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Parametrized Relativistic Quantum Theory in Curved Spacetime

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Parametrized Relativistic Quantum Theory in Curved Spacetime

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Abstract. The purpose of this paper is to present a parametrized relativistic quantum theory (pRQT) in curved spacetime. The formulation of pRQT in curved spacetime is developed and applied to free particle motion in flat and curved spacetime. It provides a theory for calculating probability amplitudes in curved spacetime. Unlike other formulations of parametrized relativistic dynamics (pRD), this work assumes that the metric tensor does not depend on the invariant evolution parameter.

1. Introduction

Contemporary formulations of parametrized relativistic dynamics (pRD) in curved spacetime [1-3] assume that the metric tensor is dependent on an invariant evolution parameter. Does the metric tensor depend on an invariant evolution parameter? Here we consider the possibility that a formulation of relativistic quantum theory in curved spacetime can be developed without requiring that the metric tensor be dependent on an invariant evolution parameter.

The primary difference between this work and previous pRD publications is the assumption that the metric tensor of parametrized relativistic quantum theory (pRQT) in curved spacetime does not depend on the invariant evolution parameter. A formulation of parameter independent general relativity was developed by Fanchi [4, 5] using a variational principle that is analogous to the Hilbert-Palatini variational principle. It shows that metrics from conventional general relativity can be used in the pRQT formulation.

The formulation of pRQT in curved spacetime is presented in Section 2. It provides equations for determining probability amplitudes. The definition of expectation value associated with pRQT in curved spacetime is discussed in Section 3. The formulation is applied in Section 4 to a free scalar particle in flat spacetime. Section 4 includes a discussion of the mass of a particle and shows that both tardyons and tachyons have real mass in pRQT [6]. The pRQT formulation is then applied to a particle moving in Schwarzschild metric in Section 5. The problem posed in Section 5 is analyzed in Section 6 for a free scalar particle at a large distance from a radially symmetric mass.

2. pRQT in Curved Spacetime

A quantum theory of the behavior of a single spinless particle moving in curved spacetime is developed here using a formulation that is analogous to an extension of the four-space formulation in flat spacetime [5, 7, 8]. We begin by assuming that the behavior of the particle can be represented by a scalar conditional probability density $\rho(x|s)$. Spacetime coordinates are denoted by the spacetime contravariant four-vector x with μ^{th} component x^μ and $\mu = 0,1,2,3$. Index 0 signifies the geometric time component $x^0 = ct$ and indices 1, 2, 3 signify space components. The scalar s is an invariant evolution parameter that conditions $\rho(x|s)$. The metric tensor of the physical system is $g_{\mu\nu}$.

According to probability theory, $\rho(x|s)$ must be positive definite and normalizable. The positive definite requirement is expressed in terms of the Born representation

$$\rho(x|s) = \Psi^*(x,s)\Psi(x,s) \geq 0 \tag{2.1}$$

where Ψ is the probability amplitude and Ψ^* is the complex conjugate of Ψ. The probability amplitude is specified to within a gauge transformation. It can be written as

$$\Psi(x,s) = \sqrt{\rho(x|s)}\, e^{i\xi(x,s)} \tag{2.2}$$

where the real scalar function ξ represents the gauge transformation.

The normalization condition is

$$\int_D \rho(x|s)\sqrt{|g(x)|}dx = 1, dx = d^4x \tag{2.3}$$

where D denotes the spacetime volume and $|g(x)|$ is the absolute value of the determinant of the metric tensor $g_{\mu\nu}$. Substituting Equation (2.1) into (2.3) gives

$$\int_D \Psi^*(x,s)\Psi(x,s)\sqrt{|g(x)|}dx = 1 \tag{2.4}$$

The probability $\Psi^*(x,s)\Psi(x,s)\sqrt{|g(x)|}dx$ is the probability that the system will be found in the volume $\sqrt{|g(x)|}dx$ in the neighborhood of spacetime point x at evolution parameter s. The normalization condition implies that the particle can be observed somewhere in space at some point in geometric time t; it does not require the particle to exist for all geometric time.

Field equations that exhibit a "minimal coupling" electromagnetic interaction are derived from the above assumptions and probability conservation. Conservation of probability in curved spacetime is expressed as an extension of the parametrized continuity equation in flat spacetime that was used to construct the Stueckelberg equation in flat spacetime [5, 7, 8]. The parametrized continuity equation in curved spacetime is

$$\frac{\partial \rho}{\partial s} + (\rho V^\mu);_\mu = 0 \tag{2.5}$$

where V^μ is contravariant four-velocity, ρV^μ is probability flux, and $(\rho V^\mu);_\mu$ is the divergence

$$(\rho V^\mu);_\mu = \frac{\partial}{\partial x^\mu}(\rho V^\mu) + \Gamma^\mu_{\nu\mu}(\rho V^\nu) \tag{2.6}$$

The tensor $\Gamma^\mu_{\nu\mu}$ is the contracted form of the Christoffel symbol of the second kind

$$\Gamma^\sigma_{\nu\mu} = \frac{1}{2}g^{\sigma\kappa}\left(\frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\kappa}}{\partial x^\nu} - \frac{\partial\partial g_{\nu\mu}}{\partial x^\kappa}\right) \tag{2.7}$$

for metric tensor $g_{\nu\mu}$. A similar continuity equation was discussed by Pavšič [9] (Chapter 1).

The divergence $(\rho V^\mu);_\mu$ of a contravariant four-vector ρV^σ is the contracted form of the covariant derivative of the contravariant four-vector

$$(\rho V^\sigma);_\mu = \frac{\partial}{\partial x^\mu}(\rho V^\sigma) + \Gamma^\sigma_{\nu\mu}(\rho V^\nu) \tag{2.8}$$

The Christoffel symbol of the second kind $\Gamma^\sigma_{\nu\mu}$ is also known as the affine connection. The contracted form of $\Gamma^\sigma_{\nu\mu}$ is

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2}g^{\mu\kappa}\frac{\partial g_{\mu\kappa}}{\partial x^\nu} \tag{2.9}$$

Substituting Equation (2.9) into (2.6) gives

$$(\rho V^\mu);_\mu = \frac{\partial}{\partial x^\mu}(\rho V^\mu) + \left(\frac{1}{2}g^{\mu\kappa}\frac{\partial g_{\mu\kappa}}{\partial x^\nu}\right)(\rho V^\nu) \tag{2.10}$$

The four-velocity V^μ can be written as

$$V^{\mu}(x,s) = \varepsilon\left[\varepsilon_1 \frac{\partial \xi(x,s)}{\partial x_{\mu}} + \varepsilon_2 A^{\mu}(x,s)\right] \tag{2.11}$$

where $\varepsilon, \varepsilon_1, \varepsilon_2$ are real, scalar constants, ξ is a real, scalar function, and A^{μ} is a contravariant four-vector. The correspondence to physical terms is provided later. Equation (2.11) can be written in the form

$$V^{\mu} = -\frac{i\varepsilon\varepsilon_1}{2\rho}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\mu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\mu}}\right] + \varepsilon\varepsilon_2 A^{\mu} \tag{2.12}$$

where the Born representation of probability density in Equation (2.1) has been used to write the term $\frac{\partial \xi}{\partial x_{\mu}}$ as

$$\frac{\partial \xi}{\partial x_{\mu}} = -\frac{i}{2\rho}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\mu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\mu}}\right] \tag{2.13}$$

The probability flux becomes

$$\rho V^{\mu} = -\frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\mu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\mu}}\right] + \varepsilon\varepsilon_2 \Psi^*\Psi A^{\mu} \tag{2.14}$$

Substituting Equations (2.1), (2.14), and (2.6) into Equation (2.5), the continuity equation, gives

$$\begin{aligned} &\frac{\partial(\Psi^*\Psi)}{\partial s} + \frac{\partial}{\partial x^{\mu}}\left\{-\frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\mu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\mu}}\right] + \varepsilon\varepsilon_2 \Psi^*\Psi A^{\mu}\right\} \\ &\qquad + \Gamma^{\mu}_{\nu\mu}\left\{-\frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\nu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\nu}}\right] + \varepsilon\varepsilon_2 \Psi^*\Psi A^{\nu}\right\} = 0 \end{aligned} \tag{2.15}$$

Expanding Equation (2.15) and using

$$\frac{\partial \Psi^*}{\partial x^{\mu}}\frac{\partial \Psi}{\partial x_{\mu}} - \frac{\partial \Psi}{\partial x^{\mu}}\frac{\partial \Psi^*}{\partial x_{\mu}} = 0 \tag{2.16}$$

gives

$$\begin{aligned} &\Psi^* \frac{\partial \Psi}{\partial s} + \frac{\partial \Psi^*}{\partial s}\Psi - \frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^* \frac{\partial}{\partial x^{\mu}}\frac{\partial \Psi}{\partial x_{\mu}} - \Psi \frac{\partial}{\partial x^{\mu}}\frac{\partial \Psi^*}{\partial x_{\mu}}\right] \\ &\qquad + \varepsilon\varepsilon_2\left[\frac{\partial \Psi^*}{\partial x^{\mu}}\Psi A^{\mu} + \Psi^* \frac{\partial \Psi}{\partial x^{\mu}} A^{\mu} + \Psi^*\Psi \frac{\partial A^{\mu}}{\partial x^{\mu}}\right] \\ &\qquad + \Gamma^{\mu}_{\nu\mu}\left\{-\frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^* \frac{\partial \Psi}{\partial x_{\nu}} - \Psi \frac{\partial \Psi^*}{\partial x_{\nu}}\right] + \varepsilon\varepsilon_2 \Psi^*\Psi A^{\nu}\right\} = 0 \end{aligned} \tag{2.17}$$

Equation (2.17) can be rewritten in the form

$$\begin{aligned} &\Psi^*\left\{\frac{\partial \Psi}{\partial s} - \frac{i\varepsilon\varepsilon_1}{2}\frac{\partial}{\partial x^{\mu}}\frac{\partial \Psi}{\partial x_{\mu}} + \varepsilon\varepsilon_2\left[A^{\mu}\frac{\partial \Psi}{\partial x^{\mu}} + \frac{1}{2}\Psi \frac{\partial A^{\mu}}{\partial x^{\mu}}\right] - \frac{i\varepsilon\varepsilon_1}{2}\Gamma^{\mu}_{\nu\mu}\frac{\partial \Psi}{\partial x_{\nu}} \right. \\ &\qquad \left. + \frac{\varepsilon\varepsilon_2}{2}\Gamma^{\mu}_{\nu\mu}\Psi A^{\nu}\right\} \\ &\qquad + \Psi\left\{\frac{\partial \Psi^*}{\partial s} + \frac{i\varepsilon\varepsilon_1}{2}\frac{\partial}{\partial x^{\mu}}\frac{\partial \Psi^*}{\partial x_{\mu}} + \varepsilon\varepsilon_2\left[A^{\mu}\frac{\partial \Psi^*}{\partial x^{\mu}} + \frac{1}{2}\Psi^*\frac{\partial A^{\mu}}{\partial x^{\mu}}\right] \right. \\ &\qquad \left. + \frac{i\varepsilon\varepsilon_1}{2}\Gamma^{\mu}_{\nu\mu}\frac{\partial \Psi^*}{\partial x_{\nu}} + \frac{\varepsilon\varepsilon_2}{2}\Gamma^{\mu}_{\nu\mu}\Psi^* A^{\nu}\right\} = 0 \end{aligned} \tag{2.18}$$

Multiply Equation (2.18) by $i\varepsilon_1$ and rearrange:

$$\begin{aligned}\Psi^*\Bigg\{ & i\varepsilon_1\frac{\partial\Psi}{\partial s}+\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial}{\partial x^\mu}\frac{\partial\Psi}{\partial x_\mu}+i\varepsilon\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{1}{2}\Psi\frac{\partial A^\mu}{\partial x^\mu}\right] \\ & +\Gamma^\mu_{\nu\mu}\left[\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial\Psi}{\partial x_\nu}+\frac{i\varepsilon\varepsilon_1\varepsilon_2}{2}\Psi A^\nu\right]\Bigg\} \\ & =\Psi\Bigg\{-i\varepsilon_1\frac{\partial\Psi^*}{\partial s}+\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial}{\partial x^\mu}\frac{\partial\Psi^*}{\partial x_\mu} \\ & -i\varepsilon\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi^*}{\partial x^\mu}+\frac{1}{2}\Psi^*\frac{\partial A^\mu}{\partial x^\mu}\right] \\ & +\Gamma^\mu_{\nu\mu}\left[\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial\Psi^*}{\partial x_\nu}-\frac{i\varepsilon\varepsilon_1\varepsilon_2}{2}\Psi^* A^\nu\right]\Bigg\}\end{aligned} \tag{2.19}$$

Equation (2.19) can be written in the form

$$\Psi^*\{F\}=\{F^*\}\,\Psi=\{\Psi^* F\}^* \tag{2.20}$$

where

$$\begin{aligned}F= & \, i\varepsilon_1\frac{\partial\Psi}{\partial s}+\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial}{\partial x^\mu}\frac{\partial\Psi}{\partial x_\mu}+i\varepsilon\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{1}{2}\Psi\frac{\partial A^\mu}{\partial x^\mu}\right] \\ & +\Gamma^\mu_{\nu\mu}\left[\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial\Psi}{\partial x_\nu}+\frac{i\varepsilon\varepsilon_1\varepsilon_2}{2}\Psi A^\nu\right]\end{aligned} \tag{2.21}$$

Equation (2.21) can be decoupled by observing that the product $\Psi^* F$ must be real since $\Psi^* F$ equals its complex conjugate. We can assure that $\Psi^* F$ is real for arbitrary Ψ if we write

$$F=W\,\Psi \tag{2.22}$$

where W is a real scalar. Combining Equations (2.21) and (2.22) gives

$$\begin{aligned} & i\varepsilon_1\frac{\partial\Psi}{\partial s}+\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial}{\partial x^\mu}\frac{\partial\Psi}{\partial x_\mu}+i\frac{\varepsilon\varepsilon_1\varepsilon_2}{2}\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{\partial A^\mu\Psi}{\partial x^\mu}\right] \\ & +\Gamma^\mu_{\nu\mu}\left[\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial\Psi}{\partial x_\nu}+\frac{i\varepsilon\varepsilon_1\varepsilon_2}{2}\Psi A^\nu\right]=W\,\Psi\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}i\varepsilon\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{1}{2}\Psi\frac{\partial A^\mu}{\partial x^\mu}\right] & =i\frac{\varepsilon\varepsilon_1\varepsilon_2}{2}\left[2A^\mu\frac{\partial\Psi}{\partial x^\mu}+\Psi\frac{\partial A^\mu}{\partial x^\mu}\right] \\ & =i\frac{\varepsilon\varepsilon_1\varepsilon_2}{2}\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{\partial A^\mu\Psi}{\partial x^\mu}\right]\end{aligned} \tag{2.24}$$

Equation (2.23) can be written in a more suggestive form by writing W as the invariant

$$W=\frac{\varepsilon\varepsilon_2^2}{2}A_\mu A^\mu+V \tag{2.25}$$

so that Equation (2.23) becomes

$$\begin{aligned} & i\varepsilon_1\frac{\partial\Psi}{\partial s}+\frac{\varepsilon}{2}\left\{\varepsilon_1^2\frac{\partial}{\partial x^\mu}\frac{\partial\Psi}{\partial x_\mu}+i\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{\partial A^\mu\Psi}{\partial x^\mu}\right]-\frac{2}{\varepsilon}\frac{\varepsilon\varepsilon_2^2}{2}A_\mu A^\mu\Psi\right\} \\ & +\Gamma^\mu_{\nu\mu}\left[\frac{\varepsilon\varepsilon_1^2}{2}\frac{\partial\Psi}{\partial x_\nu}+\frac{i\varepsilon\varepsilon_1\varepsilon_2}{2}\Psi A^\nu\right]-V\,\Psi=0\end{aligned} \tag{2.26}$$

Rearranging Equation (2.26) gives

$$\begin{aligned}i\varepsilon_1\frac{\partial\Psi}{\partial s}= & \frac{\varepsilon}{2}\Bigg\{-\varepsilon_1^2\frac{\partial}{\partial x^\mu}\frac{\partial\Psi}{\partial x_\mu}-i\varepsilon_1\varepsilon_2\left[A^\mu\frac{\partial\Psi}{\partial x^\mu}+\frac{\partial A^\mu\Psi}{\partial x^\mu}\right]+\varepsilon_2^2A_\mu A^\mu\Psi \\ & -\Gamma^\mu_{\nu\mu}\left[\varepsilon_1^2\frac{\partial\Psi}{\partial x_\nu}+i\varepsilon_1\varepsilon_2\Psi A^\nu\right]\Bigg\}+V\,\Psi\end{aligned} \tag{2.27}$$

Simplifying Equation (2.27) lets us write

$$i\varepsilon_1 \frac{\partial \Psi}{\partial s} = \frac{\varepsilon}{2}\left\{\left[\frac{\varepsilon_1}{i}\frac{\partial}{\partial x^\mu} + \varepsilon_2 A_\mu\right]\left[\frac{\varepsilon_1}{i}\frac{\partial}{\partial x_\mu} + \varepsilon_2 A^\mu\right]\Psi - \Gamma^\mu_{\nu\mu}\left[\varepsilon_1^2 \frac{\partial \Psi}{\partial x_\nu} + i\varepsilon_1\varepsilon_2 \Psi A^\nu\right]\right\} + V\,\Psi \tag{2.28}$$

The physical interpretation of the scalar parameters $\varepsilon, \varepsilon_1, \varepsilon_2$ in Equation (2.28) are

$$\varepsilon = \frac{1}{m}, \varepsilon_1 = \hbar, \varepsilon_2 = -\frac{e}{c} \tag{2.29}$$

The relationships in Equation (2.29) are obtained by comparing Equation (2.28) with the Klein-Gordon equation. Substituting Equation (2.29) into (2.28) gives

$$i\hbar \frac{\partial \Psi}{\partial s} = \frac{1}{2m}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}A^\mu\right]\Psi - \Gamma^\mu_{\nu\mu}\left[\hbar^2 \frac{\partial \Psi}{\partial x_\nu} - i\hbar\frac{e}{c}\Psi A^\nu\right]\right\} + V\,\Psi \tag{2.30}$$

Equation (2.30) is referred to here as the Stueckelberg equation in curved spacetime for a single particle.

An alternative form of Equation (2.30) is

$$i\hbar \frac{\partial \Psi}{\partial s} = \frac{1}{2m}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}A^\mu\right]\Psi - i\hbar\Gamma^\mu_{\nu\mu}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\nu} - \frac{e}{c}A^\nu\right]\Psi\right\} + V\,\Psi \tag{2.31}$$

Equation (2.31) can be written in the form

$$i\hbar \frac{\partial \Psi}{\partial s} = K\Psi \tag{2.32}$$

where K is the operator

$$K = \frac{1}{2m}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}A^\mu\right] - i\hbar\Gamma^\mu_{\nu\mu}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\nu} - \frac{e}{c}A^\nu\right]\right\} + V \tag{2.33}$$

A more compact form of Equation (2.33) is obtained by defining the operators

$$\pi_\sigma = \frac{\hbar}{i}\frac{\partial}{\partial x^\sigma} - \frac{e}{c}A_\sigma, \pi^\sigma = \frac{\hbar}{i}\frac{\partial}{\partial x_\sigma} - \frac{e}{c}A^\sigma \tag{2.34}$$

so that Equation (2.34) becomes

$$K = \frac{1}{2m}\left\{\pi_\mu \pi^\mu - i\hbar\Gamma^\mu_{\nu\mu}\pi^\nu\right\} + V \tag{2.35}$$

3. Expectation Values

The probabilistic formalism in Section 2 is connected to observable quantities using expectation values, a well-defined concept in probability theory. The form of the definition of expectation value that is well suited to the operators introduced in Section 2 is motivated by applying the probabilistic concept of expectation value to the calculation of two common physical quantities: four-position, and four-velocity

The expectation value of the μ^{th} component of particle four-position is

$$\langle y^\mu \rangle = \int_D \rho y^\mu \sqrt{|g(y)|}dy \tag{3.1}$$

Replacing ρ with Equation (2.1) gives the expectation value

$$\langle y^\mu \rangle = \int_D \Psi^* y^\mu \Psi \sqrt{|g(y)|}dy. \tag{3.2}$$

The rate of change of $\langle y^\mu \rangle$ with respect to evolution parameter s is obtained by differentiating Equation (3.2) with respect to s:

$$\frac{d\langle y^\mu \rangle}{ds} = \int_D y^\mu \left[\frac{\partial \rho}{\partial s}\right]\sqrt{|g(y)|}dy. \tag{3.3}$$

Using the continuity equation, Equation (2.5), lets us write

$$\frac{d\langle y^\mu\rangle}{ds} = -\int_D y^\mu[(\rho V^\nu)_{;\nu}]\sqrt{|g(y)|}dy. \tag{3.4}$$

Replacing the covariant derivative $(\rho V^\nu)_{;\nu}$ with Equation (2.6) gives

$$\frac{d\langle y^\mu\rangle}{ds} = -\int_D y^\mu\left[\frac{\partial}{\partial y^\nu}(\rho V^\nu) + \Gamma^\nu_{\sigma\nu}(\rho V^\sigma)\right]\sqrt{|g(y)|}dy. \tag{3.5}$$

where ν, σ are dummy indices. Replacing $y^\mu \frac{\partial}{\partial y^\nu}(\rho V^\nu)$ with

$$y^\mu \frac{\partial}{\partial y^\nu}(\rho V^\nu) = \frac{\partial \rho V^\nu y^\mu}{\partial y^\nu} - \rho V^\nu \frac{\partial y^\mu}{\partial y^\nu} \tag{3.6}$$

lets us write

$$\begin{aligned}\frac{d\langle y^\mu\rangle}{ds} = -\Bigg\{&\int_D \left[\frac{\partial \rho V^\nu y^\mu}{\partial y^\nu} - \rho V^\nu \frac{\partial y^\mu}{\partial y^\nu}\right]\sqrt{|g(y)|}dy \\ &+ \int_D y^\mu \Gamma^\nu_{\sigma\nu}(\rho V^\sigma)\sqrt{|g(y)|}dy\Bigg\}.\end{aligned} \tag{3.7}$$

Equation (3.7) can be written as

$$\begin{aligned}\frac{d\langle y^\mu\rangle}{ds} = -\Bigg\{&\int_D \frac{\partial \rho V^\nu y^\mu}{\partial y^\nu}\sqrt{|g(y)|}dy - \int_D \rho V^\nu \frac{\partial y^\mu}{\partial y^\nu}\sqrt{|g(y)|}dy \\ &+ \int_D \Gamma^\nu_{\sigma\nu}(\rho V^\sigma y^\mu)\sqrt{|g(y)|}dy\Bigg\}.\end{aligned} \tag{3.8}$$

Combining the first and third terms in the curly bracketed term gives

$$\begin{aligned}\frac{d\langle y^\mu\rangle}{ds} = -\Bigg\{&\int_D \left[\frac{\partial \rho V^\nu y^\mu}{\partial y^\nu} + \Gamma^\nu_{\sigma\nu}(\rho V^\sigma y^\mu)\right]\sqrt{|g(y)|}dy \\ &- \int_D \rho V^\nu \frac{\partial y^\mu}{\partial y^\nu}\sqrt{|g(y)|}dy\Bigg\}.\end{aligned} \tag{3.9}$$

The bracketed term on the right-hand side of Equation (3.9) is the covariant derivative $(\rho V^\nu y^\mu)_{;\nu}$ so Equation (3.9) can be written as

$$\frac{d\langle y^\mu\rangle}{ds} = -\left\{\int_D (\rho V^\nu y^\mu)_{;\nu}\sqrt{|g(y)|}dy - \int_D \rho V^\nu \frac{\partial y^\mu}{\partial y^\nu}\sqrt{|g(y)|}dy\right\}. \tag{3.10}$$

The covariant derivative term on the right-hand side of Equation (3.10) vanishes when we use the divergence theorem

$$\int_D (\rho V^\nu y^\mu)_{;\nu}\sqrt{|g(y)|}dy = \oint_S \rho V^\nu y^\mu \sqrt{|g(y)|}dS_\nu = 0 \tag{3.11}$$

to convert the integral over the hypervolume D to an integral over the hypersurface S that encloses the hypervolume D. The hypersurface integral vanishes if we assume that probability ρ goes to zero on the hypersurface S. In this case, Equation (3.10) simplifies to the form

$$\frac{d\langle y^\mu\rangle}{ds} = \int_D \rho V^\mu \sqrt{|g(y)|}dy = \langle V^\mu\rangle \tag{3.12}$$

where we have used the Kronecker delta

$$\frac{\partial y^\mu}{\partial y^\nu} = \delta^\mu_\nu. \tag{3.13}$$

A quantum mechanical expression for $\langle V^\mu\rangle$ is obtained by writing Equation (2.14) as

$$\rho V^{\mu} = -\frac{i\varepsilon\varepsilon_1}{2}\left[\Psi^*\frac{\partial\Psi}{\partial x_{\mu}} - \Psi\frac{\partial\Psi^*}{\partial x_{\mu}}\right] + \varepsilon\varepsilon_2\Psi^*\Psi A^{\mu}$$
$$= -\frac{i\varepsilon\varepsilon_1}{2}\left[2\Psi^*\frac{\partial\Psi}{\partial x_{\mu}} - \frac{\partial\Psi^*\Psi}{\partial x_{\mu}}\right] + \varepsilon\varepsilon_2\Psi^*\Psi A^{\mu} \tag{3.14}$$

where

$$\Psi\frac{\partial\Psi^*}{\partial x_{\mu}} = \frac{\partial\Psi^*\Psi}{\partial x_{\mu}} - \Psi^*\frac{\partial\Psi}{\partial x_{\mu}} \tag{3.15}$$

has been used. Equation (3.15) can be rearranged to the form

$$\rho V^{\mu} = \Psi^*\left[-i\varepsilon\varepsilon_1\frac{\partial}{\partial x_{\mu}} + \varepsilon\varepsilon_2 A^{\mu}\right]\Psi + \frac{i\varepsilon\varepsilon_1}{2}\frac{\partial\Psi^*\Psi}{\partial x_{\mu}} \tag{3.16}$$

Substituting Equation (3.16) into (3.12) gives

$$\langle V^{\mu}\rangle = \int_D \Psi^*\left[-i\varepsilon\varepsilon_1\frac{\partial}{\partial x_{\mu}} + \varepsilon\varepsilon_2 A^{\mu}\right]\Psi\sqrt{|g(y)|}dy + \int_D \frac{i\varepsilon\varepsilon_1}{2}\frac{\partial\Psi^*\Psi}{\partial x_{\mu}}\sqrt{|g(y)|}dy \tag{3.17}$$

The second integral on the left-hand side of Equation (3.17) vanishes when the divergence theorem is applied and noting that ρ vanishes on the hypersurface of the hypervolume. The result is

$$\langle V^{\mu}\rangle = \int_D \Psi^*\left[-i\varepsilon\varepsilon_1\frac{\partial}{\partial x_{\mu}} + \varepsilon\varepsilon_2 A^{\mu}\right]\Psi\sqrt{|g(y)|}dy. \tag{3.18}$$

Substituting the relationships in Equation (2.29) into Equation (3.18) gives

$$\langle V^{\mu}\rangle = \int_D \Psi^*\left[-i\frac{\hbar}{m}\frac{\partial}{\partial x_{\mu}} - \frac{e}{mc}A^{\mu}\right]\Psi\sqrt{|g(y)|}dy. \tag{3.19}$$

The bracketed term on the right-hand side of Equation (3.19) is the operator

$$\frac{\pi^{\mu}}{m} = \frac{1}{m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_{\mu}} - \frac{e}{c}A^{\mu}\right] \tag{3.20}$$

Substituting Equation (3.20) into (3.19) gives

$$\langle V^{\mu}\rangle = \int_D \Psi^*\left[\frac{\pi^{\mu}}{m}\Psi\right]\sqrt{|g(y)|}dy. \tag{3.21}$$

The expectation values $\langle y^{\mu}\rangle, \langle V^{\mu}\rangle$ are expressed in terms of operators in Equations (3.2) and (3.21). This suggests that a reasonable definition of the expectation value of an operator Ω should be

$$\langle\Omega\rangle = \int_D \Psi^*[\Omega\Psi]\sqrt{|g(y)|}dy. \tag{3.22}$$

where Ω operates only on Ψ. Equation (3.22) is applicable to both flat and curved spacetime.

4. Free Scalar Particle in Flat Spacetime

A free scalar particle moving in flat spacetime is modeled by Equation (2.31) with $\{V = 0, A^{\mu} = 0\}$ and the fundamental metric tensor with nonzero elements

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33} \tag{4.1}$$

In this case Equation (2.31) becomes

$$i\hbar\frac{\partial\psi_f}{\partial s} = \frac{1}{2m}\left\{\left[\frac{\hbar}{i}\frac{\partial}{\partial x^{\mu}}\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x_{\mu}}\right] - i\hbar\Gamma^{\mu}_{\nu\mu}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_{\nu}}\right]\right\}\psi_f \tag{4.2}$$

with probability amplitude ψ_f. The contracted form $\Gamma^{\mu}_{\nu\mu}$ of the Christoffel symbol of the second kind is found from Equation (2.9) to be

$$\Gamma^{\mu}_{\nu\mu} = \frac{1}{2}g^{\mu\kappa}\frac{\partial g_{\mu\kappa}}{\partial x^{\nu}} = 0 \tag{4.3}$$

for the fundamental metric tensor in Equation (4.1). Equation (4.2) simplifies to

$$i\hbar\frac{\partial\psi_f}{\partial s} = K_{flat}\psi_f \tag{4.4}$$

where K_{flat} is the operator

$$K_{flat} = \frac{1}{2m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu}\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\mu}\right] \tag{4.5}$$

in flat spacetime. Combining Equations (4.4) and (4.5) gives the Stueckelberg equation for the free particle

$$i\hbar\frac{\partial\psi_f}{\partial s} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\psi_f = -\frac{\hbar^2}{2m}\partial_\mu\partial^\mu\psi_f \tag{4.6}$$

Equation (4.6) has the general solution

$$\Psi_f(x,s) = \int \psi_{f\kappa}(x,s)\,dk_f = \int \eta_{f\kappa}\, exp\left[i\frac{\hbar\beta_f(k_f)}{2m}s + ik_{f\mu}x^\mu\right]dk_f \tag{4.7}$$

where the integral is over energy-momentum, $\eta_{f\kappa}$ denotes normalization coefficients, and

$$\beta_f(k_f) = k_{f\mu}k_f^\mu \tag{4.8}$$

A wave packet can be constructed using Equations (4.6) to (4.8).

4.1. The Meaning of Mass

The expectation value of the four-velocity of the free particle is

$$\left\langle V_f^\mu\right\rangle = \frac{d\left\langle x_f^\mu\right\rangle}{ds} = \frac{\left\langle \hbar k_f^\mu\right\rangle}{m} = \frac{\left\langle p_f^\mu\right\rangle}{m} \tag{4.9}$$

Integrating Equation (4.9) from s to $s+\delta s$ gives

$$\delta\left\langle x_f^\mu\right\rangle = \frac{\left\langle p_f^\mu\right\rangle}{m}\delta s \tag{4.10}$$

Equation (4.10) is the expectation value of spacetime position of the free particle. The observable world-line of the free particle is given by the inner product

$$\delta\langle x_{f\mu}\rangle\delta\left\langle x_f^\mu\right\rangle = \frac{\langle p_{f\mu}\rangle\left\langle p_f^\mu\right\rangle}{m^2}(\delta s)^2 \tag{4.11}$$

Solving for m^2 gives

$$m^2 = \frac{\langle p_{f\mu}\rangle\left\langle p_f^\mu\right\rangle}{\delta\langle x_{f\mu}\rangle\delta\left\langle x_f^\mu\right\rangle}(\delta s)^2 \tag{4.12}$$

Equation (4.12) applies to both timelike (bradyonic) motion and spacelike (tachyonic) motion. We can see this by observing that in parametrized relativistic dynamics, the invariant evolution parameter increases monotonically so that $\delta s > 0$. On the other hand, the terms $\langle p_{f\mu}\rangle\left\langle p_f^\mu\right\rangle$, $\delta\langle x_{f\mu}\rangle\delta\left\langle x_f^\mu\right\rangle$ can be either timelike $\langle p_{f\mu}\rangle\left\langle p_f^\mu\right\rangle > 0$, $\delta\langle x_{f\mu}\rangle\delta\left\langle x_f^\mu\right\rangle > 0$ or spacelike $\langle p_{f\mu}\rangle\left\langle p_f^\mu\right\rangle < 0$, $\delta\langle x_{f\mu}\rangle\delta\left\langle x_f^\mu\right\rangle < 0$. Consequently, m^2 is positive for both timelike and spacelike motion because negative signs associated with spacelike motion cancel. Free tachyons in parametrized relativistic dynamics have real mass.

The conventional classical theory imposes the on-shell mass constraint

$$m_{cl}^2c^2 = p_{f\mu}p_f^\mu \tag{4.13}$$

on allowed values of energy-momentum. Subscript cl indicates that mass m_{cl} refers to the conventional classical theory. An on-shell mass relationship between mass and energy-momentum exists in pRQT in the classical limit of negligible dispersion. Energy-momentum dispersion is

$$\Delta^2 p_f \equiv \left\langle p_{f\mu} p_f^\mu \right\rangle - \langle p_{f\mu} \rangle \left\langle p_f^\mu \right\rangle \tag{4.14}$$

Substituting Equation (4.14) into (4.11) gives

$$\delta \langle x_{f\mu} \rangle \delta \left\langle x_f^\mu \right\rangle = \frac{\left\langle p_{f\mu} p_f^\mu \right\rangle - \Delta^2 p_f}{m^2} (\delta s)^2 \tag{4.15}$$

In the classical limit, dispersion is negligible and we find that m^2 is statistically related to independent variables energy and momentum by

$$m^2 c^2 = \left\{ \frac{c^2 (\delta s)^2}{\delta \langle x_{f\mu} \rangle \delta \left\langle x_f^\mu \right\rangle} \right\} \left\langle p_{f\mu} p_f^\mu \right\rangle \tag{4.16}$$

We have inserted the term c^2 on both sides of Equation (4.16) so that the bracketed term is dimensionless. In the classical limit, the magnitude of the bracketed term is one, but we need to account for the sign of $\delta \langle x_{f\mu} \rangle \delta \left\langle x_f^\mu \right\rangle$, so Equation (4.16) becomes

$$m^2 c^2 = \frac{\left\langle p_{f\mu} p_f^\mu \right\rangle}{sgn \left\{ \delta \langle x_{f\mu} \rangle \delta \left\langle x_f^\mu \right\rangle \right\}} \geq 0 \tag{4.17}$$

where "sgn" says that we are accounting for the sign of the bracketed term. Equation (4.17) applies to both timelike and spacelike free particles. It says that m may be interpreted as on-shell mass for a free particle moving at any speed.

5. Free Scalar Particle in Schwarzschild Metric

The equation for a free scalar particle with mass m moving in the gravitational field of a spherically symmetric mass M can be modeled by Equation (2.31) with $\{V = 0, A^\mu = 0\}$ and the Schwarzschild metric [10] (Chapter 7), [11] (Chapter 8), [12] (Chapter 9). In this case Equation (2.31) becomes

$$i\hbar \frac{\partial \psi_{SM}}{\partial s} = K_{SM} \psi_{SM} = \frac{1}{2m} \left\{ \left[\frac{\hbar}{i} \frac{\partial}{\partial y^\mu} \right] \left[\frac{\hbar}{i} \frac{\partial}{\partial y_\mu} \right] - i\hbar \Gamma^\mu_{\nu\mu} \left[\frac{\hbar}{i} \frac{\partial}{\partial y_\nu} \right] \right\} \psi_{SM} \tag{5.1}$$

where the operator K_{SM} for the Schwarzschild metric is

$$K_{SM} = \frac{1}{2m} \left\{ \left[\frac{\hbar}{i} \frac{\partial}{\partial y^\mu} \right] \left[\frac{\hbar}{i} \frac{\partial}{\partial y_\mu} \right] - i\hbar \Gamma^\mu_{\nu\mu} \left[\frac{\hbar}{i} \frac{\partial}{\partial y_\nu} \right] \right\} \tag{5.2}$$

The contravariant four-vector for the Schwarzschild metric is $y^\mu = \{ct, r, \theta, \phi\}$ where $\{r, \theta, \phi\}$ are spherical coordinates. The invariant interval between points y^μ and $y^\mu + dy^\mu$ is

$$ds^2 = g_{\mu\nu} dy^\mu dy^\nu \tag{5.3}$$

where the nonzero elements of the Schwarzschild metric are

$$g_{00} = -A(r), g_{11} = 1/A(r), g_{22} = r^2, g_{33} = r^2 sin^2\theta, A(r) = 1 - \frac{r_S}{r} \tag{5.4}$$

The term r_S is the Schwarzschild radius

$$r_S = \frac{2MG_N}{c^2} \tag{5.5}$$

and G_N is Newton's gravitational constant. Nonzero elements of the contravariant metric tensor $g^{\mu\nu}$ of the Schwarzschild metric $g_{\mu\nu}$ in Equation (5.4) are the inverse values

$$g^{00}=-\frac{1}{A(r)},g^{11}=A(r),g^{22}=\frac{1}{r^2},g^{33}=\frac{1}{r^2sin^2\theta},A(r)=1-\frac{r_S}{r} \tag{5.6}$$

Combining Equations (5.4) and (5.6) shows that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$:

$$g_{\mu\sigma}g^{\sigma\nu}=\delta_\mu^\nu \tag{5.7}$$

The contracted form of the Christoffel symbol of the second kind $\Gamma^\mu_{\nu\mu}$ in Equation (5.1) is calculated using the relationship

$$\Gamma^\mu_{\nu\mu}=\frac{1}{2g}\frac{\partial g}{\partial x^\nu} \tag{5.8}$$

where the determinant of the Schwarzschild metric g is

$$\mathrm{g}=-r^4sin^2\theta \tag{5.9}$$

Combining Equations (5.8) and (5.9) gives

$$\Gamma^\mu_{0\mu}=0,\Gamma^\mu_{1\mu}=\frac{2}{r},\Gamma^\mu_{2\mu}=2cot\theta,\Gamma^\mu_{3\mu}=0 \tag{5.10}$$

Rearranging Equation (5.1) and writing the term $\Gamma^\mu_{\nu\mu}\left[\frac{\partial}{\partial y_\nu}\right]$ as $\Gamma^\mu_{\nu\mu}\left[g^{\nu\sigma}\frac{\partial}{\partial y^\sigma}\right]$ gives

$$\begin{aligned}i\hbar\frac{\partial\psi_{SM}}{\partial s}&=\frac{-\hbar^2}{2m}\left\{\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}-\Gamma^\mu_{\nu\mu}\left[\frac{\partial}{\partial y_\nu}\right]\right\}\psi_{SM}\\&=\frac{-\hbar^2}{2m}\left\{\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}-\Gamma^\mu_{\nu\mu}\left[g^{\nu\sigma}\frac{\partial}{\partial y^\sigma}\right]\right\}\psi_{SM}\end{aligned} \tag{5.11}$$

Using Equations (5.6) and (5.10) in (5.11) gives

$$i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}\left\{\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}+\left[\frac{2}{r}A(r)\frac{\partial}{\partial r}+\frac{cot\theta}{r^2}\frac{\partial}{\partial\theta}\right]\right\}\psi_{SM} \tag{5.12}$$

The term $\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}$ can be written in the Schwarzschild metric as

$$\begin{aligned}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}\psi_{SM}&=\frac{\partial}{\partial y^\mu}g^{\mu\sigma}\frac{\partial}{\partial y^\sigma}\psi_{SM}\\&=\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right.\\&\quad\left.+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi_{SM}\end{aligned} \tag{5.13}$$

Substituting Equation (5.13) into (5.12) gives

$$\begin{aligned}i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}&\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right.\\&\left.+\left[\frac{2}{r}A(r)\frac{\partial}{\partial r}+\frac{cot\theta}{r^2}\frac{\partial}{\partial\theta}\right]\right\}\psi_{SM}\end{aligned} \tag{5.14}$$

Equation (5.14) can be written in terms of Laplacian and D'Alembertian operators by rearranging terms as follows:

$$\begin{aligned}i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}&\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]+\frac{2}{r}A(r)\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right.\\&\left.+\frac{cos\theta}{r^2sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi_{SM}\end{aligned} \tag{5.15}$$

or

$$i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]+\frac{1}{r^2sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi_{SM} \tag{5.16}$$

The bracketed term in Equation (5.16) can be written in terms of the D'Alembertian operator for the Schwarzschild metric

$$\Box_{SM}=-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\triangle_{SM} \tag{5.17}$$

where the Laplacian operator for the Schwarzschild metric is

$$\triangle_{SM}=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]+\frac{1}{r^2sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{5.18}$$

Equation (5.16) becomes

$$i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}\Box_{SM}\,\psi_{SM}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\triangle_{SM}\right\}\psi_{SM} \tag{5.19}$$

In flat spacetime, $A(r)\to 1$and Equation (5.16) becomes

$$i\hbar\frac{\partial\psi_{flat}}{\partial s}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]+\frac{1}{r^2sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi_{flat} \tag{5.20}$$

or

$$i\hbar\frac{\partial\psi_{flat}}{\partial s}=\frac{-\hbar^2}{2m}\Box_{flat}\,\psi_{flat}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\triangle_{flat}\right\}\psi_{flat} \tag{5.21}$$

with D'Alembertian $\Box_{flat}$ and Laplacian $\triangle_{flat}$ operators in flat spacetime.

6. Solution of the Free Scalar Particle in the Schwarzschild Metric

The equation for a free scalar particle with mass m moving in the gravitational field of a spherically symmetric mass M is Equation (5.16), written here in terms of the operator $L\cdot L$

$$i\hbar\frac{\partial\psi_{SM}}{\partial s}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{L\cdot L}{r^2}\right\}\psi_{SM} \tag{6.1}$$

where $L\cdot L$ is the operator

$$L\cdot L=-\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{6.2}$$

The nonzero elements of the Schwarzschild metric are

$$g_{00}=-A(r),g_{11}=1/A(r),g_{22}=r^2,g_{33}=r^2sin^2\theta,A(r)=1-\frac{r_S}{r} \tag{6.3}$$

where the contravariant four-vector is $y^\mu=\{ct,r,\theta,\phi\}$ and $\{r,\theta,\phi\}$ are spherical coordinates.

A trial solution for separating variables in Equation (6.1) is

$$\psi_{SM}=T(t)R(r)Y_{\ell m}(\theta,\phi)exp\left[i\frac{\hbar\beta_{SM}}{2m}s\right] \tag{6.4}$$

The term $Y_{\ell m}(\theta,\phi)$ represents orthonormal spherical harmonics and satisfies the eigenvalue equation

$$L\cdot LY_{\ell m}(\theta,\phi)=\ell(\ell+1)Y_{\ell m}(\theta,\phi)\ for\ \left\{\begin{matrix}\ell=0,1,2\ldots\\ m=0,1,\ldots,|\ell|\end{matrix}\right\} \tag{6.5}$$

Substituting Equation (6.4) into (6.1) and rearranging lets us write

$$i\hbar\frac{i\hbar\beta_{SM}}{2m}\psi_{SM}=\frac{-\hbar^2}{2m}\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}\right\}\psi_{SM} \tag{6.6}$$

Canceling $\frac{-\hbar^2}{2m}$ gives

$$\beta_{SM}\psi_{SM}=\left\{-\frac{1}{A(r)}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}\right\}\psi_{SM} \tag{6.7}$$

Multiplying Equation (6.7) by $1/\psi_{SM}$ and simplifying gives

$$\beta_{SM}=-\frac{1}{A(r)}\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}+\frac{1}{R}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}\right\}R \tag{6.8}$$

The $T(t)$ factor can be written as

$$T(t)=\alpha exp(\pm i\omega t) \tag{6.9}$$

where α,ω are constants. Substituting Equation (6.9) into (6.8) gives

$$\beta_{SM}=\frac{1}{A(r)}\frac{\omega^2}{c^2}+\frac{1}{R}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}\right\}R \tag{6.10}$$

since

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=-\omega^2 \tag{6.11}$$

Multiplying Equation (6.10) by R gives

$$\beta_{SM}R=\frac{1}{A(r)}\frac{\omega^2}{c^2}R+\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]-\frac{\ell(\ell+1)}{r^2}\right\}R \tag{6.12}$$

Equation (6.12) can be written as the radial equation

$$\left[\beta_{SM}-\frac{1}{A(r)}\frac{\omega^2}{c^2}+\frac{\ell(\ell+1)}{r^2}\right]R=\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2A(r)\frac{\partial}{\partial r}\right]\right\}R \tag{6.13}$$

upon rearrangement. Expanding the right-hand side of Equation (6.13) gives

$$\left[\beta_{SM}-\frac{1}{A(r)}\frac{\omega^2}{c^2}+\frac{\ell(\ell+1)}{r^2}\right]R=A(r)\frac{\partial^2 R}{\partial r^2}+\left[\frac{r_S}{r^2}+\frac{2A(r)}{r}\right]\frac{\partial R}{\partial r} \tag{6.14}$$

Rearranging Equation (6.14) gives the homogeneous ordinary differential equation

$$A(r)\frac{d^2R}{dr^2}+\left[\frac{r_S}{r^2}+\frac{2A(r)}{r}\right]\frac{dR}{dr}-\left[\beta_{SM}-\frac{1}{A(r)}\frac{\omega^2}{c^2}+\frac{\ell(\ell+1)}{r^2}\right]R=0 \tag{6.15}$$

Multiplying by $1/A(r)$ gives

$$\frac{d^2R}{dr^2}+p(r)\frac{dR}{dr}+q(r)R=0 \tag{6.16}$$

where

$$p(r)=\frac{1}{A(r)}\frac{r_S}{r^2}+\frac{2}{r},q(r)=\frac{1}{A(r)}\left[\frac{1}{A(r)}\frac{\omega^2}{c^2}-\beta_{SM}-\frac{\ell(\ell+1)}{r^2}\right] \tag{6.17}$$

6.1. The Large Distance Approximation

Equation (6.16) can be solved in the large distance approximation corresponding to $r\gg r_S$ and $A(r)\rightarrow 1$. In this case the particle is at a large distance from the spherically symmetric mass M. The nonzero elements of the metric become

$$g_{00}=-1,g_{11}=1,g_{22}=r^2,g_{33}=r^2sin^2\theta \tag{6.18}$$

Contrasting Equation (6.18) with (4.1) shows that the two metrics differ in spatial geometry and sign convention. The spatial geometry in Equation (4.1) is Cartesian while it is spherically symmetric in Equation (6.18).

The terms $p(r), q(r)$ in Equation (6.17) simplify to

$$p(r) \to \frac{2}{r}, q(r) \to \frac{\omega^2}{c^2} - \beta_{SM} \equiv Q \tag{6.19}$$

in the large distance approximation. The corresponding radial equation has the simplified form

$$\frac{d^2 R_{Q\ell}(r)}{dr^2} + \frac{2}{r}\frac{dR_{Q\ell}(r)}{dr} + QR_{Q\ell}(r) - \frac{\ell(\ell+1)}{r^2}R_{Q\ell}(r) = 0 \tag{6.20}$$

where the subscripts of the radial solution $R_{Q\ell}$ specify solution parameters. Multiplying by r^2 gives

$$r^2\frac{d^2 R_{Q\ell}(r)}{dr^2} + 2r\frac{dR_{Q\ell}(r)}{dr} + [Qr^2 - \ell(\ell+1)]R_{Q\ell}(r) = 0 \tag{6.21}$$

The variable transformation

$$\chi = \kappa r \tag{6.22}$$

with constant κ lets us write Equation (6.21) in the form

$$\chi^2\frac{d^2 R_{Q\ell}(\chi)}{d\chi^2} + 2\chi\frac{dR_{Q\ell}(\chi)}{d\chi} + [\chi^2 - \ell(\ell+1)]R_{Q\ell}(\chi) = 0 \tag{6.23}$$

where

$$Q = \kappa^2 \tag{6.24}$$

Solutions of Equation (6.23) that are regular at $r = 0$ are spherical Bessel functions of the first kind [13] (Chapter 9), [14] (Section 11.7), [15] (Chapter 10):

$$R_{Q\ell}(\chi) = j_\ell(\chi) = \sqrt{\frac{\pi/2}{\chi}}J_{\ell+\frac{1}{2}}(\chi) \tag{6.25}$$

The term $J_{\ell+\frac{1}{2}}(\chi)$ is the Bessel function of the first kind. As an illustration, the lowest order term is

$$R_{Q0}(\chi) = j_0(\chi) = \frac{sin\chi}{\chi}, \chi = \kappa r = \sqrt{Q}\, r \tag{6.26}$$

7. Conclusions

A formulation of parametrized relativistic quantum theory (pRQT) in curved spacetime has been developed. The formulation is based on the simplifying assumption that the metric is independent of the evolution parameter. Given this assumption, the resulting formulation was applied to the case of a free particle in flat spacetime and the case of a particle in the Schwarzschild metric. The differential equations for the Schwarzschild metric case were presented and solved in the large distance approximation. The formulation developed here provides a basis for future applications.

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