluate

$$\begin{bmatrix} T_z & 1 - T_z T_{\bar{z}} \\ 0 & T_{\bar{z}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_z & 1 - T_z T_{\bar{z}} \\ 0 & T_{\bar{z}} \end{bmatrix}^* = \begin{bmatrix} T_z T_{\bar{z}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 - F_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The last equality follows from Example 1.1.2.

By the standard definition of the index map, $\delta_1([z]_1)$ will be this last matrix with any constant terms removed, so $\delta_1([z]_1) = -[F_1]_0$. Since $[F_1]$ was a generator of $K_0(\mathcal{K})$, δ_1 is surjective. But a surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ must be an isomorphism. Therefore δ_1 is an isomorphism.

Returning to the six-term exact sequence, δ_1 being an isomorphism implies that $K_1(\sigma)$ is the zero-map. Since $K_1(\sigma)$ was also injective, $K_1(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) \cong 0$. Further, the map from $K_0(\mathcal{K})$ to $K_0(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n))$ must be the zero-map, implying that $K_0(\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)) \cong K_0(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}$. □

It follows from the above argument that $[z]_1$ generates $K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, because it is mapped through an isomorphism to a generator of $K_0(\mathcal{K})$. We will use this fact in the next section.

2.4 Index Formula

The index formula will be similar to the standard formula for winding number mentioned in Section 1.1, but it will require the additional map ν from Definition 2.1.4 to restrict a 1-form on $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ to a 1-form on $C(S^1)$, allowing us to integrate. We begin with two necessary, if technical, lemmas.

Lemma 2.4.1. *Let X be an element of $M_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, and let Y be a 1-form from $M_m(\Omega^1(C(S^1) \rtimes_\alpha \mathbb{Z}_n))$. If $\sum_{-\infty}^{\infty} a_j h_j$ is the power series expansion of $\nu(tr(XY)) - \nu(tr(YX))$, then whenever $j \equiv -1 \mod n$, a_j will be 0. In particular, the difference $\nu(tr(XY)) - \nu(tr(YX))$ will be an exact form.*

Proof. We show the result holds when $m = 1$. The result will follow for $m > 1$ as $\nu(tr(XY)) - \nu(tr(YX)) = \sum_{i=0,j=0}^{m-1,m-1} (\nu(X_{i,j}Y_{j,i}) - \nu(Y_{j,i}X_{i,j}))$ where $X_{i,j}$ is the (i,j)th entry of X and $Y_{j,i}$ is the (j,i)th entry of Y. Thus the $m \times m$ case reduces to the one-dimensional case.

For $X = \sum_{w=0}^{n-1} f_w V^w$ in $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ and $Y = \sum_{j=0}^{n-1} g_j dz V^j$ in $\Omega^1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, it is clear that $\nu(XY)$ will be the sum of the products $(f_w V^w)(g_j dz V^j) = f_w(\alpha^w \cdot (g_j dz))$ where $w + j = n$ or 0. Similarly $\nu(YX)$ will be the sum of the products $(g_j dz V^j)(f_w V^w) = g_j dz(\alpha^j \cdot f_w)$ where $w + j = n$ or 0.

When $w + j = 0$ then $w = j = 0$ and the relevant terms are $f_0 V^0$ and $g_0 V^0$. But clearly, $(f_0 V^0)(g_0 V^0) = f_0 g_0 = g_0 f_0 = (g_0 V^0)(f_0 V^0)$.

We next examine any particular w and j such that $w + j = n$.

Let f_w have the power series expansion $\sum_{-\infty}^{\infty} a_k z^k$ and g_j the expansion $\sum_{-\infty}^{\infty} b_l z^l$, where each a_k and b_l is a constant. Since this lemma is concerned only with $-1 \bmod n$ terms, and we will be multiplying z^k and z^l, the terms we are concerned with are precisely those where $k + l \equiv -1 \bmod n$. But using Example 2.1.2 and noting that $w = -j + n$, we see that for each such k and l,

$$
\begin{aligned}
(a_k z^k V^w)(b_l z^l dz V^j) &= a_k z^k (\alpha^w \cdot (b_l z^l dz)) V^{w+j} \\
= a_k b_l z^k (\alpha^w \cdot (z^l))(\alpha^w \cdot (dz)) &= a_k b_l z^{k+l} e^{2\pi l w i/n} e^{2\pi w i/n} dz \\
= a_k b_l z^{k+l} e^{(l+1) 2\pi w i/n} dz &= a_k b_l z^{k+l} e^{(-k \bmod n) 2\pi w i/n} dz \\
= a_k b_l z^{k+l} e^{(-k) 2\pi w i/n} dz &= a_k b_l z^{k+l} e^{(jk) 2\pi i/n} dz.
\end{aligned}
$$

But $(b_l z^l dz V^j)(a_k z^k V^w) = a_k b_l z^l dz (\alpha^j \cdot (z^k)) = a_k b_l z^{k+l} e^{(jk) 2\pi i/n} dz$ as well. Thus the $-1 \bmod n$ terms of $\nu(XY)$ are identical to those of $\nu(YX)$ since they match for each relevant selection of k and l.

We conclude by noting that the expansion of $\nu(XY) - \nu(YX)$ will have 0 as the constant for its $z^{-1} dz$ term. Since every other term integrates to 0 over the circle, $\nu(XY) - \nu(YX)$ is an exact form. □

Lemma 2.4.2. *If X and Y are elements of $GL_m(C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n)$ and X is homotopic to Y in $GL_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, then X is also homotopic to Y in $GL_m(C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n)$.*

Proof. For $t \in [0,1]$, let $X_t = \sum_{i=0}^{n-1} (f_i)_t V^i$ (where each $(f_i)_t$ is in $M_m(C(S^1))$ and V^i applies to each entry) be a homotopy from X to Y in $GL_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$. Define $\tilde{\nu} : M_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n) \to M_m(C(S^1))$ by $\tilde{\nu}\left(\sum_{i=0}^{n-1} h_i V^i\right) = h_0$. Since $\tilde{\nu}$ is continuous, $\tilde{\nu}(X_t) = (f_0)_t$ will vary continuously in $M_m(C(S^1))$. Similarly, for each i, $\tilde{\nu}(X_t V^{n-i}) = (f_i)_t$, so each $(f_i)_t$ will also vary continuously in $M_m(C(S^1))$. However, we can approximate each path $(f_i)_t$ in $M_m(C(S^1))$ with a path $(g_i)_t$ in $M_m(C^\infty(S^1))$ as closely as needed to make $\sum_{i=0}^{n-1} (g_i)_t V^i$ a homotopy between X and Y in $GL_m(C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n)$. □

We define the map Ch and then use it to define and verify our index formula.

Definition 2.4.3. Let the map $Ch : GL_m(C^1(S^1) \rtimes_\alpha \mathbb{Z}_n) \to \Omega^1(C(S^1))$ be defined by $Ch(X) = \nu(tr(X^{-1}dX))$. ♢

Theorem 2.4.4. *If X is in $GL_m(C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n)$, then the index of T_X is equal to* $-\frac{1}{2\pi i}\int_{S^1} Ch(X) = -\frac{1}{2\pi i}\int_{S^1} \nu(tr(X^{-1}dX))$.

Proof. The following argument has a similar outline to that of the standard proof for the index formula for Toeplitz operators with continuous symbols mentioned in Section 1.1. We begin by demonstrating that our formula has certain index like properties, although only Property 2 is actually required to complete the proof.

Property 1: $\int_{S^1} Ch(XY) = \int_{S^1} Ch(X) + \int_{S^1} Ch(Y)$.

Compute:

$$\begin{aligned}\int_{S^1}(Ch(XY)) &= \int_{S^1}\nu(tr((Y^{-1}X^{-1})d(XY)))\\ &= \int_{S^1}\nu(tr((Y^{-1}X^{-1})((dX)Y+XdY)))\\ &= \int_{S^1}\nu(tr(Y^{-1}X^{-1}(dX)Y)) + \int_{S^1}\nu(tr(Y^{-1}dY))\\ &= \int_{S^1}\nu(tr(Y^{-1}X^{-1}(dX)Y)) + \int_{S^1}Ch(Y).\end{aligned}$$

But, by Lemma 2.4.1,

$$\int_{S^1}\nu(tr(Y^{-1}X^{-1}(dX)Y)) = \int_{S^1}\nu(tr(YY^{-1}X^{-1}dX)) = \int_{S^1}Ch(X),$$

verifying Property 1.

Property 2: If X is homotopic to Y in $GL_m(C^\infty(S^1)\rtimes_\alpha \mathbb{Z}_n)$, then $\int_{S^1}Ch(X) = \int_{S^1}Ch(Y)$.

Let X_t be a differentiable homotopy between X and Y. We will show that, for each $t\in[0,1]$, $\frac{\partial}{\partial t}(Ch(X_t))$ is an exact form in $\Omega^1(C(S^1))$.

Compute:

$$\frac{\partial}{\partial t}(Ch(X_t)) = \frac{\partial}{\partial t}(\nu(tr(X_t^{-1}dX_t)))$$

$$= \nu\left(tr\left(\frac{\partial}{\partial t}(X_t^{-1}dX_t)\right)\right)$$
$$= \nu\left(tr\left(\left(\frac{\partial X_t^{-1}}{\partial t}\right)(dX_t) + X_t^{-1}\left(\frac{\partial}{\partial t}dX_t\right)\right)\right)$$
$$= \nu\left(tr\left((-X_t^{-1})\frac{\partial X_t}{\partial t}(X_t^{-1})(dX_t) + X_t^{-1}\frac{\partial}{\partial t}(dX_t)\right)\right).$$

But, using the fact that $\frac{\partial}{\partial t}(dX_t) = d\left(\frac{\partial X_t}{\partial t}\right)$ and Lemma 2.4.1, we see that

$$\nu\left(tr\left((-X_t^{-1})\frac{\partial X_t}{\partial t}(X_t^{-1})(dX_t)\right)\right) - \nu\left(tr\left((X_t^{-1})(dX_t)(-X_t^{-1})\frac{\partial X_t}{\partial t}\right)\right)$$

is an exact form. Let this difference be called J. We conclude that

$$\begin{aligned}
&\nu\left(tr\left((-X_t^{-1})\frac{\partial X_t}{\partial t}(X_t^{-1})(dX_t) + X_t^{-1}\frac{\partial}{\partial t}(dX_t)\right)\right)\\
=&\,\nu\left(tr\left((X_t^{-1})(dX_t)(-X_t^{-1})\frac{\partial X_t}{\partial t} + X_t^{-1}d\left(\frac{\partial X_t}{\partial t}\right)\right)\right) + J\\
=&\,\nu\left(tr\left(d(X_t^{-1})\frac{\partial X_t}{\partial t} + X_t^{-1}d\left(\frac{\partial X_t}{\partial t}\right)\right)\right) + J\\
=&\,\nu\left(tr\left(d\left(X_t^{-1}\frac{\partial X_t}{\partial t}\right)\right)\right) + J\\
=&\,d\left(\nu\left(tr\left(X_t^{-1}\frac{\partial X_t}{\partial t}\right)\right)\right) + J,
\end{aligned}$$

is an exact form.

Let π be the quotient map from $\Omega^1(C(S^1))$ to $\Omega^1(C(S^1))/Z$, where we let Z be the space of exact forms. We see that for every $t \in [0,1]$, $\frac{\partial}{\partial t}(Ch(X_t))$ is in Z, so $\pi \circ Ch$ is homotopy invariant. Thus $\pi(Ch(X)) = \pi(Ch(Y))$. We conclude that $Ch(X) = Ch(Y) +$ an exact form, so $\int_{S^1} Ch(X) = \int_{S^1} Ch(Y)$, proving the second property.

With these properties established, we can now argue that $-\frac{1}{2\pi i}\int_{S^1} Ch(X)$ computes the index of T_X.

We know from Example 1.1.2 that the index of T_z is -1. Further, it is clear that $\frac{1}{2\pi i}\int_{S^1} Ch(z) = \frac{1}{2\pi i}\int_{S^1} \nu(z^{-1}dz) = \frac{1}{2\pi i}\int_{S^1} \frac{dz}{z} = 1$. We also know that, for any Fredholm T_X with X in $M_m(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, X must also be invertible since $T_X + \mathcal{K}$ is invertible and the isomorphism φ from Theorem 2.2.2 maps X to $T_X + \mathcal{K}$. Thus, X is homotopic through invertible elements to a unitary [8, Proposition 2.1.7], and further, since $[z]_1$ generates $K_1(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$, X is homotopic through invertible elements to

$$Y = \begin{bmatrix} z^k & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$$

for some integer k. Further, by Lemma 2.4.2, we can construct this homotopy to be in $GL_m(C^\infty(S^1) \rtimes_\alpha \mathbb{Z}_n)$ so that Ch will be defined along the entire homotopy. A straightforward calculation shows that $Ch(Y) = kz^{-1}dz$.

Applying Property 2, we get

$$-\frac{1}{2\pi i}\int_{S^1} Ch(X) = -\frac{1}{2\pi i}\int_{S^1} Ch(Y) = -\frac{k}{2\pi i}\int_{S^1} z^{-1}dz = -k.$$

Since T_Y has index $-k$ and index is homotopy invariant in Fredholm operators, T_X also has an index of $-k$, so $-\frac{1}{2\pi i}\int_{S^1} Ch(X)$ is the index formula. □

Example 2.4.5. Let $X = 3z^2 + z^{-4}V^2$ in $C(S^1) \rtimes_\alpha \mathbb{Z}_3$. From Example 2.1.7 we know that $Ch(X) = \nu(X^{-1}dX) = \frac{54z^5 - 4z^{-13}}{27z^6 + z^{-12}}dz$. Evaluating, we get that $-\frac{1}{2\pi i}\int_{S^1} Ch(X) = -2$. We conclude that the index of T_X is -2. ♢

3 Toeplitz Operators with Symbols from $C(S^3) \rtimes_\alpha \mathbb{Z}_n$

The argument in this final chapter will follow the basic structure of the previous chapter; however, the proofs will be more involved and the resulting index formula more interesting. The additional complexity stems from the facts that $z\bar{z}$ is not equal to 1 in $C(S^3)$ and that the set of points on S^3 that are fixed by rotation is the set $\{(w, 0) \mid w \in S^1\}$, not the empty set. We will rely heavily on the map ϕ from $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ to a subalgebra of $M_n(C(S^3))$ from Definition 1.3.3 and the K-theory discussed in Section 1.4.

3.1 Definitions and Examples

In this section we will update the definitions from the previous chapter to apply in the case of the 3-sphere.

Definition 3.1.1. Having selected any positive integer n, let α be the homomorphism from the group $\mathbb{Z}_n$ to the automorphism group of $C(S^3)$ where $\alpha(1)$ acts by $\alpha(1) \cdot f(w, z) = f(w, e^{2\pi i/n} z)$. As in previous chapters, we will use just α to stand for $\alpha(1)$ as well. Note that with $w = e^{i\theta_1} \sin\eta$ and $z = e^{i\theta_2} \cos\eta$, we could define α by $\alpha \cdot f(\theta_1, \theta_2, \eta) = f(\theta_1, \theta_2 + 2\pi i/n, \eta)$. In particular, $\alpha \cdot (e^{i\theta_1}) = e^{i\theta_1}$ and $\alpha \cdot (e^{i\theta_2}) = e^{2\pi i/n} e^{i\theta_2}$.

The crossed product algebra that results, $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, can be viewed as the subalgebra of $B(L^2(S^3))$ generated by multiplication operators with symbols from $C(S^3)$ and the operator, V, which sends $f(w, z)$ to $f(w, e^{2\pi i/n} z)$ for every f in $L^2(S^3)$.

Again we will need differential forms for our index formula. As before we define a linear map d on the crossed product algebra by $d\left(\sum_{i=0}^{n-1} f_i V^i\right) = \sum_{i=0}^{n-1}(df_i)V^i$, where d on the right hand side is the usual exterior derivative. We decree that $d(V) = 0$. With $w = e^{i\theta_1}\sin\eta$ and $z = e^{i\theta_2}\cos\eta$, for any $f \in C(S^3)$ set $V(fd\theta_1) = (\alpha \cdot f)d\theta_1 V$, $V(fd\theta_2) = (\alpha \cdot f)d\theta_2 V$, and $V(fd\eta) = (\alpha \cdot f)d\eta V$. ♢

Example 3.1.2. Note that $dz = d(e^{i\theta_2}\cos\eta) = ie^{i\theta_2}\cos\eta d\theta_2 - e^{i\theta_2}\sin\eta d\eta$. Thus $V(dz) = \alpha \cdot (ie^{i\theta_2}\cos\eta d\theta_2 - e^{i\theta_2}\sin\eta d\eta)V = e^{2\pi i/n}(ie^{i\theta_2}\cos\eta d\theta_2 - e^{i\theta_2}\sin\eta d\eta)V = e^{2\pi i/n}dzV$.

Similarly, $dw = d(e^{i\theta_1}\sin\eta) = ie^{i\theta_1}\sin\eta d\theta_1 + e^{i\theta_1}\cos\eta d\eta$. And so $V(dw) = \alpha \cdot (ie^{i\theta_1}\sin\eta d\theta_1 + e^{i\theta_1}\cos\eta d\eta)V = (ie^{i\theta_1}\sin\eta d\theta_1 + e^{i\theta_1}\cos\eta d\eta)V = dwV$. ♢

The product rule for d follows from Lemma 2.1.3.

Definition 3.1.3. For $l = 1$, 2, or 3, let $\Omega^l(C(S^3))$ be the space of l-forms on the 3-sphere, and let $\Omega^l(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ be the space of l-forms on the crossed product algebra; that is, $\Omega^l(C(S^3) \rtimes_\alpha \mathbb{Z}_n) = \left\{\sum_{i=0}^{n-1} \omega_i V^i \,|\, \text{each } \omega_i \in \Omega^l(C(S^3))\right\}$.

We redefine the map $\nu : \Omega^3(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \to \Omega^3(C(S^3))$ by $\nu\left(\sum_{j=0}^{n-1} \omega_j V^j\right) = \omega_0$, where each ω_j is a 3-form. Further, let π be the quotient map from $\Omega^3(C(S^3))$ to $\Omega^3(C(S^3))/Z$ where Z is the subspace of exact forms.

For $l = 1$, 2, or 3 and an integer k such that $0 \leq k < n$, we define $\Omega^l(A_k) = \{\omega \in \Omega^l(C(S^3)) \,|\, V(\omega) = e^{2\pi i/n}\omega\}$. It follows from Example 3.1.2 that, for appropriate integers l_1, l_2, k_1, and k_2, if $\omega_1 \in \Omega^{l_1}(A_{k_1})$ and $\omega_2 \in \Omega^{l_2}(A_{k_2})$ then $\omega_1\omega_2 \in \Omega^{l_1+l_2}(A_{(k_1+k_2) \bmod n})$. We allow l_1 or l_2 to be 0 here with the understanding that $\Omega^0(A_k) = A_k$.

As in Section 1.2, let P be the projection onto $H^2(S^3)$, and let $T_X = PX$. Define $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ as the algebra generated by the set of Toeplitz operators $\{T_X \,|\, X \in C(S^3) \rtimes_\alpha \mathbb{Z}_n\}$.

Note that V acts invariantly on $H^2(S^3)$, so $PV - VP = 0$. $\diamondsuit$

Definition 3.1.4. We also redefine the map Ch so that we can use it in this chapter. Let $Ch : GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n) \to \Omega^3(C(S^3))$ be defined by $Ch(X) = \nu(tr((X^{-1}dX)^3))$. $\diamondsuit$

We present a modification of Lemma 2.4.1 for 3-forms on the 3-sphere.

Lemma 3.1.5. *Let X be a 2-form element of $M_m(\Omega^2(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ and Y a 1-form from $M_m(\Omega^1(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$. The difference of $\int_{S^3} \nu(tr(XY))$ and $\int_{S^3} \nu(tr(YX))$ will be an exact form.*

Proof. We will prove that the statement is true when $m = 1$. The more general result will then follow from the property of the trace that for matrices X and Y with entries $X_{i,j}$

and $Y_{i,j}$ respectively, $tr(XY) = \sum_{i,j} X_{i,j}Y_{j,i}$ and $tr(YX) = \sum_{i,j} Y_{i,j}X_{j,i}$. So the difference between these traces is a sum of one-dimensional $X_{i,j}Y_{j,i} - Y_{j,i}X_{i,j}$.

In a manner similar to that of Definition 1.3.3, for $0 \leq k < n$ and $l = 1, 2$, or 3, let E_k^l be the projection from $\Omega^l(C(S^3))$ to $\Omega^l(A_k)$. Note that $\sum_{k=0}^{n-1} E_k^l$ is the identity on $\Omega^l(C(S^3))$.

Suppose $\omega \in \Omega^3(C(S^3))$. Then, using the coordinates $(\theta_1, \theta_2, \eta)$, and the volume form $\sin\eta\cos\eta d\theta_2 d\eta d\theta_1$, we get

$$\omega = f(\theta_1, \theta_2, \eta)\sin\eta\cos\eta d\theta_2 d\eta d\theta_1$$

for some function $f(\theta_1, \theta_2, \eta) \in C(S^3)$. Further, the integral over S^3 can be written as the triple integral $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\pi} f(\theta_1, \theta_2, \eta)\sin\eta\cos\eta d\theta_2 d\eta d\theta_1$. We will only need to focus on the first part of this integral: $\int_0^{2\pi} f(\theta_1, \theta_2, \eta) d\theta_2$.

Since the action α takes θ_2 to $(\theta_2 + 2\pi/n) \bmod 2\pi$, it is clear that, for any function $g \in C(S^3)$, $\int_0^{2\pi} V(g d\theta_2) = \int_0^{2\pi} g d\theta_2$. However, when $k \neq 0$, we get that $\int_0^{2\pi} E_k^1(f d\theta_2) = \int_0^{2\pi} V(E_k^1(f d\theta_2)) = e^{2\pi k i/n}\int_0^{2\pi} E_k^1(f d\theta_2)$. It follows that $\int_0^{2\pi} E_k^1(f d\theta_2) = 0$. We conclude that $\int_0^{2\pi} f d\theta_2 = \int_0^{2\pi} E_0^1(f d\theta_2)$, and therefore $\int_{S^3} \omega = \int_{S^3} E_0^3(\omega)$.

We next show that $E_0^3(\nu(XY) - \nu(YX)) = 0$.

For $X = \sum_{w=0}^{n-1} f_w V^w$, where each f_w is a 2-form, and $Y = \sum_{j=0}^{n-1} g_j V^j$, where each g_j is a 1-form, it is clear that $\nu(XY)$ will be the sum of the products $(f_w V^w)(g_j V^j)$ where $w + j = n$ or 0. Similarly, $\nu(YX)$ will be the sum of the products $(g_j V^j)(f_w V^w)$ where $w + j = n$ or 0.

If $w + j = 0$ then $w = j = 0$ and the relevant terms are $f_0 V^0$ and $g_0 V^0$. But clearly, $(f_0 V^0)(g_0 V^0) = f_0 g_0 = g_0 f_0 = (g_0 V^0)(f_0 V^0)$.

We next examine any particular w and j such that $w + j = n$.

Note that $\alpha^j \cdot (f_w(\alpha^w \cdot g_j)) = (\alpha^j \cdot f_w) g_j$. Thus $E_0^3((f_w V^w)(g_j V^j) - (g_j V^j)(f_w V^w)) = E_0^3((f_w V^w)(g_j V^j) - \alpha^j \cdot ((f_w V^w)(g_j V^j)))$. But we also have that $E_0^3(\alpha^j \cdot ((f_w V^w)(g_j V^j))) = E_0^3((f_w V^w)(g_j V^j))$, so $E_0^3((f_w V^w)(g_j V^j) - (g_j V^j)(f_w V^w)) = 0$. So $E_0^3(\nu(XY) - \nu(YX)) = 0$, and thus $\int_{S^3} (\nu(XY) - \nu(YX)) = \int_{S^3} E_0^3(\nu(XY) - \nu(YX)) = 0$.

In conclusion, by de Rham's theorem, integration over S^3 is an isomorphism between the de Rham cohomology group $H^3(S^3; \mathbb{C})$ and $\mathbb{C}$. Thus any 3-form on S^3 which integrates to 0 is an exact form. □

Corollary 3.1.6. *If E_0^3 is the projection from $\Omega^3(C(S^3))$ onto $\Omega^3(A_0)$, then for any $\omega \in \Omega^3(C(S^3))$, $\int_{S^3} \omega = \int_{S^3} E_0^3(\omega)$. Further, if $\omega \in \Omega^3(A_l)$ where $l \neq 0$, then ω is an exact form and $\int_{S^3} \omega = 0$.*

Proof. Immediate from the argument of Lemma 3.1.5. □

3.2 The Symbol Map

Considering Toeplitz operators with symbols from $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, we will show that there exists the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \xrightarrow{\sigma} C(S^3) \rtimes_\alpha \mathbb{Z}_n \to 0$$

where σ is the symbol map. This sequence extends naturally to the short exact sequence of $m \times m$ matrices

$$0 \to M_m(\mathcal{K}) \to M_m(\mathcal{T}(C(S^3))) \xrightarrow{\sigma} M_m(C(S^3)) \to 0.$$

We establish the first short exact sequence by showing that $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ is isomorphic to $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$.

Lemma 3.2.1. *The map $\varphi : C(S^3) \rtimes_\alpha \mathbb{Z}_n \to \mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$ defined by $\varphi(X) = T_X + \mathcal{K}$, is an isomorphism.*

Proof. We will show that φ is a $*$-map, multiplicative, and bijective.

It is already known from [2, Theorem 1] that φ restricted to $C(S^3)$ is a $*$-isomorphism from $C(S^3)$ to $\mathcal{T}(C(S^3))/\mathcal{K}$. We consider those Toeplitz operators involving the operator

V. For any functions f, h in $H^2(S^3)$, any function g in $C(S^3)$, and any integer l such that $0 \leq l < n$ note that $\langle T_{gV^l} f, h\rangle = \langle Pg(\alpha^l \cdot f), h\rangle = \langle \alpha^l \cdot f, \bar{g}h\rangle = \langle f, P(\alpha^{-l} \cdot (\bar{g}h))\rangle = \langle f, T_{(\alpha^{-l} \cdot \bar{g})V^{n-l}} h\rangle$. Thus $T^*_{gV^l} = T_{(\alpha^{-l}\cdot\bar{g})V^{n-l}} = T_{(gV^l)^*}$, and, by linearity, for any X in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, $\varphi(X)^* = (T_X + \mathcal{K})^* = (T_X)^* + \mathcal{K} = T_{X^*} + \mathcal{K} = \varphi(X^*)$.

Next, with P the projection onto the Hardy space, note that $PV = VP$ since V acts invariantly on the Hardy space. Further, from Coburn [2, Theorem 1], semi-commutators in $\mathcal{T}(C(S^3))$ are compact. So, following a nearly identical argument to Lemma 2.2.1, for any $X = \sum_{l=0}^{n-1} f_l V^l, Y = \sum_{k=0}^{n-1} g_k V^k$ in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$,

$$\begin{aligned}
T_{XY} - T_X T_Y &= PXY - PXPY \\
&= P\left(\sum_{l=0}^{n-1} f_l V^l\right)\left(\sum_{k=0}^{n-1} g_k V^k\right) - P\left(\sum_{l=0}^{n-1} f_l V^l\right) P \left(\sum_{k=0}^{n-1} g_k V^k\right) \\
&= P\left(\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} f_l V^l g_k V^k - f_l P V^l g_k V^k\right) \\
&= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1} (T_{f_l(\alpha^l\cdot g_k)} - T_{f_l(\alpha^l\cdot g_k)})V^{l+k}.
\end{aligned}$$

But each $T_{f_l(\alpha^l\cdot g_k)} - T_{f_l(\alpha^l\cdot g_k)}$ is compact from [2, Theorem 1], so the sum is compact as well. It follows that $\varphi(XY) = T_{XY} + \mathcal{K} = T_X T_Y + \mathcal{K} = \varphi(X)\varphi(Y)$, establishing that φ is multiplicative.

To show that φ is injective, suppose T_X is compact for some $X = \sum_{l=0}^{n-1} f_l V^l$ in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$. Then for positive integers a and b such that $a + b = n$ we have that $T_{z^a} T_X$ is compact. Using the compactness of semi-commutators, we get that $T_{z^a X}$ is compact. This implies that $T_{z^a X} T_{z^b}$ is compact, which, applying the compactness of semi-commutators again, implies that $T_{z^a X z^b}$ is also compact.

But $z^a X z^b = z^a \sum_{l=0}^{n-1} f_l V^l z^b = \sum_{l=0}^{n-1} e^{2\pi bl/n} z^n f_l V^l$.

Letting a range from 0 to $n-1$, we get n compact operators $T_{z^a X z^b}$ which we can sum into another compact operator T_W where $W = \sum_{a=0}^{n-1} \sum_{l=0}^{n-1} e^{2\pi i(n-a)l/n} z^n f_l V^l = \sum_{l=0}^{n-1} \sum_{a=0}^{n-1} e^{-2\pi ial/n} z^n f_l V^l$. It is a standard result for roots of unity that $\sum_{a=0}^{n-1} e^{-2\pi ial/n} = 0$ unless $l = 0$. So $W = \sum_{a=0}^{n-1} z^n f_0 = nz^n f_0$. Thus T_W is a compact operator in $\mathcal{T}(C(S^3))$ and so a result from Coburn [2, Lemma 2] informs us that $W = nz^n f_0 = 0$. Since f_0 is continuous and $z^n = 0$ only on a measure 0 subset of S^3, we conclude that $f_0 = 0$.

Similarly, for each non-negative integer k less than n, we can construct n compact operators $T_{e^{2\pi ika/n} z^a X z^b}$, whose sum is the compact operator T_{W_k} where $W_k = \sum_{a=0}^{n-1} \sum_{l=0}^{n-1} e^{2\pi i(ka-la)/n} z^n f_l V^l = nz^n f_k V^k$. So $T_{W_k} V^{n-k} = T_{nz^n f_k}$ is also compact. As before, this requires that $f_k = 0$. Thus φ is injective.

The surjectivity of φ is clear as the image of φ will contain all the generators of $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$. □

3.3 K-Theory

We now look at the K-theory of $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$. We will use the results from Lemma 1.4.6 and the fact that $\mathcal{T}(C(S^3)\rtimes_\alpha\mathbb{Z}_n)$ is a subalgebra of the algebra of bounded operators on $H^2(S^3)$, so $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$ is a subalgebra of the related Calkin algebra.

Theorem 3.3.1. *$K_1(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ is isomorphic to $\mathbb{Z}^{n-1}$ and $K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ is isomorphic to $\mathbb{Z}^n$.*

Proof. We turn our attention to the six-term exact sequence

$$\begin{array}{ccccc} K_1(\mathcal{K}) & \longrightarrow & K_1(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) & \xrightarrow{K_1(\sigma)} & K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \\ \uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_n) & \xleftarrow[K_0(\sigma)]{} & K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) & \longleftarrow & K_0(\mathcal{K}) \end{array}$$

arising from the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \xrightarrow{\sigma} C(S^3) \rtimes_\alpha \mathbb{Z}_n \to 0.$$

Using the results from Lemma 1.4.6 — that $K_i(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}^n$ when $i = 0$ or 1 — and the facts that $K_0(\mathcal{K}) \cong \mathbb{Z}$ and $K_1(\mathcal{K}) \cong 0$, the six-term sequence becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) & \xrightarrow{K_1(\sigma)} & \mathbb{Z}^n \cong K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \\ \uparrow & & & & \downarrow{\scriptstyle \delta_1} \\ \mathbb{Z}^n & \xleftarrow[K_0(\sigma)]{} & K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) & \longleftarrow & \mathbb{Z}. \end{array}$$

Clearly, $K_0(\sigma)$ is surjective and $K_1(\sigma)$ is injective. If δ_1 is surjective, then the rest of the sequence will be known. We make a brief argument that this is the case using the result of Lemma 3.2.1 that $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ is isomorphic to $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K}$. It is clear from the contruction of the maps φ and σ that, for any $T_Y \in \mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$, $\varphi(\sigma(Y)) = T_Y + \mathcal{K}$.

Consider $[T_X + \mathcal{K}]_1$ in $K_1(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)/\mathcal{K})$ where X is the element from Example 1.2.4. We select any unitary v in $M_2(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ such that $\varphi(\sigma(v)) = \begin{bmatrix} T_X + \mathcal{K} & 0 \\ 0 & T_{X^*} + \mathcal{K} \end{bmatrix}$. Using the definition of the connecting map δ_1 found in [8, Proposition 9.1.4], we know that $\delta_1([T_X + \mathcal{K}]_1)$ will be determined by v. However, the element v is also in $M_2(B(H^2(S^3)))$, while $T_X + \mathcal{K}$ and $T_{X^*} + \mathcal{K}$ are also in the Calkin algebra $B(H^2(S^3))/\mathcal{K}$. Letting ρ be the quotient map from $B(H^2(S^3))$ to $B(H^2(S^3))/\mathcal{K}$ and $\tilde{\delta}$ the new connecting map, we have the six term sequence

$$\begin{array}{ccccc} K_1(\mathcal{K}) & \longrightarrow & K_1(B(H^2(S^3))) & \xrightarrow{K_1(\rho)} & K_1(B(H^2(S^3))/\mathcal{K}) \\ \uparrow & & & & \downarrow \tilde{\delta} \\ K_0(B(H^2(S^3))/\mathcal{K}) & \longleftarrow & K_0(B(H^2(S^3))) & \longleftarrow & K_0(\mathcal{K}). \end{array}$$

However, for any $T_Y \in \mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$, ρ and $\varphi \circ \sigma$ will be equivalent, since both will send T_Y to $T_Y + \mathcal{K}$. It follows from this fact, and from the definition of the connecting map from [8, Proposition 9.1.4], that $\tilde{\delta}([T_X + \mathcal{K}]_1)$ is determined by v in the exact same manner as $\delta_1([T_X + \mathcal{K}]_1)$ was, so $\delta_1([T_X + \mathcal{K}]_1) = \tilde{\delta}([T_X + \mathcal{K}]_1)$. But for any Fredholm operator $W \in B(H^2(S^3))$, $K_0(Tr) \circ \tilde{\delta}([\rho(W)]_1)$ gives the index of W [8, Proposition 9.4.2], and we have shown in Example 1.2.4 that the index of T_X is 1. Thus $[T_X + \mathcal{K}]_1$ is mapped by both $\tilde{\delta}$ and δ_1 to a generator of $K_0(\mathcal{K})$. Since $K_0(\mathcal{K}) \cong \mathbb{Z}$, which only requires one generator, δ_1 is surjective.

Returning to the six-term exact sequence, δ_1 being surjective and $K_1(\sigma)$ being injective requires that $K_1(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{n-1}$. Further, the map from $K_0(\mathcal{K})$ to $K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ must be the 0-map, implying that $K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)) \cong K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}^n$. □

It is clear that the generators of $K_0(\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n))$ will map to the generators of $K_0(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$. Refer to the note after Lemma 1.4.3 for the generating projections when $n = 2$.

In the last few sections we will find generating unitaries, X_i, of $C(S^3) \rtimes_\alpha \mathbb{Z}_n$. It is clear from the K-theory that all except one of these will be related to a generating unitary in $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$. In fact, there must exist generating unitaries $T_{X_i} + L_i$ where each L_i is some compact operator. Precisely which compact operators should be used for each unitary, however, is unknown.

3.4 Maps for the Index Formula

$K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}^n$, so it is not too shocking that our index formula will require n different parts added together. We will view each part as corresponding to a generator of one copy of $\mathbb{Z}$ in the K_1 group. It will be necessary to show that each of these parts has the same two index-like properties we saw in Theorem 2.4.4. The first part of our formula simply uses the map ν to apply the formula of Theorem 1.2.5 to the crossed product algebra. We begin by proving that this formula still has the necessary index-like properties in this new context.

Lemma 3.4.1. *If X and Y are in $GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$, then*

$$\int_{S^3} Ch(XY) = \int_{S^3} Ch(X) + \int_{S^3} Ch(Y).$$

Proof. Compute: $\int_{S^3}(Ch(XY)) = \int_{S^3} \nu(tr[(Y^{-1}X^{-1}d(XY))^3])$

$$\begin{aligned}
&= \int_{S^3} \nu(tr[(Y^{-1}X^{-1}((dX)Y + X(dY)))^3]) \\
&= \int_{S^3} \nu(tr[(Y^{-1}X^{-1}(dX)Y + Y^{-1}dY)^3]) \\
&= \int_{S^3} \nu(tr[(Y^{-1}X^{-1}(dX)Y)^3]) \\
&\quad + \int_{S^3} \nu(tr[3(Y^{-1}X^{-1}(dX)Y)^2(Y^{-1}dY) + 3(Y^{-1}dY)(Y^{-1}X^{-1}(dX)Y)(Y^{-1}dY)]) \\
&\quad + \int_{S^3} \nu(tr[(Y^{-1}dY)^3]).
\end{aligned}$$

The last equality here applies Lemma 3.1.5 to collect the terms in the middle integral. It is further noted that

$$\begin{aligned}
&3(Y^{-1}X^{-1}(dX)Y)^2(Y^{-1}dY) + 3(Y^{-1}dY)(Y^{-1}X^{-1}(dX)Y)(Y^{-1}dY) \\
&= 3Y^{-1}X^{-1}(dX)X^{-1}dXdY + 3Y^{-1}(dY)Y^{-1}X^{-1}dXdY \\
&= 3d(-Y^{-1}X^{-1}dXdY).
\end{aligned}$$

Since this is an exact form, and ν, tr, and d are all linear maps, we get that

$$\nu(tr(3d(-Y^{-1}X^{-1}dXdY))) = d(\nu(tr(-3Y^{-1}X^{-1}dXdY)))$$

is also an exact form. Thus

$$\int_{S^3} \nu(tr[3(Y^{-1}X^{-1}(dX)Y)^2(Y^{-1}dY) + 3(Y^{-1}dY)(Y^{-1}X^{-1}(dX)Y)(Y^{-1}dY)]) = 0.$$

Applying Lemma 3.1.5 one more time we get

$$\begin{aligned}\int_{S^3} \nu(tr[(Y^{-1}X^{-1}(dX)Y)^3]) &= \int_{S^3} \nu(tr[Y^{-1}X^{-1}(dX)X^{-1}(dX)X^{-1}(dX)Y]) \\ &= \int_{S^3} \nu(tr[(X^{-1}(dX))^3]) \\ &= \int_{S^3} Ch(X).\end{aligned}$$

Thus $\int_{S^3} Ch(XY) = \int_{S^3} Ch(X) + \int_{S^3} Ch(Y)$. □

Lemma 3.4.2. *If X and Y are in $GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and X is homotopic to Y via invertibles, then* $\int_{S^3} Ch(X) = \int_{S^3} Ch(Y)$.

Proof. Let X_t be a homotopy via invertibles from X to Y. We will show that $\frac{\partial}{\partial t}(Ch(X_t))$ is an exact form. To simplify expressions, let $B = X_t^{-1}dX_t$.

Compute:

$$\begin{aligned}\frac{\partial}{\partial t}(Ch(X_t)) &= \frac{\partial}{\partial t}(\nu(tr[B^3])) \\ &= \nu\left(tr\left[\frac{\partial}{\partial t}(B^3)\right]\right) \\ &= \nu\left(tr\left[\left(\frac{\partial B}{\partial t}B^2 + B\frac{\partial B}{\partial t}B + B^2\frac{\partial B}{\partial t}\right)\right]\right).\end{aligned}$$

Now Lemma 3.1.5 implies that $\nu\left(tr\left(\frac{\partial B}{\partial t}B^2\right)\right)$ and $\nu\left(tr\left(B\frac{\partial B}{\partial t}B\right)\right)$ differ by an exact form which we label J_1. Further, $\nu\left(tr\left(\frac{\partial B}{\partial t}B^2\right)\right)$ and $\nu\left(tr\left(B^2\frac{\partial B}{\partial t}\right)\right)$ also differ by an exact form which we label J_2. Thus $\nu\left(tr\left[\left(\frac{\partial B}{\partial t}B^2 + B\frac{\partial B}{\partial t}B + B^2\frac{\partial B}{\partial t}\right)\right]\right) = 3\nu\left(tr\left[\frac{\partial B}{\partial t}B^2\right]\right) + J_1 + J_2$.

We replace $\frac{\partial B}{\partial t}$ with $-X_t^{-1}\frac{\partial X_t}{\partial t}X_t^{-1}dX_t + X_t^{-1}\frac{\partial}{\partial t}(dX_t)$, giving us

$$\frac{\partial}{\partial t}(Ch(X_t)) = 3\nu\left(tr\left[\left(-X_t^{-1}\frac{\partial X_t}{\partial t}X_t^{-1}dX_t + X_t^{-1}\frac{\partial}{\partial t}(dX_t)\right)B^2\right]\right) + J_1 + J_2.$$

We apply Lemma 3.1.5 one more time to notice that $\nu\left(tr\left[-X_t^{-1}\frac{\partial X_t}{\partial t}X_t^{-1}dX_tB^2\right]\right)$ and $\nu\left(tr\left[-X_t^{-1}dX_tX_t^{-1}\frac{\partial X_t}{\partial t}B^2\right]\right)$ differ by an exact form, which we label J_3. Using the facts that $\frac{\partial}{\partial t}(dX_t) = d\left(\frac{\partial X_t}{\partial t}\right)$ and that, by the product rule for 1-forms, $d(B^2) = 0$, we conclude that

$$3\nu\left(tr\left[\left(-X_t^{-1}\frac{\partial X_t}{\partial t}X_t^{-1}dX_t + X_t^{-1}\frac{\partial}{\partial t}(dX_t)\right)B^2\right]\right) + J_1 + J_2$$

$$
\begin{aligned}
=&3\nu \left(tr \left[\left(-X_t^{-1} dX_t X_t^{-1} \frac{\partial X_t}{\partial t} B^2 + X_t^{-1} \frac{\partial}{\partial t}(dX_t) \right) B^2 \right] \right) + J_1 + J_2 + J_3 \\
=&3\nu \left(tr \left[\left(-X_t^{-1} dX_t X_t^{-1} \frac{\partial X_t}{\partial t} B^2 + X_t^{-1} d\left(\frac{\partial X_t}{\partial t} \right) \right) B^2 \right] \right) + J_1 + J_2 + J_3 \\
=&3\nu \left(tr \left[d \left(X_t^{-1} \frac{\partial X_t}{\partial t} B^2 \right) \right] \right) + J_1 + J_2 + J_3 \\
=&d \left(3\nu \left(tr \left[X_t^{-1} \frac{\partial X_t}{\partial t} B^2 \right] \right) \right) + J_1 + J_2 + J_3,
\end{aligned}
$$

an exact form.

Letting π be the quotient map from $\Omega^3(C(S^3))$ to $\Omega^3(C(S^3))/Z$, where Z is the space of exact forms, we see that for each $t \in [0,1]$, $\frac{\partial}{\partial t}(Ch(X_t))$ is in Z, so $\pi \circ Ch$ is homotopy invariant. Thus $\pi(Ch(X)) = \pi(Ch(Y))$. We conclude that $Ch(X) = Ch(Y) +$ an exact form, so $\int_{S^3} Ch(X) = \int_{S^3} Ch(Y)$. □

The other parts of the index formula we are constructing will use the isomorphism ϕ from Definition 1.3.3 to represent elements of $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ as matrices in a subalgebra of $M_n(C(S^3))$ and the algebra homomorphisms ψ_i, $0 < i < n$, from Definition 1.4.4 to map from these matrices into A_0/I_{i-1}. Recall from Lemmas 1.3.6 and 1.4.6 that for each i, $A_0/I_{i-1} \cong C(S^1)$.

Definition 3.4.3. For any positive integer m, extend the map ϕ from Definition 1.3.3 to the map $\tilde{\phi} : M_m(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \to M_{mn}(C(S^3))$ defined by sending each $m \times m$

matrix $\begin{bmatrix} x_{1,1} & \dots & x_{1,m} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \dots & x_{m,m} \end{bmatrix}$ with entries from $C(S^3) \rtimes_\alpha \mathbb{Z}_n$, to the $mn \times mn$ matrix $\begin{bmatrix} \phi(x_{1,1}) & \dots & \phi(x_{1,m}) \\ \vdots & \ddots & \vdots \\ \phi(x_{m,1}) & \dots & \phi(x_{m,m}) \end{bmatrix}$. Note that each $\phi(x_{i,j})$ is an $n \times n$ block. It is clear that $\tilde{\phi}$ is an algebra homomorphism.

Extend each ψ_i from Definition 1.4.4 to $\tilde{\psi}_i$ from $M_{mn}(C(S^3))$ to $M_m(C(S^1))$ by considering the $mn \times mn$ matrix as an $m \times m$ matrix of $n \times n$ blocks and applying ψ_i to each block to make an $m \times m$ matrix of elements from $C(S^1)$. Note that $\tilde{\phi}_i$ is also an algebra homomorphism.

To simplify notation, let Ψ_i be the composition $\tilde{\psi}_i \tilde{\phi}$.

Define $Ch_i : GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n) \to C^\infty(S^1)$ by $Ch_i(X) = tr[(\Psi_i(X))^{-1} d(\Psi_i(X))]$ for $0 < i < n$. $\diamondsuit$

Lemma 3.4.4. *For invertible elements X and Y in $M_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and any integer i such that $0 < i < n$, $Ch_i(XY) = Ch_i(X) + Ch_i(Y)$. Further, if X is homotopic to Y through invertible elements, then $\int_{S^1} Ch_i(X) = \int_{S^1} Ch_i(Y)$.*

Proof. We have that $\Psi_i = \psi_i \phi$ is an algebra homomorphism into $C(S^1)$.

Compute:

$$\begin{aligned} Ch_i(XY) &= tr(\Psi_i(XY))^{-1} d(\Psi_i(XY))) \\ &= tr((\Psi_i(Y))^{-1}\Psi_i(X)^{-1}((d\Psi_i(X))\Psi_i(Y) + \Psi_i(X)d(\Psi_i(Y)))) \\ &= tr(\Psi_i(Y)^{-1}\Psi_i(X)^{-1}d\Psi_i(X)\Psi_i(Y)) + tr(\Psi_i(Y)^{-1}d\Psi_i(Y)) \\ &= Ch_i(X) + Ch_i(Y). \end{aligned}$$

This last step uses the cyclic property of the trace, which applies here as $C(S^1)$ is commutative.

Further, $\int_{S^1} Ch_i(X)$ is homotopy invariant since $\Psi = \psi_i \phi$ preserves homotopies and if $\tilde{X}$ is homotopic to $\tilde{Y}$ via invertible elements of $M_m(C^\infty(S^1))$, then $T_{\tilde{X}}$ and $T_{\tilde{Y}}$ have the same index, so $\int_{S^1} tr(\tilde{X}^{-1}d\tilde{X}) = \int_{S^1} tr(\tilde{Y}^{-1}d\tilde{Y})$. (See [4, Theorem 7.26] for when $m = 1$ and [9, Exercise 4.4.30(2)] for the more general case.) □

Lemma 3.4.5. *If X and Y are elements of $GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and X is homotopic to Y in $GL_m(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$, then X is also homotopic to Y in $GL_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$.*

Proof. The argument is nearly identical to the argument of Lemma 2.4.2. □

3.5 Index Formula and Calculations

In this final section we will find a generating invertible element for each copy of $\mathbb{Z}$ in $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and show that the corresponding Toeplitz operators all have index one. We will then see that the symbols of each of these operators will correspond to either $\int_{S^3} Ch$ or a particular $\int_{S^1} Ch_i$ from Section 3.4. We will then weight these integrals as necessary to discover a general index formula.

We choose the following invertible elements:

$$X_0 = \begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix},$$

$$X_1 = \phi^{-1} \begin{bmatrix} \bar{w} & 0 & \dots & z^{n-1} \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ -\bar{z}^{n-1} & 0 & \dots & w \end{bmatrix},$$

and, for each integer k such that $1 < k < n$,

$$X_k = \phi^{-1} \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & w & & \bar{z}^k \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & -z^k & \dots & \bar{w} \end{bmatrix}.$$

Each of these is invertible as they all have a determinant of $|w|^2 + |z|^{2k}$, which is never 0.

While it is usually easier to represent and work with these elements as matrices, to show that their associated Toeplitz operators are index one we will need to view them in the crossed product algebra. Using the formula for ϕ^{-1} from the end of Definition 1.3.3, and letting $\xi = e^{2\pi i/n}$, note that

$$X_k = \frac{1}{n}\Bigg(\sum_{c=0}^{n-1} V^c + \sum_{c=0}^{n-1} \xi^c V^c + \sum_{c=0}^{n-1} \xi^{2c} V^c + \cdots + (w - z^k)\sum_{c=0}^{n-1} \xi^{kc} V^c \\ + \sum_{c=0}^{n-1} \xi^{(k+1)c} V^c + \cdots + (\bar{w} + \bar{z}^k)\sum_{c=0}^{n-1} \xi^{(n-1)c} V^c\Bigg).$$

Further, note that for positive integers $B < n$ and l

$$\frac{1}{n}\sum_{c=0}^{n-1} \xi^{Bc} V^c(z^l) = \frac{1}{n}\sum_{c=0}^{n-1} \xi^{Bc}\xi^{cl}(z^l) = \frac{z^l}{n}\sum_{c=0}^{n-1} \xi^{c(B+l)}.$$

When $(B+l) \bmod n \equiv 0$, this sum is equal to z^l since $\sum_{c=0}^{n-1} \xi^{c(B+l)} = \sum_{c=0}^{n-1} 1 = n$. And when $(B+l) \bmod n \not\equiv 0$, the sum will be 0 since $\sum_{c=0}^{n-1} \xi^{c(B+l)} = 0$.

Lemma 3.5.1. *For $k > 1$, the kernel of T_{X_k} will be the linear span of $\{z\}$ while the kernel of the adjoint, $T^*_{X_k}$, will be $\{0\}$. Thus the index of T_{X_k} will be one.*

Proof. Applying the fact mentioned before this lemma, we see that $\frac{1}{n}\sum_{c=0}^{n-1} \xi^{Bc} V^c(z) = z$ when $B = n-1$ and 0 when $0 < B < n-1$. Thus it follows from our expression for X_k that

$$T_{X_k}(z) = PX_k(z) = P((\bar{w} + \bar{z}^k)(z)) = P(\bar{w}z) + P(|z|^2\bar{z}^{k-1}).$$

By an argument similar to the one in Lemma 1.2.3, $\bar{w}z$ and $|z|^2\bar{z}^{k-1}$ will be orthogonal to the Hardy space, so $T_{X_k}(z) = 0$.

Suppose $g = \sum_{i,j=0}^{\infty} a_{i,j} w^i z^j$ is in the kernel. Compute:

$$\begin{aligned}
T_{X_k}(g) =& P\Bigg(\sum_{i,j=0}^{\infty} a_{i,nj} w^i z^{nj} + \sum_{i=0,j=1}^{\infty} a_{i,nj-1} w^i z^{nj-1} + \cdots \\
&+ (w - z^k) \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^i z^{nj-k+1} + \cdots + (\bar{w} + \bar{z}^k) \sum_{i=0,j=1}^{\infty} a_{i,nj-(n-1)} w^i z^{nj-(n-1)} \Bigg) \\
&= \sum_{i,j=0}^{\infty} a_{i,nj} w^i z^{nj} + \sum_{i=0,j=1}^{\infty} a_{i,nj-1} w^i z^{nj-1} + \cdots \\
&+ \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^{i+1} z^{nj-k+1} - \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^i z^{nj+1} + \cdots \\
&+ P\Bigg(|w|^2 \sum_{i,j=0}^{\infty} a_{i,nj+1} w^{i-1} z^{nj+1} \Bigg) + P\Bigg(|z|^{2k} \sum_{i,j=0}^{\infty} a_{i,nj+1} w^i z^{nj+1-k} \Bigg).
\end{aligned}$$

Applying Lemma 1.2.3, these last two projections can be rewritten as the two sums $\sum_{i=1,j=0}^{\infty} A_{i,nj+1} a_{i,nj+1} w^{i-1} z^{nj+1}$ and $\sum_{i=0,j=1}^{\infty} B_{i,nj+1} a_{i,nj+1} w^i z^{nj+1-k}$ respectively, where each $A_{i,nj+1}$ and $B_{i,nj+1}$ is a positive constant.

Since we are assuming $T_{X_k}(g) = 0$, we examine the sums termwise and conclude that any $w^i z^J$ term, where $J \neq nj - k + 1$ and $J \neq nj + 1$, must be 0. Further,

$$\sum_{i=1,j=0}^{\infty} A_{i,nj+1} a_{i,nj+1} w^{i-1} z^{nj+1} = \sum_{i=1,j=1}^{\infty} a_{i-1,nj-k+1} w^{i-1} z^{nj+1}$$

and

$$\sum_{i=0,j=1}^{\infty} B_{i,nj+1}a_{i,nj+1}w^i z^{nj+1-k} = -\sum_{i=1,j=1}^{\infty} a_{i-1,nj-k+1}w^i z^{nj-k+1}.$$

This leads us to conclude that, for $i, j > 0$,

$$a_{i-1,nj-k+1} = A_{i,nj+1}a_{i,nj+1} = -B_{i,nj+1}a_{i,nj+1}.$$

But all $B_{i,j}$ and $A_{i,j}$ were strictly positive, so $a_{i-1,nj-k+1} = 0$ and $a_{i,nj+1} = 0$.

When $i = 0$ and $j > 0$ we have that $B_{0,nj+1}a_{0,nj+1} = 0$. So $a_{0,nj+1} = 0$ also.

Finally, when $i = j = 0$, $a_{0,1}$ is free. So the kernel of T_{X_k} is the linear span of $\{z\}$.

We now make a similar calculation for the adjoint which, applying ϕ^{-1} again, has the expression

$$\begin{aligned} T^*_{X_k} = \frac{1}{n}\Bigg(\sum_{c=0}^{n-1} V^c + \sum_{c=0}^{n-1} \xi^c V^c + \sum_{c=0}^{n-1} \xi^{2c} V^c + \cdots + (\bar{w} + z^k) \sum_{c=0}^{n-1} \xi^{kc} V^c + \cdots \\ + \sum_{c=0}^{n-1} \xi^{(k+1)c} V^c + \cdots + (w - \bar{z}^k) \sum_{c=0}^{n-1} \xi^{(n-1)c} V^c \Bigg). \end{aligned}$$

Suppose $g = \sum_{i,j=0}^{\infty} a_{i,j} w^i z^j$ is in the kernel. Compute:

$$\begin{aligned}
T_{X_k}(g) =& P\Bigg(\sum_{i,j=0}^{\infty} a_{i,nj} w^i z^{nj} + \sum_{i=0,j=1}^{\infty} a_{i,nj-1} w^i z^{nj-1} + \cdots \\
&+ (\bar{w} + z^k) \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^i z^{nj-k+1} + \cdots + (w - \bar{z}^k) \sum_{i=0,j=1}^{\infty} a_{i,nj-(n-1)} w^i z^{nj-(n-1)} \Bigg) \\
&= \sum_{i,j=0}^{\infty} a_{i,nj} w^i z^{nj} + \sum_{i=0,j=1}^{\infty} a_{i,nj-1} w^i z^{nj-1} + \cdots \\
&+ P\Bigg(|w|^2 \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^{i-1} z^{nj-k+1} \Bigg) + \sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^i z^{nj+1} + \cdots \\
&+ \sum_{i,j=0}^{\infty} a_{i,nj+1} w^{i+1} z^{nj+1} - P\Bigg(|z|^{2k} \sum_{i,j=0}^{\infty} a_{i,nj+1} w^i z^{nj+1-k} \Bigg).
\end{aligned}$$

Applying Lemma 1.2.3, the two projections in this last expression can be rewritten as the sums $\sum_{i=1,j=1}^{\infty} A_{i,nj-k+1} a_{i,nj-k+1} w^{i-1} z^{nj-k+1}$ and $\sum_{i=0,j=1}^{\infty} B_{i,nj+1} a_{i,nj+1} w^i z^{nj+1-k}$ respectively, where each $A_{i,nj+1}$ and $B_{i,nj+1}$ is a positive constant.

Examining all the sums termwise, we conclude that any $w^i z^J$ term, where $J \neq nj - k + 1$ and $J \neq nj + 1$, must be 0. Further,

$$\sum_{i=1,j=1}^{\infty} A_{i,nj-k+1} a_{i,nj-k+1} w^{i-1} z^{nj-k+1} = \sum_{i=1,j=1}^{\infty} B_{i-1,nj+1} a_{i-1,nj+1} w^{i-1} z^{nj+1-k}$$

and

$$\sum_{i=0,j=1}^{\infty} a_{i,nj-k+1} w^i z^{nj+1} = - \sum_{i=1,j=0}^{\infty} a_{i-1,nj+1} w^i z^{nj+1}.$$

We know from this last equation that all $a_{0,nj-k+1} = 0$ and all $a_{i-1,1} = 0$. And together the equations allow us to conclude that, for $i, j > 0$,

$$A_{i,nj-k+1}a_{i,nj-k+1} = B_{i-1,nj+1}a_{i-1,nj+1}$$

and

$$a_{i,nj-k+1} = -a_{i-1,nj+1}.$$

But all $B_{i,j}$ and $A_{i,j}$ were strictly positive, so $a_{i,nj-k+1} = 0$ and $a_{i-1,nj+1} = 0$.

Thus kernel of the adjoint is $\{0\}$, and the index of T_{X_k} is one. □

A very similar argument shows that the index of T_{X_1} is one.

We next find each $Ch_j(X_k)$ for $0 < j < n$ and $0 < k < n$.

Lemma 3.5.2. *If $j \neq k$, then $Ch_j(X_k) = 0$. If $j = k = 1$, then $\frac{1}{2\pi i}\int_{S^1} Ch_1(X_1) = -1$. If $j = k > 1$, then $\frac{1}{2\pi i}\int_{S^1} Ch_j(X_k) = 1$*

Proof. First note that

$$X_1^{-1} = \phi^{-1}\left(\frac{1}{|w|^2 + |z|^{2n-2}}\begin{bmatrix} w & 0 & \dots & -z^{n-1} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \bar{z}^{n-1} & 0 & \dots & \bar{w} \end{bmatrix}\right)$$

and

$$X_k^{-1} = \phi^{-1}\left(\frac{1}{|w|^2+|z|^{2k}}\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \bar{w} & & -\bar{z}^k \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & z^k & \dots & w \end{bmatrix}\right),$$

while

$$d(X_1) = \phi^{-1}\begin{bmatrix} d\bar{w} & 0 & \dots & (n-1)z^{n-2}dz \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ -(n-1)\bar{z}^{n-2}d\bar{z} & 0 & \dots & dw \end{bmatrix}$$

and

$$d(X_k) = \phi^{-1}\begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & dw & & k\bar{z}^{k-1}d\bar{z} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & -kz^{k-1}dz & \dots & d\bar{w} \end{bmatrix}.$$

So $X_1^{-1}dX_1 =$

$$\phi^{-1}\left(\frac{1}{|w|^2+|z|^{2n-2}}\begin{bmatrix} wd\bar{w}+(n-1)|z|^{2n-4}zd\bar{z} & 0 & \dots & w(n-1)z^{2n-2}dz - z^{n-1}dw \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ \bar{z}^{n-1}d\bar{w}-(n-1)\bar{w}\bar{z}^{n-2}d\bar{z} & \dots & \dots & \bar{w}dw+(n-1)|z|^{2n-4}\bar{z}dz \end{bmatrix}\right),$$

while $X_k^{-1}dX_k =$

$$\phi^{-1}\left(\frac{1}{|w|^2+|z|^{2k}}\begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \bar{w}dw + k|z|^{2k-2}\bar{z}dz & & \bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w} \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & z^k dw - wkz^{k-1}dz & \dots & wd\bar{w} + k|z|^{2k-2}zd\bar{z} \end{bmatrix}\right).$$

Note that for any positive integer B and every integer j such that $0 < j < n$,

$$\psi_j\left(\frac{1}{|w|^2+|z|^{2B}}\right) = \frac{1}{|w|^2}$$

as $1 = \psi_j(1) = \psi_j\left(\frac{1}{|w|^2+|z|^{2B}}\right)\psi_j(|w|^2+|z|^{2B}) = \psi_j\left(\frac{1}{|w|^2+|z|^{2B}}\right)|w|^2$.

We evaluate each $\displaystyle\int_{S^1} Ch_j(X_k)$ in the three possible cases.

Case 1: $j \neq k$

In this case $Ch_j(X_k) = 0$ as the (j,j)th entry of $\phi(X_k^{-1}dX_k)$ is 0.

Case 2. $j = k = 1$

Here $Ch_1(X_1) = \dfrac{wd\bar{w}}{|w|^2}$. On the circle, with $w = e^{i\theta}$, $\dfrac{wd\bar{w}}{|w|^2}$ is equal to $-id\theta$, so $\displaystyle\frac{1}{2\pi i}\int_{S^1} Ch_1(X_1) = -1$.

Case 3. $j = k > 1$

Here $Ch_j(X_k) = \dfrac{\bar{w}dw}{|w|^2}$. On the circle, with $w = e^{i\theta}$, $\dfrac{\bar{w}dw}{|w|^2}$ is equal to $id\theta$, so $\dfrac{1}{2\pi i}\displaystyle\int_{S^1} Ch_j(X_k) = 1$.

Since we know from Lemma 3.4.4 that each Ch_j is homotopy invariant, we conclude that each X_k is in a separate homotopy class. These classes will, in fact, each be a distinct generator of $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$. □

The only thing left is to calculate the integral over S^3 of $Ch(X_k)$ for each $0 < k < n$. Note first that for a matrix of 1-forms

$$A = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & a & & b \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & c & \dots & d \end{bmatrix},$$

we have

$$A^3 = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 2abc - bcd & & abd \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & -acd & \dots & -2bcd + abc \end{bmatrix}.$$

Also, note that for any Y in $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ or $\Omega^3(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$, $\nu(Y)$ is equal to $\frac{1}{n}$ multiplied by the sum of the entries of $\phi(Y)$. In particular, for A^3 from above, $\nu(\phi^{-1}(A^3)) = \frac{1}{n}(3abc - 3bcd + abd - acd)$.

Lemma 3.5.3. *For* $1 < k < n$, $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_k) = \frac{k}{m}$, *while* $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_1) = \frac{n-1}{n}$.

Proof. For $1 < k < n$, we see from the previous example and the above notes that

$$\int_{S^3} Ch(X_k) = \int_{S^3} \nu((X_k^{-1} dX_k)^3) =$$

$$\int_{S^3} \nu \left(\phi^{-1} \left(\frac{1}{|w|^2 + |z|^{2k}} \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \bar{w}dw + k|z|^{k-1}\bar{z}dz & & \bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & z^k dw - wkz^{k-1}dz & \cdots & wd\bar{w} + k|z|^{k-1}zd\bar{z} \end{bmatrix} \right)^3 \right)$$

$$\begin{aligned} = \int_{S^3} \frac{1}{n} \frac{1}{(|w|^2 + |z|^{2k})^3} [&3(\bar{w}dw + k|z|^{2k-2}\bar{z}dz)(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz) \\ &-3(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz)(wd\bar{w} + k|z|^{2k-2}zd\bar{z}) \\ &+(\bar{w}dw + k|z|^{k-1}\bar{z}dz)(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(wd\bar{w} + k|z|^{k-1}zd\bar{z}) \\ &-(\bar{w}dw + k|z|^{k-1}\bar{z}dz)(z^k dw - wkz^{k-1}dz)(wd\bar{w} + k|z|^{k-1}zd\bar{z})]. \end{aligned}$$

However, the last two lines are in $\Omega^3(A_{n-k})$ and $\Omega^3(A_k)$ respectively, so by Corollary 3.1.6, they will be exact forms. Thus we can simplify to

$$\int_{S^3} \frac{1}{n} \frac{1}{(|w|^2 + |z|^{2k})^3} [3(\bar{w}dw + k|z|^{2k-2}\bar{z}dz)(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz)$$
$$-3(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz)(wd\bar{w} + k|z|^{2k-2}zd\bar{z})].$$

We calculate:

$$\begin{aligned}
&(\bar{w}dw + k|z|^{2k-2}\bar{z}dz)(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz) \\
=& - k^2\bar{w}|w|^2|z|^{2k-2}dwd\bar{z}dz + k|w|^2|z|^{2k-2}\bar{z}dwd\bar{w}dz \\
& - k^2|z|^{2k-2}|z|^{2k}\bar{w}dwd\bar{z}dz + k|z|^{2k-2}|z|^{2k}\bar{z}dwd\bar{w}dz \\
=&k|z|^{2k-2}(|w|^2 + |z|^{2k})(k\bar{w}dwdzd\bar{z} + \bar{z}dwd\bar{w}dz),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&(\bar{w}k\bar{z}^{k-1}d\bar{z} - \bar{z}^k d\bar{w})(z^k dw - wkz^{k-1}dz)(wd\bar{w} + k|z|^{2k-2}zd\bar{z}) \\
=&k|z|^{2k-2}(|w|^2 + |z|^{2k})(zdwd\bar{w}d\bar{z} + kwd\bar{w}dzd\bar{z}).
\end{aligned}$$

So $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_k) =$

$$\frac{3k}{12n} \cdot \frac{1}{2\pi^2} \int_{S^3} \frac{|z|^{2k-2}}{(|w|^2 + |z|^{2k})^2}(k\bar{w}dwdzd\bar{z} + \bar{z}dwd\bar{w}dz - zdwd\bar{w}d\bar{z} - kwd\bar{w}dzd\bar{z})$$

$$
\begin{aligned}
=&\frac{k}{8n\pi^2}\int_{S^3}\frac{|z|^{2k-2}}{(|w|^2+|z|^{2k})^2}[(k\sin\eta(i\sin\eta d\theta_1+\cos\eta d\eta)(-2i\cos\eta\sin\eta d\theta_2 d\eta)\\
&\qquad +\cos\eta(i\cos\eta d\theta_2-\sin\eta d\eta)(2i\cos\eta\sin\eta d\theta_1 d\eta)\\
&\qquad -\cos\eta(-i\cos\eta d\theta_2-\sin\eta d\eta)(2i\cos\eta\sin\eta d\theta_1 d\eta)\\
&\qquad -k\sin\eta(-i\sin\eta d\theta_1+\cos\eta d\eta)(-2i\cos\eta\sin\eta d\theta_2 d\eta)]\\
=&\frac{k}{8n\pi^2}\int_{S^3}\frac{\cos^{2k-2}(\eta)}{(\sin^2(\eta)+\cos^{2k}(\eta))^2}(4\cos^3(\eta)\sin\eta+4k\cos\eta\sin^3(\eta))d\theta_1 d\theta_2 d\eta\\
=&\frac{k}{2n\pi^2}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos^{2k-1}(\eta)\sin\eta}{(\sin^2(\eta)+\cos^{2k}(\eta))^2}(\cos^2(\eta)+k\sin^2(\eta))d\theta_1 d\theta_2 d\eta\\
=&\frac{2k}{n}\int_0^{\pi/2}\frac{\cos^{2k-1}(\eta)\sin\eta}{(\sin^2(\eta)+\cos^{2k}(\eta))^2}(\cos^2(\eta)+k\sin^2(\eta))d\eta\\
=&\frac{2}{n}\cdot\frac{1}{2}\cdot\frac{\sin^2(\eta)}{\sin^2(\eta)+\cos^{2k}(\eta)}\Big|_0^{\pi/2}\\
=&\frac{2k}{n}\cdot\frac{1}{2}=\frac{k}{n}.
\end{aligned}
$$

By a very similar calculation we get $\frac{1}{12}\cdot\frac{1}{2\pi^2}\int_{S^3}Ch(X_1)=\frac{n-1}{n}$. □

Theorem 3.5.4. *Let X be in $GL_m(C^\infty(S^3)\rtimes_\alpha \mathbb{Z}_n)$. Define*

$$
\Upsilon(X)=\frac{1}{12}\cdot\frac{1}{2\pi^2}\int_{S^3}Ch(X)-\frac{1}{n}\cdot\frac{1}{2\pi i}\int_{S^1}Ch_1(X)+\sum_{k=2}^{n-1}\frac{n-k}{n}\cdot\frac{1}{2\pi i}\int_{S^1}Ch_k(X).
$$

The index of T_X is equal to $\Upsilon(X)$.

Proof. Using Lemmas 3.5.2 and 3.5.3 it is clear that, for $1 < j < n$,

$$\begin{aligned}\Upsilon(X_j) =& \frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_j) - \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_1(X_j) + \sum_{k=2}^{n-1} \frac{n-k}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_k(X_j) \\ =& \frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_j) + \frac{n-j}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_j(X_j) = \frac{j}{n} + \frac{n-j}{n} = 1.\end{aligned}$$

And, when $j = 1$,

$$\begin{aligned}\Upsilon(X_1) =& \frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_1) - \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_1(X_1) + \sum_{k=2}^{n-1} \frac{n-k}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_k(X_1) \\ =& \frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_1) - \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{S^1} Ch_1(X_1) = \frac{n-1}{n} - \frac{-1}{n} = 1.\end{aligned}$$

And for X_0 a calculation verifies that, for $0 < k < n$,

$$Ch_k(X_0) = tr \begin{bmatrix} \bar{w}dw & 0 \\ 0 & wd\bar{w} \end{bmatrix} = \bar{w}dw + wd\bar{w},$$

so $\int_{S^1} Ch_k(X_0) = \int_{S^1} (\bar{w}dw + wd\bar{w}) = 0$. So $\Upsilon(X_0) = \frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_0)$, as the other terms will be 0. But the argument of Theorem 1.2.5 includes a calculation showing that $\frac{1}{12} \cdot \frac{1}{2\pi^2} \int_{S^3} Ch(X_0) = 1$.

Further, if X and Y in $M_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$ are homotopic via invertibles, then $\Upsilon(X) = \Upsilon(Y)$, since Lemmas 3.4.2 and 3.4.4 inform us that each summand in the formula is homotopy invariant. This also allows us to conclude that X_0 is not homotopic to any other X_j and is a generator for $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$.

And, for any X and Y in $M_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$, $\Upsilon(XY) = \Upsilon(X) + \Upsilon(Y)$ since Lemmas 3.4.2 and 3.4.4 again inform us that each summand in the formula has a similar property for products.

Finally, since $K_1(C(S^3) \rtimes_\alpha \mathbb{Z}_n) \cong \mathbb{Z}^n$ and has X_j $(0 \leq j < n)$ as independent generators, for any Y in $M_m(C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n)$ there exists an integer M such that $Y \oplus 1_M$ will be homotopic via invertibles to some $X = (X_0)^l \oplus \Pi_{j=1}^{n-1}(X_j)^{k_j} \oplus 1_{M+m-3}$ where l and each k_j are integers. By Lemma 3.4.5 we can construct this homotopy to be through elements of $C^\infty(S^3) \rtimes_\alpha \mathbb{Z}_n$ so that Ch and each Ch_k will be defined along the entire homotopy. We conclude that $ind(T_Y) = ind(T_{Y \oplus 1_M}) = ind(T_X) = \Upsilon(X) = \Upsilon(Y \oplus 1_M) = \Upsilon(Y)$. $\square$

Example 3.5.5. Let $X = \phi^{-1} \begin{bmatrix} \bar{w}^2 & 0 & -z^2 & 0 \\ 0 & w & 0 & \bar{z}^2 \\ \bar{z}^2 & 0 & w^2 & 0 \\ 0 & -z^2 & 0 & \bar{w} \end{bmatrix}$. We will find the index of T_X.

Let $A = |w|^4 + |z|^4$ and $B = |w|^2 + |z|^4$. By inspection we see that

$$X^{-1} = \phi^{-1} \begin{bmatrix} \frac{1}{A}w^2 & 0 & \frac{1}{A}z^2 & 0 \\ 0 & \frac{1}{B}\bar{w} & 0 & -\frac{1}{B}\bar{z}^2 \\ -\frac{1}{A}\bar{z}^2 & 0 & \frac{1}{A}\bar{w}^2 & 0 \\ 0 & \frac{1}{B}z^2 & 0 & \frac{1}{B}w \end{bmatrix},$$

and that

$$dX = \phi^{-1}\begin{bmatrix} 2\bar{w}d\bar{w} & 0 & -2zdz & 0 \\ 0 & dw & 0 & 2\bar{z}d\bar{z} \\ 2\bar{z}d\bar{z} & 0 & 2wdw & 0 \\ 0 & -2zdz & 0 & d\bar{w} \end{bmatrix}.$$

Thus $X^{-1}dX =$

$$\phi^{-1}\begin{bmatrix} \frac{2}{A}(|w|^2wd\bar{w} + |z|^2zd\bar{z}) & 0 & \frac{2}{A}(-w^2zdz + wz^2dw) & 0 \\ 0 & \frac{1}{B}(\bar{w}dw + 2|z|^2\bar{z}dz) & 0 & \frac{1}{B}(2\bar{w}\bar{z}d\bar{z} - \bar{z}^2d\bar{w}) \\ \frac{2}{A}(-\bar{w}\bar{z}^2d\bar{w} + \bar{w}^2\bar{z}d\bar{z}) & 0 & \frac{2}{A}(|w|^2\bar{w}dw + |z|^2\bar{z}dz) & 0 \\ 0 & \frac{1}{B}(z^2dw - 2wzdz) & 0 & \frac{1}{B}(wd\bar{w} + 2|z|^2zd\bar{z}) \end{bmatrix}.$$

At this point it is clear that $Ch_1(X) = 2wd\bar{w}$, $Ch_2(X) = \bar{w}dw$, and $Ch_3(X) = 2\bar{w}dw$.

To calculate $Ch(X)$, we see by inspection that for 1-forms a, b, c, d, e, f, g, and h,

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & e & 0 & f \\ c & 0 & d & 0 \\ 0 & g & 0 & h \end{bmatrix}^3 = \begin{bmatrix} 2abc - bcd & 0 & abd & 0 \\ 0 & 2efg - fgh & 0 & efh \\ -acd & 0 & abc - 2bcd & 0 \\ 0 & -egh & 0 & efg - 2fgh \end{bmatrix}.$$

If we let $\phi(X^{-1}dX) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & e & 0 & f \\ c & 0 & d & 0 \\ 0 & g & 0 & h \end{bmatrix}$, then, recalling that, for $Y \in \Omega^3(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$, $\nu(Y)$ is equal to $\frac{1}{n}$ times the sum of the entries of $\phi(Y)$, we get that $\nu((X^{-1}dX)^3) = \frac{1}{4}(3abc - 3bcd + abd - acd + 3efg - 3fgh + efh - egh)$. However, from Lemma 3.1.5, it is clear that abd, acd, efh, and egh will all integrate to 0 over the 3-sphere, as they are orthogonal to $\Omega^3(A_0)$. So, after some simplification, we get

$$\frac{1}{2\pi^2}\int_{S^3} Ch(X) = \frac{1}{2\pi^2}\int_{S^3} \frac{1}{4}(3abc - 3bcd + 3efg - 3egh)$$
$$= \frac{1}{2\pi^2}\int_{S^3} \left(\frac{48}{4A^2}|w|^2|z|^2(wd\bar{w}d\bar{z}dz + zd\bar{w}dwd\bar{z}) + \frac{6}{4B^2}|z|^2(4\bar{w}dwdzd\bar{z} + \bar{z}dwd\bar{w}dz)\right)$$
$$= \frac{1}{2\pi^2}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi} \left(\frac{24}{A^2}\sin^3\eta\cos^3\eta + \frac{6}{B^2}(2\sin^3\eta\cos^3\eta + \sin\eta\cos^5\eta)\right) d\theta_1 d\theta_2 d\eta$$
$$= 2(6+3) = 18.$$

Thus

$$\Upsilon(X) = \frac{1}{12}\cdot\frac{1}{2\pi^2}\int_{S^3} Ch(X) - \frac{1}{4}\cdot\frac{1}{2\pi i}\int_{S^1} Ch_1(X) + \frac{1}{2}\cdot\frac{1}{2\pi i}\int_{S^1} Ch_2(X) + \frac{1}{4}\cdot\frac{1}{2\pi i}\int_{S^1} Ch_3(X),$$

which yields $\Upsilon(X) = \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3$.

We conclude that the index of T_X is 3. $\diamondsuit$

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ABSTRACT

TOEPLITZ OPERATORS WITH SYMBOLS FROM CERTAIN ROTATION ALGEBRAS AND THEIR INDEX FORMULAS

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Let $C(S^1) \rtimes_\alpha \mathbb{Z}_n$ be the crossed product algebra where α enacts a rotation on the complex coordinate z by rotating it to $e^{2\pi i/n}z$. Let $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ be the algebra of Toeplitz operators with symbols from the crossed product algebra $C(S^1) \rtimes_\alpha \mathbb{Z}_n$. We find the K-theory of $\mathcal{T}(C(S^1) \rtimes_\alpha \mathbb{Z}_n)$ and a formula to calculate the Fredholm index of an operator from this algebra.

Similarly, let $C(S^3) \rtimes_\alpha \mathbb{Z}_n$ be the crossed product algebra where α enacts a rotation on one of the complex coordinates of S^3 by rotating (w, z) to $(w, e^{2\pi i/n}z)$. Let $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ be the algebra of Toeplitz operators with symbols from the crossed product algebra $C(S^3) \rtimes_\alpha \mathbb{Z}_n$. We find the K-theory of $\mathcal{T}(C(S^3) \rtimes_\alpha \mathbb{Z}_n)$ and a formula to calculate the Fredholm index of an operator from this algebra. The index formula is the sum of n integrals which can be used to determine when two operators are in different homotopy classes, even if they have the same index.