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# Can particle appearance or disappearance be described by a quantum mechanical theory?

To cite this article: John R. Fanchi 2021 *J. Phys.: Conf. Ser.* **1956** 012007

# Can particle appearance or disappearance be described by a quantum mechanical theory?

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**Abstract**. A common justification for replacing quantum mechanics with quantum field theory (QFT) is that the appearance or disappearance of particles cannot be described using quantum mechanics. We show that this justification for QFT is not generally true by presenting a counterexample: parametrized relativistic quantum mechanics (pRQM). We begin by outlining a pioneering formulation of QFT that includes an invariant evolution parameter. The introduction of an invariant evolution parameter helped guide the development of QFT and is a characteristic feature of pRQM. We then present a probabilistic formulation of pRQM that highlights features of the theory that make it suitable for modelling particle stability. Two examples of particle stability are then presented within the context of pRQM to show that a quantum mechanical theory can be applied to particle stability. The examples considered in this paper are exponential particle decay and neutrino oscillations.

**Keywords.** quantum field theory, relativistic quantum mechanics, parametrized, particle stability, particle decay, neutrino oscillations

## 1. Introduction

The Particle Data Group [1] periodically compiles experimental results and a selection of theoretical attempts to understand the results. Precision tests of quantum electrodynamics (QED), such as the calculation of the anomalous magnetic moment of the electron, agree with measured values to an accuracy of one part per billion or better.

QED is an example of a quantum field theory (QFT) and is part of the Standard Model. Despite its record of success, many anomalies can be identified in the Standard Model. The Standard Model hypothesizes the existence of quarks, leptons, and the Higgs particle. Quarks and leptons are the building blocks of the particle zoo, yet no quark has ever been observed. The existence of quarks is inferred from a sizable body of indirect experimental evidence, and the Higgs particle was discovered in 2012 [2]. Masses of quarks, leptons, and the Higgs particle are some of the parameters that must be entered into the Standard Model.

The Standard Model assumes that neutrinos are massless, yet neutrino oscillation experiments imply that neutrinos have mass. Neutrinos with mass can be included in an extension of the Standard Model, but this requires the use of more unexplained parameters.

The Standard Model does not account for gravity. A theory of gravity is needed that is consistent with QFT and is able to yield general relativity as the classical approximation. In a related context, proponents of the conventional paradigm are continuing to seek a better understanding of cosmological issues such as inflation, dark matter, and dark energy.

In addition to these anomalies, there are some practical issues associated with the formalism of the conventional paradigm. Historically, relativistic QFT has had conceptual and technical difficulties, such as divergences and infinities that require renormalization, and the need to use perturbative techniques to solve problems. For example, it is difficult to solve relativistic bound state problems using the formalism of the conventional paradigm.

Proponents of the conventional paradigm view the anomalies as problems that will eventually be solved within the general framework of the conventional paradigm. Difficulties with the formalism are issues that can be overcome by technological advances in areas such as mathematics or computing.

The purpose of this paper is to challenge a common misconception that QFT must be used to model particle stability, which refers here to either particle appearance or disappearance. The misconception is the view that quantum mechanics in either its non-relativistic or relativistic form cannot model particle stability. We show by example that this view is incorrect. The example used here is parametrized relativistic quantum mechanics (pRQM).

Parametrization helped Feynman [3], Schwinger [4, 5] and other researchers develop quantum field theory (QFT). The historical context for QFT prior to 1945 was provided by Schweber [6]. Here we begin by outlining a pioneering formulation of QFT with an evolution parameter: Feynman's path integral formulation [7]. The invariant evolution parameter is a characteristic feature of pRQM.

A review of parametrized theories prior to 1990 was presented by Fanchi [8]. Introductions to pRQM are presented by Fanchi [9, 10], Pavšič [11], and Horwitz [12]. We then outline a probabilistic formulation of pRQM that highlights features of the theory that make it suitable for modelling particle stability. Two examples of particle stability are then discussed within the context of pRQM: the law of exponential particle decay, and neutrino oscillations.

## 2. How did the common misconception arise?

Quantum field theory (QFT) is designed to integrate quantum mechanics and special relativity. In QFT, the wave function is considered a field and the wave equation is the field equation. The success of QED and other QFT examples has led to the misconception that quantum mechanics cannot model particle instability in either its non-relativistic or relativistic form. The origin of the misconception can be seen by briefly reviewing the history of the transition from quantum mechanics to QFT.

The wave equation for non-relativistic quantum mechanics is the Schroedinger equation

$$i\hbar\frac{\partial\psi_S}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_S \tag{2.1}$$

for a free particle of mass $m$ and wave function $\psi_S$. The quantity $\rho_S = \psi_S^*\psi_S$ associated with $\psi_S$ and its complex conjugate $\psi_S^*$ was postulated to be the probability density that satisfies the normalization condition over space

$$\int \rho_S d^3x = 1 \tag{2.2}$$

The normalization condition implies that the particle can be found somewhere in space at all times.

Experimental observations since the 1920s have shown that particles can appear and disappear. Several processes have been observed, including exponential decay, particle creation and annihilation, mass state transitions, mass-energy transformation, and the mass-energy uncertainty principle.

The need to extend non-relativistic quantum mechanics to include particle instability helped motivate the integration of quantum mechanical concepts with special relativity. One of the first single particle wave equations to be proposed was the Klein-Gordon equation

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - \frac{e}{c}A^\mu\right]\left[\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right]\Psi = m_0^2c^2\Psi \tag{2.3}$$

for a particle of rest mass $\boldsymbol{m_0}$, electric charge $\boldsymbol{e}$, four-vector potential $\boldsymbol{A^\mu}$, and wave function $\boldsymbol{\psi_{KG}}$. The nonzero elements of the metric are

$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33} \tag{2.4}$$

Several issues arose in relativistic quantum mechanics based on equations like the Klein-Gordon and Dirac equation [13; 14, Chap. 1]. For example, how do we interpret negative energy solutions and negative probability densities? Do we need Dirac's hole theory (the Dirac sea) which works for fermions but not for bosons? Is it correct to apply a single particle formalism to particle creation and annihilation? According to Peskin and Schroeder [15] (pg. 13), "We have no right to assume that any relativistic process can be explained in terms of a single particle, since the Einstein relation $\boldsymbol{E = mc^2}$ allows for the creation of particle-antiparticle pairs. Even when there is not enough energy for pair creation, multiparticle states appear…"

Zee supported this point of view [16] (pg. 3) by observing that "It is in the peculiar confluence of special relativity and quantum mechanics that a new set of phenomena arises: Particles can be born and particles can die. It is this matter of birth, life, and death that requires the development of a new subject in physics, that of quantum field theory." From this perspective, nonrelativistic quantum mechanics, expressed in terms of the Schroedinger equation and associated expectation value over space, is incapable of describing the birth, life, and death of a particle.

In a related context, Tong [17] (pg. 2) pointed out that "the combination of quantum mechanics and special relativity implies that particle number is not conserved." Furthermore, Tong [17] (pg. 3) added that "There is no mechanism in standard non-relativistic quantum mechanics to deal with changes in particle number." Tong argued that "once we enter the relativistic regime we need a new formalism in order to treat states with an unspecified number of particles. This formalism is quantum field theory" [17] (pg. 3).

By contrast, Padmanabhan [18] (pg. 1) recognized that most QFT textbooks said that QFT was needed "because any theory, which incorporates quantum mechanics and relativity, has to be a theory in which number of particles (and even the identity of particles) is not conserved." Padmanabhan suggested that a multi-particle formulation of relativistic quantum mechanics could be developed in principle which would allow a variable number of particles. This idea has been around for decades. For example, Droz-Vincent pointed out that "our understanding of N-body relativistic dynamics has undergone substantial progresses in the recent years." [19] (pg. 101) He suggested that the development of relativistic dynamics of directly interacting particles should be considered complementary to QFT rather than a conflicting point of view. [19] (pg. 102) An example of a multiparticle, relativistic quantum mechanical theory that includes statistical mechanical features is parametrized Relativistic Quantum Mechanics (pRQM) [for example, see 9, 11, 12, 20, 21, 22].

Wilczek outlined a simple procedure for developing a quantum field theory [23] (pg. 4). First specify a continuum field theory (CFT) that includes Poisson brackets, and then apply rules for quantizing the CFT. Quantization is achieved by replacing Poisson brackets with commutators for bosonic fields and anticommutators for fermionic fields. The Standard Model is an example of a QFT, and the QFT procedure can be used with parametrized theories [see, for example, 11].

This brief history illustrates the conceptual conflict between particles and fields. If we consider the example of an electron interacting with a photon, our goal is to treat the electron and the photon on an equal footing: they are either both particles or both fields. If we assume that particles are fundamental, then the electromagnetic field arises from a collection of photons. By contrast, if we assume that fields are fundamental, then each photon arises from quantization of the electromagnetic field, and each electron arises from quantization of a matter field.

Particles are fundamental in relativistic quantum mechanics. In this context, the eigenfunction $\Psi$ is interpreted as the probability of seeing a particle at some location. By contrast, fields are fundamental in QFT. The eigenfunction $\Psi$ is a field, and the value of $\Psi$ is the probability of seeing a particular configuration of a field. In the context of QFT, particles correspond to the quantization of fields and may be considered excitations of fields which can appear and disappear.

Path integrals can be used to derive field equations for both nonrelativistic and relativistic quantum systems. Non-relativistic path integrals can be extended to relativistic systems by introducing an invariant evolution parameter $\boldsymbol{s}$ to parametrize the space-time path of a particle. Feynman's [7] path integral formulation helped Feynman [3], Schwinger [4, 5] and others develop QFT. We show in Section 3 how the invariant evolution parameter $\boldsymbol{s}$ arises in Feynman's path integral formulation and then illustrate the formulation by applying it to the behavior of a free particle. The result is a field equation with an invariant evolution parameter that is also a characteristic feature of pRQM. In later sections we show that pRQM is a quantum mechanical theory that can be used to model particle instability.

## 3. Feynman's path integral formulation

We illustrate Feynman's path integral approach [7] by applying it to an experiment for measuring the space-time trajectory of a relativistic particle. Space-time measurements can be made at positions A and C in Figure 1 using detectors and clocks. Invariant evolution parameter values $\boldsymbol{s_A}$ and $\boldsymbol{s_C}$ are associated with measurements at positions A and C respectively.

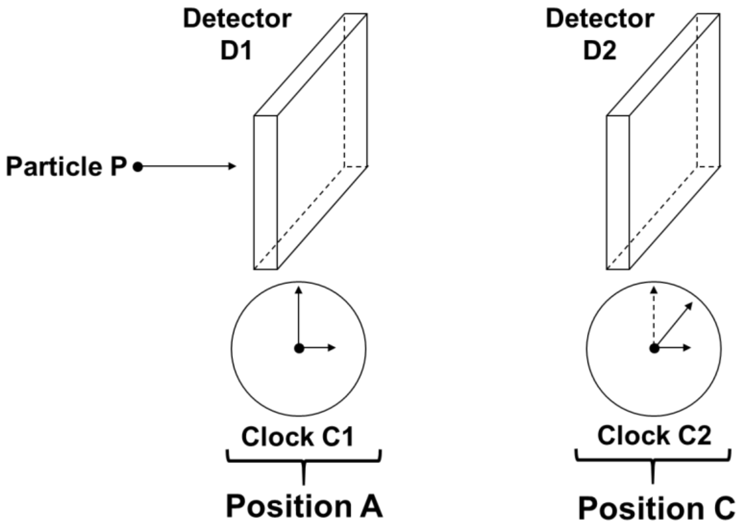


**Figure 1.** Measuring the World-line of a particle (after [9])

Feynman [7] assumed that the difference between classical and quantum physics could be shown by imagining the existence of a third detector and clock at a position $\boldsymbol{B}$ between positions $\boldsymbol{A}$ and $\boldsymbol{C}$. Measurements at the three positions $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are denoted by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ respectively.

Predictions of measurements at position $\boldsymbol{C}$ depend on outcome $\boldsymbol{a}$ measured at position $\boldsymbol{A}$ and outcome $\boldsymbol{b}$ at position $\boldsymbol{B}$. The probability $\boldsymbol{P_{ab}}$ is the probability that if measurement $\boldsymbol{A}$ gave outcome $\boldsymbol{a}$, then measurement $\boldsymbol{B}$ will give outcome $\boldsymbol{b}$. Similar definitions apply to probabilities $\boldsymbol{P_{bc}}$ and $\boldsymbol{P_{ac}}$. The probability that all three values $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ occur is $\boldsymbol{P_{abc}}$. Assuming that events between $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent of those between $\boldsymbol{b}$ and $\boldsymbol{c}$, we know from probability theory that

$$\boldsymbol{P_{abc}} = \boldsymbol{P_{ab}}\boldsymbol{P_{bc}} \tag{3.1}$$

If we sum (or integrate for a continuous variable) over all mutually exclusive outcomes of $\boldsymbol{b}$, we obtain

$$\boldsymbol{P_{ac}} = \sum\nolimits_{\boldsymbol{b}} \boldsymbol{P_{abc}} \tag{3.2}$$

According to Feynman [7] (pg. 369), Eq. (3.2) is “the essential difference between classical and quantum physics” because Eq. (3.2) is always true in classical mechanics, and often false in quantum mechanics. The classical probability $P_{ac}$ is obtained by combining Eqs. (3.1) and (3.2) to get a superposition of probabilities

$$P_{ac} = \sum_b P_{ab} P_{bc} \tag{3.3}$$

The quantum mechanical probability $P_{ac}^q$ that a measurement at $C$ has the outcome $c$ if a measurement at $A$ has the outcome $a$ is obtained from a superposition of probability amplitudes:

$$P_{ac}^q = |\phi_{ac}|^2 = \left|\sum_b \phi_{ab}\phi_{bc}\right|^2 \tag{3.4}$$

where we have replaced probabilities with complex probability amplitudes, namely

$$P_{ab} = |\phi_{ab}|^2, P_{bc} = |\phi_{bc}|^2, P_{ac}^q = |\phi_{ac}|^2 \tag{3.5}$$

Equation (3.3) is applicable if a measurement of $b$ at $B$ is made. If the path of a particle traveling from $A$ to $C$ is not verified with a measurement at $B$, then Eq. (3.4) applies. Each possible path of a particle is represented by a complex amplitude. This led to Feynman’s two postulates:

F-I. “If an ideal measurement is performed to determine whether a particle has a path lying in a region of space-time, then the probability that the result will be affirmative is the absolute square of a sum of contributions, one from each path in the region” [7] (pg. 371).

F-II. “The paths contribute equally in magnitude, but the phase of their contribution is the classical action” [7] (pg. 371).

Feynman’s first postulate F-I tells us how to calculate probabilities from probability amplitudes. His second postulate F-II tells us how to calculate probability amplitudes from all possible paths, including non-local paths. The classical path of the particle in Feynman’s path integral formulation is the path yielding the extremum action.

### *3.1. Illustration: relativistic free particle*

The following application of Feynman’s path integral formulation to a relativistic free particle shows how to find an equation that describes the evolution of a relativistic free particle between position $A$ and position $C$ if we do not make measurements at position $B$. The illustration shows the role of an invariant evolution parameter. For simplicity, we work in one space dimension, adopt natural units ($\hbar = c = 1$), and follow the procedure presented in Fanchi [9].

Suppose that space-time measurements can be made at space-time positions i and i+1 using detectors and clocks. Position i corresponds to position $A$ in Figure 1 and position i+1 corresponds to position $B$. Invariant evolution parameter values $s_i$ and $s_{i+1}$ are associated with measurements at positions i and i+1 respectively.

We apply Feynman’s postulates F-I and F-II to predict the evolution of a relativistic particle from position $i$ to position $i+1$. The probability amplitude $\phi(x_{i+1}, t_{i+1}, s+\varepsilon)$ at space-time point $x_{i+1}, t_{i+1}$ and invariant evolution parameter $s_{i+1} = s + \varepsilon$ with infinitesimal $\varepsilon$ is calculated from the equation

$$\phi(x_{i+1}, t_{i+1}, s+\varepsilon) = \frac{1}{\eta}\int e^{iS(x_i,t_i)}\phi(x_i, t_i, s)\, dx_i dt_i \tag{3.6}$$

$\eta$ is a normalization constant, $\phi(x_i, t_i, s)$ is the probability amplitude at $x_i, t_i, s$ , and action $S$ is given by

$$S(x_i, t_i) = \int L(\dot{x}_i, \dot{t}_i)\, ds \tag{3.7}$$

The dot above a variable indicates differentiation with respect to evolution parameter $s$, for example, $\dot{x} = dx/ds$. The Lagrangian for a free particle with mass $m$ may be written as

$$\boldsymbol{L(\dot{x},\dot{t}) = \frac{m}{2}\left[\left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2\right]} \tag{3.8}$$

The integral is over all possible paths.

The action is approximated as

$$\boldsymbol{S(x_i,t_i) = \varepsilon\frac{m}{2}\left[\left(\frac{t_{i+1}-t_i}{\varepsilon}\right)^2 - \left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2\right]} \tag{3.9}$$

where we have assumed the trapezoidal rule is sufficiently accurate to solve the action integral. Defining the variables

$$\boldsymbol{\delta_x = x_{i+1} - x_i,\ \delta_t = t_{i+1} - t_i} \tag{3.10}$$
$$\boldsymbol{x = x_{i+1},\ t = t_{i+1}}$$

lets us write Eq. (3.6) in the form

$$\boldsymbol{\phi(x,t,s+\varepsilon) = \frac{1}{\eta}\int e^{i\frac{m}{2\varepsilon}[(\delta_t)^2-(\delta_x)^2]}\phi(x-\delta_x,t-\delta_t,s)d\delta_x d\delta_t} \tag{3.11}$$

We use the Taylor series to expand the probability amplitudes to first order in $\boldsymbol{\varepsilon}$ and second order in $\boldsymbol{\delta_x}$, $\boldsymbol{\delta_t}$ to obtain

$$\boldsymbol{\phi(x,t,s) + \varepsilon\frac{\partial\phi}{\partial s} = \frac{1}{\eta}\int e^{i\frac{m}{2\varepsilon}[(\delta_t)^2-(\delta_x)^2]}d\delta_x d\delta_t}$$
$$\boldsymbol{\left[\phi(x,t,s) - \delta_t\frac{\partial\phi}{\partial t} - \delta_x\frac{\partial\phi}{\partial x} - \delta_x\delta_t\frac{\partial^2\phi}{\partial x\partial t} + \frac{{\delta_t}^2}{2}\frac{\partial^2\phi}{\partial t^2} + \frac{{\delta_x}^2}{2}\frac{\partial^2\phi}{\partial x^2}\right]} \tag{3.12}$$

Equation (3.12) is simplified by evaluating the integral on the right-hand side to yield

$$\boldsymbol{\phi(x,t,s) + \varepsilon\frac{\partial\phi}{\partial s} = -\frac{2\varepsilon\pi}{m\eta}\phi(x,t,s) - \frac{i\varepsilon}{2m}\left(\frac{2\varepsilon\pi}{m\eta}\right)\left[\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}\right]} \tag{3.13}$$

To assure agreement in the limit as ε goes to zero (the normalized case), we set $\boldsymbol{\eta = -2\varepsilon\pi/m}$ so that Eq. (3.13) becomes

$$\boldsymbol{\phi(x,t,s) + \varepsilon\frac{\partial\phi}{\partial s} = \phi(x,t,s) + \frac{i\varepsilon}{2m}\left[\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}\right]} \tag{3.14}$$

Equation (3.14) is satisfied when the coefficients of $\boldsymbol{\varepsilon}$ are equal, yielding the equation

$$\boldsymbol{\frac{\partial\phi}{\partial s} = \frac{i}{2m}\left[\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2}\right]} \tag{3.15}$$

Multiplying Eq. (3.15) by $\boldsymbol{i}$ and introducing four-vector notation gives the field equation

$$\boldsymbol{i\frac{\partial\phi}{\partial s} = -\frac{1}{2m}\partial^\mu\partial_\mu\phi} \tag{3.16}$$

where

$$\boldsymbol{\partial_\mu = \partial/\partial x^\mu,\ \partial^\mu = \partial/\partial x_\mu = g^{\mu\nu}\partial_\nu,} \tag{3.17}$$

and $\boldsymbol{g_{\mu\nu}}$ is the metric tensor with nonzero elements given in Eq. (2.4). Equation (3.16) is the Stueckelberg equation for a noninteracting, relativistic, spinless particle [24, 25, 26] and establishes a connection to pRQM.

## 4. Probabilistic formulation of parametrized relativistic quantum mechanics

The introduction of an invariant evolution parameter in Feynman's path integral formulation helped guide the development of QFT and is a characteristic feature of pRQM. The role of time changed when an invariant evolution parameter was included with coordinate time in pRQM. Two times were recognized [27]: Minkowski time corresponding to the temporal coordinate of a space-time four-vector, i.e. Einstein's time coordinate; and historical time corresponding to the invariant evolution parameter. Historical time is an evolution (ordering) parameter for a relativistic system.

The presence of two independent temporal variables in field equations meant that solutions of the field equations contained both temporal variables. Consequently, it was necessary to modify the definition of terms such as probability amplitude, normalization condition, and expectation value. The ability to model particle stability in pRQM is due to the modifications made to probabilistic terms. We show this by outlining a probabilistic formulation of pRQM that highlights features of the theory that make it suitable for modelling particle stability.

We begin by assuming that we can find a conditional probability density $\boldsymbol{\rho(x|s)}$ for a physical system of interest. The symbol $\boldsymbol{x}$ denotes the set of space-time coordinates $\boldsymbol{x^0, x^1, x^2, x^3}$ where the index 0 signifies Minkowski time and the indices 1, 2, 3 signify space components. The invariant evolution parameter $\boldsymbol{s}$ conditions the probability density $\boldsymbol{\rho(x|s)}$.

The conditional probability distribution $\boldsymbol{\rho(x|s)}$ can be expressed as the product $\boldsymbol{\rho(x^1, x^2, x^3|x^0, s)\rho(x^0|s)}$. The distribution $\boldsymbol{\rho(x^0|s)}$ is the marginal probability density in time and is conditioned by the evolution parameter s. It can be used to model particle appearance and disappearance. The probability $\boldsymbol{\rho(x^0|s)}$ of observing a particle at time $\boldsymbol{x^0}$ given parameter s is zero when $\boldsymbol{\rho(x^0|s) = 0}$. In this case, the particle cannot be detected anywhere in space at time $\boldsymbol{x^0}$ and parameter s. By contrast, when $\boldsymbol{\rho(x^0|s) \neq 0}$, there is a nonzero probability of observing a particle at time $\boldsymbol{x^0}$ given parameter s.

The conditional probability density $\boldsymbol{\rho(x^0|s)}$ must be positive definite and normalizable. The Born representation of the positive definite requirement is

$$\boldsymbol{\rho(x|s) = \Psi^*(x,s)\Psi(x,s) \geq 0} \tag{4.1}$$

where $\boldsymbol{\Psi(x,s)}$ is the probability amplitude and $\boldsymbol{\Psi^*(x,s)}$ is its complex conjugate. The probability amplitude $\boldsymbol{\Psi(x,s)}$ can be written in terms of the conditional probability distribution $\boldsymbol{\rho(x|s)}$ as

$$\boldsymbol{\Psi(x,s) = \sqrt{\rho(x|s)}\, e^{i\xi(x,s)}} \tag{4.2}$$

where $\boldsymbol{\Psi(x,s)}$ is specified to within a gauge transformation represented by the scalar function $\boldsymbol{\xi(x,s)}$.

The normalization condition for the relativistic formulation is

$$\int_{\boldsymbol{D}} \boldsymbol{\rho(x|s) d^4x = 1;\ d^4x = dx^0 dx^1 dx^2 dx^3} \tag{4.3}$$

where the integral is over the space-time hypervolume $\boldsymbol{D}$ for a space-time interval $\boldsymbol{d^4x}$ and the metric in Eq. (2.4). The normalization integral in Eq. (4.2) is significantly different from the 3-space normalization of nonrelativistic quantum mechanics. The integration over 4-space was a motivation for calling the probabilistic formulation described here the four-space formalism (FSF) by Fanchi [28] and Fanchi and Collins [29, 30]. The 4-space normalization condition implies that the particle can be observed somewhere in space at some point in time. Furthermore, the particle does not have to exist all of the time.

The following outline of a procedure presented by Fanchi [9] can be used to derive field equations from the above assumptions and the conservation of probability represented by the continuity equation:

$$\frac{\partial\rho}{\partial s}+\frac{\partial}{\partial x_\mu}(\rho V^\mu)=0 \tag{4.4}$$

Combining the continuity equation, minimal coupling with the electromagnetic four-vector potential $A^\mu$, and the expression of probability density in terms of probability amplitudes gives the probability flux

$$\rho V^\mu=-\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x_\mu}-\Psi\frac{\partial\Psi^*}{\partial x_\mu}\right]-\frac{eA^\mu}{mc}\Psi^*\Psi \tag{4.5}$$

with four-velocity

$$V^\mu(x,s)=\frac{\hbar}{m}\frac{\partial\xi(x,s)}{\partial x_\mu}-\frac{e}{mc}A^\mu(x,s) \tag{4.6}$$

Equations (4.4) through (4.6) can be used to derive the parametrized field equation

$$i\hbar\frac{\partial\Psi}{\partial s}=K\Psi \tag{4.7}$$

with mass operator

$$K=\frac{\pi^\mu\pi_\mu}{2m}+V \tag{4.8}$$

where V is potential energy. The four-vector potential $A^\mu$ is contained in the four-momentum operator $\pi^\mu$ with minimal coupling

$$\pi^\mu=\frac{\hbar}{i}\frac{\partial}{\partial x_\mu}-\frac{e}{c}A^\mu \tag{4.9}$$

Equation (4.7) is the Stueckelberg equation for a single particle.

The definition of expectation value $\langle\Omega\rangle$ of an observable $\Omega$ is defined as an extension of the normalization condition; thus

$$\langle\Omega\rangle=\int\Psi^*\Omega\Psi dx \tag{4.10}$$

The space-time uncertainty principle is the uncertainty principle for both energy and three-momentum

$$|\Delta x_\mu|\,|\Delta p_\mu|\geq\frac{\hbar}{2} \tag{4.11}$$

where summation over repeated indices is not implied in this equation.

### *4.1. The meaning of mass*

The Stueckelberg equation for a free particle is

$$i\hbar\frac{\partial\psi_f}{\partial s}=-\frac{\hbar^2}{2m}\partial_\mu\partial^\mu\psi_f \tag{4.12}$$

with general solution

$$\Psi_f(x,s)=\int\psi_{f\kappa}(x,s)\,dk_f \tag{4.13}$$

$$= \int \boldsymbol{\eta_{f\kappa}\, exp\left[i\kappa_f(k_f)s + ik_{f\mu}x^\mu\right] dk_f}$$

The integral in Eq. (4.13) is over energy-momentum, the term $\boldsymbol{\kappa_f(k_f)}$ is

$$\boldsymbol{\kappa_f(k_f) = -\frac{\hbar^2}{2m} k_{f\mu}k_f^\mu} \tag{4.14}$$

and $\boldsymbol{\eta_{f\kappa}}$ denotes normalization coefficients.

Applying the definition of expectation value to free particle four-space, four-momentum, and four velocity observables gives

$$\boldsymbol{\left\langle V_f^\mu \right\rangle = \frac{d\left\langle x_f^\mu \right\rangle}{ds} = \frac{\left\langle p_f^\mu \right\rangle}{m}} \tag{4.15}$$

The most probable trajectory of the free particle is found by integrating Eq. (4.15) from $\boldsymbol{s}$ to $\boldsymbol{s + \delta s}$. The result is

$$\boldsymbol{\delta\left\langle x_f^\mu \right\rangle = \frac{\left\langle p_f^\mu \right\rangle}{m}\delta s} \tag{4.16}$$

The observable free particle world-line satisfies the inner product

$$\boldsymbol{\delta\left\langle x_f^\mu \right\rangle \delta\langle x_{f\mu}\rangle = \frac{\left\langle p_f^\mu \right\rangle \langle p_{f\mu}\rangle}{m^2}(\delta s)^2} \tag{4.17}$$

We illustrate the meaning of mass $\boldsymbol{m}$ in this formulation by rearranging Eq. (4.17) and solving for $\boldsymbol{m^2}$:

$$\boldsymbol{m^2 = \left\langle p_f^\mu \right\rangle \langle p_{f\mu}\rangle \frac{(\delta s)^2}{\delta\left\langle x_f^\mu \right\rangle \delta\langle x_{f\mu}\rangle}} \tag{4.18}$$

The square of $\boldsymbol{m}$ expresses $\boldsymbol{m}$ in terms of energy-momentum and space-time observables. Equation (4.18) is consistent with Einstein's [31] view that the "mass of a body is a measure of its energy content."

The terms $\boldsymbol{\left\langle p_f^\mu \right\rangle \langle p_{f\mu}\rangle}$, $\boldsymbol{\delta\left\langle x_f^\mu \right\rangle \delta\langle x_{f\mu}\rangle}$ in Eq. (4.18) can be either time-like $\boldsymbol{\left\langle p_f^\mu \right\rangle \langle p_{f\mu}\rangle > 0}$, $\boldsymbol{\delta\left\langle x_f^\mu \right\rangle \delta\langle x_{f\mu}\rangle > 0}$ or space-like $\boldsymbol{\left\langle p_f^\mu \right\rangle \langle p_{f\mu}\rangle < 0}$, $\boldsymbol{\delta\left\langle x_f^\mu \right\rangle \delta\langle x_{f\mu}\rangle < 0}$. By contrast, $\boldsymbol{\delta s > 0}$ in all cases because the invariant evolution parameter increases monotonically in pRQM. Consequently, $\boldsymbol{m^2}$ is positive for both time-like and space-like motion because negative signs associated with space-like motion cancel. Free tachyons in parametrized relativistic dynamics have real mass [for more details, see 9 and references therein].

## 5. Exponential particle decay

The conditional probability distribution $\boldsymbol{\rho(x|s)}$ discussed in Section 4 is capable of representing the disappearance or reappearance of a particle in space-time. In this section the probabilistic formulation is applied to exponential decay of a set of particles. We begin by showing how the decay of a collection of unstable particles can be empirically described using an exponential (Poisson) probability distribution.

A distribution for particle decay is obtained by assuming that the number of particles decaying between $\boldsymbol{t'}$ and $\boldsymbol{t' + dt'}$ is

$$\boldsymbol{dN(t') = -\lambda N(t')dt'} \tag{5.1}$$

where $N(t')$ is the number of particles at $t'$, and $\lambda$ is the constant probability of particle decay per unit time. Integrating Eq. (5.1) from $t' = 0$ to $t' = t$ gives

$$N(t) = N(0)e^{-\lambda t} \tag{5.2}$$

where $N(0)$ is the number of particles at $t' = 0$. The probability of observing a particle at time *t* is

$$P_{obs} = \frac{N(t)}{N(0)} = e^{-\lambda t}. \tag{5.3}$$

Although Eq. (5.3) is commonly used for describing phenomenological results, it implies a lack of probability conservation within the context of conventional quantum theories using a normalization condition over spacial volume only. These problems are avoided in pRQM by extending probability concepts to space-time with an invariant evolution parameter as shown below.

### *5.1. pRQM model of particle decay*

The probability that a single unstable particle with mass *m* in a collection of identical unstable particles will decay can be modelled in pRQM by beginning with the Stückelberg equation

$$i\frac{\partial\Psi(x,s)}{\partial s} = \left[\frac{\pi^\mu\pi_\mu}{2m} + V_I\right]\Psi(x,s). \tag{5.4}$$

where the kinetic four-momenta $\{\pi^\mu\}$ include the four-vector potential $\{A^\mu\}$ and an interaction potential $V_I$ for other interactions. We assume that $\{A^\mu\}$ and $V_I$ do not depend on $s$ so that we can seek a stationary mass state solution, thus

$$\Psi(x,s) = \psi(x)e^{-iM^2 s/2m}. \tag{5.5}$$

The term $M^2$ is constant with respect to $(x, s)$ and is determined as part of the solution to the problem. The probability density and normalization condition become

$$\begin{aligned}\rho(x|s) &= \Psi^*(x,s)\ \Psi(x,s)\\ &= \psi^*(x)\ \psi(x)\\ &\equiv \rho_S(x)\end{aligned} \tag{5.6}$$

and

$$\int \rho(x|s)d^4x = \int \rho_S(x)d^4x = 1 \tag{5.7}$$

respectively.

The multivariable probability density $\rho_S(x)$ can be written as a product of conditional probability density $\rho_C(\vec{x}|t)$ and marginal probability density in time $\rho_T(t)$:

$$\rho_S(x) = \rho_C(\vec{x}|t)\rho_T(t) \tag{5.8}$$

where we have adopted the notation

$$\begin{aligned}x^0 &= x_0 = t,\\ k^0 &= k_0 = \omega.\end{aligned} \tag{5.9}$$

The conditional and marginal probability densities must satisfy the normalization conditions

$$\int \rho_C(\vec{x}|t)d^3x = 1 \tag{5.10}$$

and

$$\int \rho_T(t)dt = 1 \tag{5.11}$$

respectively. The spacial volume element is denoted by $d^3x$. Integrating $\rho_S$ over the spacial volume and using Eq. (5.10) gives

$$\int \rho_S(x)d^3x = \int \rho_C(\vec{x}|t)\rho_T(t)d^3x \tag{5.12}$$

$$= \rho_T(t) \int \rho_C(\vec{x}|t) d^3x$$
$$= \rho_T(t).$$

Unlike conventional nonrelativistic quantum theory, Eq. (5.12) shows that integration over all space does not have to equal 1 in pRQM because of Eq. (4.3).

Substituting the stationary mass state given by Eq. (5.5) into Eq. (5.4) gives

$$M^2 \psi(x,s) = [\pi^\mu \pi_\mu + 2mV_I]\ \psi(x,s). \tag{5.13}$$

Expanding the kinetic four-momentum operators $\{\pi^\mu\}$ and regrouping terms gives

$$M^2 \psi\ (x,s) = [p^\mu p_\mu + U_I]\ \psi(x,s), \tag{5.14}$$

where the interaction terms are collected in the operator

$$U_I = -\ eA^\mu p_\mu -\ ep^\mu A_\mu + e^2 A^\mu A_\mu + 2mV_I, \tag{5.15}$$

and

$$p^\mu = \frac{\hbar}{i}\frac{\partial}{\partial x_\mu} \tag{5.16}$$

The operator $p^\mu$ is the 4-momentum operator. To simplify the problem, we assume the interaction operator $U_I$ satisfies an equation of the form

$$U_I \psi(x) = u_I \psi(x) \tag{5.17}$$

where $u_I$ changes slowly in the space-time region of interest. It may be possible to solve more complex problems, but the adiabatic-type approximation in Eq. (5.17) is sufficient for our purposes. The resulting field equation is

$$[M^2\ -\ u_I]\psi(x) = p^\mu p_\mu \psi(x) = \left[-\ \frac{\partial^2}{\partial t^2} + \nabla^2\right]\psi(x). \tag{5.18}$$

Our interest is in calculating both $\rho_C$ and $\rho_T$. To get the correct dependence on space and time variables, we write the state function as

$$\psi(x) = \zeta(\vec{x},t)\ \xi(t) \tag{5.19}$$

so that

$$\rho_C(\vec{x}|t) = \zeta^*(\vec{x},t)\ \zeta(\vec{x},t),$$
$$\rho_T(t) = \xi^*(t)\ \xi(t). \tag{5.20}$$

Substituting Eq. (5.19) into (5.18), dividing by $\xi(t)$ and rearranging yields

$$[M^2 - u_I]\zeta = \left[-\ \frac{\partial^2}{\partial t^2} + \nabla^2\right]\zeta\ -\ \frac{\zeta}{\xi}\frac{d^2\xi}{dt^2}\ -\ \frac{2}{\xi}\frac{d\xi}{dt}\frac{\partial\zeta}{\partial t}. \tag{5.21}$$

Further simplification of Eq. (5.21) is motivated by recognizing that $\rho_T$ should have the form

$$\rho_T(t) \sim e^{-\lambda t}, 0 \le t \le \infty. \tag{5.22}$$

Introducing a real, constant phase factor $\varepsilon$, we combine Eq. (5.22) and (5.20) to get the trial solution

$$\xi(t) \sim e^{i\varepsilon t - \lambda t/2}. \tag{5.23}$$

Substituting Eq. (5.23) into (5.21) gives

$$\begin{aligned}[M^2 - u_I]\zeta = \left[-\ \frac{\partial^2}{\partial t^2} + \nabla^2\right]\zeta\ -\ \left(i\varepsilon\ -\ \frac{\lambda}{2}\right)^2 \zeta \\ -\ (2i\varepsilon - \lambda)\ \frac{\partial \zeta}{\partial t}.\end{aligned} \tag{5.24}$$

Equation (5.24) is a field equation for $\zeta$. Substituting the plane wave solution

$$\zeta(\vec{x},t) \sim e^{i\omega t - i\vec{k}\cdot\vec{x}}. \tag{5.25}$$

into Eq. (5.24) yields the characteristic equation

$$-\omega^2 - i\lambda\omega - 2\varepsilon\omega + \frac{\lambda^2}{4} - \varepsilon^2 - i\lambda\varepsilon + M^2 + \vec{k}\cdot\vec{k} - u_I = 0. \quad (5.26)$$

The real and imaginary terms of Eq. (5.26) are separated to give

$$-\omega^2 - 2\varepsilon\omega + \frac{\lambda^2}{4} - \varepsilon^2 + M^2 + \vec{k}\cdot\vec{k} - u_I = 0, \quad (5.27)$$

$$-i\lambda\omega - i\lambda\varepsilon = 0. \quad (5.28)$$

Using $\omega = -\varepsilon$ from Eq. (5.28) in Eq. (5.27) lets us express the physically significant transition rate $\lambda$ in the form

$$\lambda^2 = 4[u_I - (M^2 + \vec{k}\cdot\vec{k})]. \quad (5.29)$$

If $\lambda$ is allowed to be imaginary, the resulting marginal probability distribution would be uniform instead of exponentially declining. Therefore, the interaction must satisfy

$$u_I > (M^2 + \vec{k}\cdot\vec{k}) \quad (5.30)$$

so that $\lambda^2 > 0$.

Finally, by combining all of the above results, the probability amplitude $\Psi(x, s)$ can be written as

$$\Psi(x,s) \sim exp\left[-i\frac{M^2 s}{2m} - \frac{\lambda}{2}t + i\vec{k}\cdot\vec{x}\right]. \quad (5.31)$$

The probability amplitude $\Psi(x, s)$ decays in time and the marginal probability density $\rho_T$ has the exponential decay form needed to reproduce empirical results.

## 6. Neutrino oscillations

Our goal here is to show how pRQM can be used to model neutrinos changing from one flavor state to another. pRQM models of neutrino oscillation by flavor mixing of up to four neutrino flavor states have been presented [32]. In this section we review two-state flavor mixing. The calculation shows that the probability of transition from one neutrino flavor to another has similarities with the results of the conventional theory, but the details differ. The difference in results may be useful for experimentally comparing the conventional theory and pRQM.

The disappearance of an electron antineutrino $\bar{v}_e$ has been observed in the production of a positron $e^+$ and a neutron $n$ when $\bar{v}_e$ interacts with a proton $p$ in the process $\bar{v}_e + p \rightarrow e^+ + n$. The flavor-mixing hypothesis says that an electron antineutrino $\bar{v}_e$ may transform into a different neutrino flavor as it propagates. In principle, detectors can be placed along the path of the electron antineutrino to determine the probability of disappearance of electron antineutrinos as a function of distance *L* from their source.

The evolution equation for a state may be written in terms of the evolution operator as

$$i\hbar\frac{\partial}{\partial s}|v_j\rangle = K_j|v_j\rangle \quad (6.1)$$

where $K_j$ is the eigenvalue of the mass operator for mass state j. We restrict our discussion to two mass states $\{|v_j\rangle\}$ and two neutrino flavor states $\{|v_\varepsilon\rangle\}$. Mass states and flavor states are written as 2-component column vectors:

$$\{|v_j\rangle\} = \begin{bmatrix}|v_1\rangle\\|v_2\rangle\end{bmatrix}, \{|v_\alpha\rangle\} = \begin{bmatrix}|v_e\rangle\\|v_\mu\rangle\end{bmatrix} \quad (6.2)$$

The mass basis $|v_j\rangle$ and the flavor basis $|v_\alpha\rangle$ are related by a unitary transformation U so that

$$|v_\alpha\rangle = U|v_j\rangle \quad (6.3)$$

where

$$\boldsymbol{U} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{6.4}$$

The angle $\boldsymbol{\theta}$ is the mixing angle of mass states in vacuum. We invoke the conventional hypothesis that the mixing angle is not zero so that it makes sense to discuss flavor state mixing.

The solution of Eq. (6.1) in the mass basis depends on the invariant evolution parameter $\boldsymbol{s}$ and is given by

$$\begin{bmatrix} |\nu_1(s)\rangle \\ |\nu_2(s)\rangle \end{bmatrix} = \begin{bmatrix} e^{-iK_1 s/\hbar} & 0 \\ 0 & e^{-iK_2 s/\hbar} \end{bmatrix} \begin{bmatrix} |\nu_1(0)\rangle \\ |\nu_2(0)\rangle \end{bmatrix} \tag{6.5}$$

where

$$K_j = \hbar^2 k_j^{\mu} k_{j\mu}/2m_j = \hbar^2 \left[ (\omega_j/c)^2 - k_j \cdot k_j \right]/2m_j \tag{6.6}$$

The energy-momentum four-vector is $\boldsymbol{k_j^{\mu}}$ and $\boldsymbol{m_j}$ is the mass of state j.

As an illustrative example, we consider the oscillation between an electron neutrino $\boldsymbol{\nu_e}$ and a muon neutrino $\boldsymbol{\nu_\mu}$. We start with a pure beam of $\boldsymbol{\nu_e}$ particles and calculate the probability of forming muon neutrino particles. The pRQM result for the probability of forming the final state $\boldsymbol{\nu_\mu}$ from initial state $\boldsymbol{\nu_e}$ is

$$\begin{aligned} P_{pRQM}(\nu_e \to \nu_\mu) &= \sin^2 2\,\theta \sin^2 \left\{ \frac{(m_2 - m_1)c^2}{4\hbar} s \right\} \\ &\equiv \sin^2 2\,\theta \sin^2 \alpha_{pRQM} \end{aligned} \tag{6.7}$$

where the invariant evolution parameter $\boldsymbol{s}$ is measured by an evolution parameter clock [32]. The result for the conventional theory is

$$\begin{aligned} P_{con}(\nu_e \to \nu_\mu) &= \sin^2 2\,\theta \sin^2 \left\{ \frac{(m_2^2 - m_1^2)c^4}{4\hbar} \frac{L}{cE_\nu} \right\} \\ &\equiv \sin^2 2\,\theta \sin^2 \alpha_{con} \end{aligned} \tag{6.8}$$

The energy of the incident neutrino is $\boldsymbol{E_\nu}$.

The ratio of probabilities in Eqs. (6.7) and (6.8) is

$$\frac{P_{con}}{P_{pRQM}} = \frac{\sin^2 \alpha_{con}}{\sin^2 \alpha_{PRQM}} \tag{6.9}$$

The ratio $\boldsymbol{P_{pRQM}/P_{con}}$ shows that the pRQM model and the conventional theory have the same dependence on the flavor mixing angle $\boldsymbol{\theta}$. However, the dynamical factors $\boldsymbol{\alpha_{pRQM}}$ and $\boldsymbol{\alpha_{con}}$ differ significantly.

Rusov and Vlasenko [33] used the difference between the standard and PRD models of two-state vacuum flavor mixing to conduct an experimental test of the two models. They estimated neutrino masses for the electron, muon, and tau neutrinos based on data for solar and atmospheric neutrinos. The neutrino masses were then used to estimate the diameter of a neutrino cloud and the results supported a preference for PRD model predictions. According to Rusov and Vlasenko [33, Conclusion], direct experimental measurement of neutrino mass can justify reconsideration of quantum theoretical issues and a "holistic understanding of the nature of physical reality."

The Karlsruhe Tritium Neutrino (KATRIN) Collaboration is a direct method of measuring neutrino mass. KATRIN uses beta decay of tritium into helium-3, an electron, and an electron antineutrino. The KATRIN method has the advantage of being model independent, which makes it suitable for comparing neutrino mass predictions based on different models. Lahav and Thomas [34, Section 7.6] and Brugnera [35] reported that KATRIN does not have the sensitivity to measure neutrino mass if neutrino mass is less than 0.2 eV. The 0.2 eV limit is slightly greater than the neutrino masses reported in Table 1 of Rusov and Vlasenko [33]. The standard model masses are about 0.13 eV, while the pRQM masses are about 0.18 eV. If the masses calculated by Rusov and Vlasenko [33] are correct, it may be necessary to develop more sensitive methods of directly measuring neutrino masses.

## 7. Summary

We observed that Feynman's path integral formulation is a pioneering formulation of parametrized relativistic quantum mechanics (pRQM) and QFT. The role of time changed when an invariant evolution parameter was included with coordinate time as a characteristic feature of pRQM. Two times were recognized in pRQM: Minkowski time corresponding to the temporal coordinate of a space-time four-vector; and historical time corresponding to the invariant evolution parameter.

We then presented a probabilistic formulation of pRQM that highlighted features of the theory that made it suitable for modelling particle stability. The appearance of both Minkowski time and historical time in pRQM field equations and their solutions made it necessary to modify the definition of terms such as probability amplitude, normalization condition, and expectation value. The ability to model particle stability in pRQM is due to the modifications made to probabilistic terms. We show this by outlining two examples of particle stability within the context of pRQM to show that a quantum mechanical theory can be applied to particle stability.

pRQM is presented as a counterexample to the claim that only quantum field theory (QFT) can be used to describe the appearance or disappearance of particles. pRQM is a method of analyzing physical systems as interactions between particles and fields. The pRQM models presented here are relatively simple models that illustrate concepts. More sophisticated studies could be developed using multiparticle formulations of pRQM [for example, see 9 – 12]. By removing a misconception about relativistic quantum mechanics, it is possible to consider pRQM an alternative method for solving problems of physical interest. pRQM can help us better understand the agreement between QFT and experiments such as scattering experiments.

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