ON LINKING MULTIPLE LINES

by

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Submitted to the Graduate Faculty of the
College of Science and Engineering
Texas Christian University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

May 2018

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ACKNOWLEDGEMENTS

I thank my family who have always supported my academic career. I thank Chantel Mendoza for all her patience and support she has given me during graduate school. I thank Dr. Mohan Kumar for providing a sketch of the proof for the classification of quadric surfaces in characteristic equal two. I thank my fellow graduate students for their contribution to my learning, especially Fazle Rabby Shaikat as a fellow algebraic geometry graduate student. I thank Dr. Ken Richardson for his support as the Graduate Director. Finally, I thank my advisor Dr. Scott Nollet for the countless hours of help and guidance he has provided me, not only for my dissertation but throughout my graduate career.

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1 Introduction

Algebraic geometry studies the roots of polynomials and their generalizations. The earliest origins of algebraic geometry can be traced back to work by the Greeks in the 5th Century BCE. They studied the geometry of conics, lines, and circles. A more proper birth of algebraic geometry can be traced back to the independent invention of Fermat and Descartes of analytic geometry during the 17th century. Many concepts used in this paper were developed during this time for the real number system. A few examples include the degree of plane curves, classification of curves, and curves given as an intersection of two surfaces.

The end of the 18th century saw the introduction of points at infinity and complex numbers. The move to the complex projective plane and the complex projective three-dimensional space immensely benefited the theory of conics, quadrics, and linear families of conics and quadrics. For example, of the 72 types of cubics classified by Newton, only three remain distinct in projective space up to coordinate change.

Many new developments occurred during the 19th and 20th century, but probably none greater than the introduction of scheme theory by Grothendieck in 1960 (9). Schemes revolutionized algebraic geometry by working in the most general context possible and through the extensive use of commutative algebra. Not only do schemes allow us to keep track of geometric information as mathematicians have been doing for centuries, but also keep track of algebraic information.

One major theme in algebraic geometry is the classification of various mathematical objects, whether they be curves, varieties, schemes or vector bundles. Linkage theory, sometimes called liaison theory[1], studies links between curves. Roughly speaking, two curves in projective space are directly linked if they complement each other, that is, if their scheme-theoretic union is in a desired classification of curves. Scheme theory allows us to use the same curve twice, which we call self-linkage. Charles Walter showed that if we use arithmetically Cohen-Macaulay curves to link, then any two curves are in the same linkage class (27). The most commonly used classification for linkage is complete intersections. A complete intersection of dimension one in $\mathbb{P}^3$ is a curve whose ideal is generated by exactly two elements: geometrically this means an intersection of two surfaces meeting properly. Linkage is very important because it allows us to study a curves by linking it to a better understood curve, and draw conclusions for our original curve. For example, linkage theory provides many formulas for numeric invariants of schemes such as degree and genus. A commonly used practice in linkage theory is to create the linkage class of a curve, which allows us to study a family of curves instead of one or two curves at a time. Two curves are in the same linkage class if there exist a chain of links between them. The linkage equivalence class defines a classification of curves.

[1]Linkage is an Americanization of the French “liaison”. Migliore coined the term “linkage” in the 1980s.

We explore a different classification question: self-linkage. Many algebraic geometers have worked on questions of self-linkage, including the following: Rao (24) characterized self-linkage for local complete intersection curves; Franco, Kleiman, and Lascu (6) analyzed schemes of codimension two over a field of characteristic 0; Kleiman and Ulrich (13) characterized self-linked perfect ideals of grade 2; and Ellia (5) analyzed smooth, integral curves. Casnati and Catanese (3) characterized when a scheme of pure codimension two in $\mathbb{P}^n$ is self-linked over a field of characteristic not equal to 2. They proved that, under minor restrictions, self-linkage of scheme C is equivalent to the existence of a resolution

$$0 \longrightarrow \mathcal{E}^\vee(-d-m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-m) \oplus \mathcal{E} \longrightarrow \mathcal{I}_C \longrightarrow 0$$

where $\mathcal{I}_C$ is the ideal sheaf of C, $\mathcal{E}$ is a vector bundle over $\mathbb{P}^n$, and $\mathcal{E}^\vee$ is the dual of $\mathcal{E}$. Furthermore, the map $\alpha : \mathcal{E}^\vee(-d-m) \to \mathcal{E}$ is symmetric. The degrees of the hypersurfaces giving the self-linkage are d, m.

We study non-reduced locally Cohen-Macaulay quasi-primitive curves in projective three space whose support consist of a line, or multiple lines for short. If we need to specify the degree d of the multiple line, we will refer to it as a d-fold line, or prime-fold line if d is prime.

Problem 1. Let $W \subset \mathbb{P}^3$ be a d-line.

(a) Under what conditions is W directly linked to another d-line?

(b) Under what conditions is W self-linked?

The answers are easy when $d = 1$: two lines in three dimensional projective space meeting at a point are linked, but a pair of skew lines are never linked because complete intersections are connected. For $d = 2$, Migliore (18) proved that, with a few exceptions, double lines are self-linked only in characteristic two. For multiple lines of higher degrees, we give further results on self-linkage in characteristic two, where the methods of Casnati and Catanese do not apply. Martin-Deschamps and Perrin proved upper bounds on the values of the dimensions of the graded pieces of the Rao module $h^1(\mathcal{I}_C(n))$ for a curve $C \subset \mathbb{P}^3$ in terms of the degree and genus of C (15), and classified the curves giving equality for all n, which they called extremal. A characterization of extremal curves is that they are non-planar curves that contain a planar curve of degree one less. Every non-planar double line is automatically extremal because it contains a line. We extend Migliore's result to extremal prime-fold lines.

Theorem 1. *(Corollary 4.3.2) An extremal prime-fold line in $\mathbb{P}^3$ is self-linked if and only if the characteristic of the ground field is two.*

Roughly speaking, the fact that $2d$ only has two factorizations when d is prime forces self-linkage of extremal prime-fold lines to occur using a quadric, if possible. Theorem 1 follows from a more general analysis of multiple lines.

Theorem 2. *(Theorem 4.3.1) Consider a multiple line W in $\mathbb{P}^3$ of degree $d \geq 2$.*

1. *W is self-linked if one of the following occurs. The multiple line:*
   - (a) *is a complete intersection,*
   - (b) *up to coordinate change has ideal of the form $(x^2, xy, xq - y^{d-1})$,*
   - (c) *is extremal and the characteristic of the ground field is two,*
   - (d) *lies on a quadric cone.*
2. *W can't be self-linked by a quadric if W:*
   - (a) *is an extremal prime-fold line and the characteristic of the ground field is not two,*
   - (b) *lies on the smooth quadric and no other quadric,*
   - (c) *lies on the reducible quadric and no other quadric.*

Greco and Paxia observed that if a curve of degree at least five is contained in two independent quadric surfaces, then it is extremal (8). Therefore, the classifications of quadrics drives the organization of the statement and proof of Theorem 2. In characteristic not equal to two, it is well known that there are four categories of quadrics in $\mathbb{P}^3$: the smooth quadric, the quadric cone, the reducible quadric and the double plane (10). With help by Dr. Mohan Kumar, we provide a proof that the result extends to characteristic two. The alert reader might notice that the double plane is missing from the statement of Theorem 2. At this point, self-linkage of multiple lines that lie on the double plane and no other quadric is still an open question.

We also provide a more detailed analysis of triple lines which includes direct linkage as well as self-linkage.

Theorem 3. *(Theorem 5.5.1) Consider d-fold lines W, W' in $\mathbb{P}^3$ with $d = 3$. They are directly linked if and only if one of the following occurs:*

1. *They are complete intersections whose ideals share a generator.*

2. *They are supported on opposite rulings of a smooth quadric.*

3. *They lie on a quadric cone.*

4. *Up to coordinate change, the ideal of W is $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$ and one of the following:*

   (a) The ideal of W' is $I_{W'} = (x^2, xy, y^d, xq' - y^{d-1}p')$, where for some $\alpha \in k^*$ and some $g \in S_{d-2}$ we have with $p' = \alpha p$ and $q' = -\alpha(gp + q)$.

   (b) The ideal of W' is $I_{W'} = (y^2, xy, x^d, yq - x^{d-1}p)$.

Similar constructions generalize for all $d \geq 3$, but whether these are the only possibilities is an open question (see Theorem 4.3.3). The following example uses part $4b$ of Theorem 3 to show that characteristic two is needed for self-linkage.

Example 1. Consider the triple line $W \subset \mathbb{P}^3$ with ideal $I_W = (x^2, xy, y^3, xw^{N+1} - y^2z^N)$ where $N \geq 1$ and the complete intersection ideal $I_X = (x^2, y^3)$. Then X links W to the triple line W' with ideal $(x^2, xy, y^3, xw^{N+1} + y^2z^N)$. In particular, this complete intersection yields self-linkage only in characteristic two.

2 Background

We will assume the reader is familiar with basic algebraic geometry (10). This chapter is dedicated to covering well established definitions in algebraic geometry relevant to linkage theory of space curves. We start by introducing definitions and results in linkage theory that will be relevant to our arguments. We will end this chapter by introducing our main object of interest, extremal curves. First we will talk about their constructions and properties, and then we will discuss the extremal multiplicity structures on a line.

Notation 2.0.1. *Let k be an algebraically closed field of arbitrary characteristic. We denote by S the ring $k[x,y,z,w]$ and the associated projective space by $\mathbb{P}^3 = \operatorname{Proj} S$. The graded decomposition of S is $\bigoplus_{d\geq 0} S_d$ where S_d is the vector space of homogenous polynomials of degree d. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^3$, then $H^i_*(\mathcal{F})$ denotes the graded S-module $\bigoplus_{n\in Z} H^i(\mathbb{P}^3, \mathcal{F}(n))$. For convenience we will write $H^i(\mathcal{F}(n))$ for $H^i(\mathbb{P}^3, \mathcal{F}(n))$.*

Definition 2.0.2. We define a *curve* C to be a one dimensional proper scheme all of whose local rings $A = \mathcal{O}_{C,p}$ are Cohen-Macaulay, which means $\operatorname{depth} A = \dim A = 1$. In particular, any curve in $\mathbb{P}^3$ is closed.

The following lemma will be used throughout this thesis.

Lemma 2.0.3. *Let $C \subset C' \subset \mathbb{P}^3$ be two curves of degree d. Then $C = C'$.*

Proof. The Hilbert polynomials of C and C' take the following form:

$$P_C(t) = dt + 1 - p_a(C) \text{ and } P_{C'}(t) = dt + 1 - p_a(C').$$

The containment $C \subset C'$ implies we have a surjective map $\Phi : \mathcal{O}_{C'} \to \mathcal{O}_C$. Thus we have the following short exact sequence:

$$0 \longrightarrow \operatorname{Ker}\Phi \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

From the short exact sequence we get that the Hilbert polynomial

$$P_{\operatorname{Ker}\Phi}(t) = P_C(t) - P_{C'}(t) = (dt + 1 - p_a(C)) - (dt + 1 - p_a(C')) = p_a(C') - p_a(C)$$

is constant. Hence $\operatorname{Ker}\Phi$ has finite support, say $\{p_i\}$. Consider the following short exact sequence

$$0 \longrightarrow \operatorname{Ker}\Phi_{p_i} \longrightarrow \mathcal{O}_{C',p_i} \longrightarrow \mathcal{O}_{C,p_i} \longrightarrow 0.$$

By our definition of curve, we have that $\mathcal{O}_{C,p_i}$ and $\mathcal{O}_{C',p_i}$ are Cohen-Macaulay rings of dimension and depth 1. By (16, Theorem 17.6), they have no embedded primes. Furthermore, by (26, Lemma 2.5) $\operatorname{Ker}\Phi_{p_i} = 0$ for all p_i. Therefore $\operatorname{Ker}\Phi = 0$ since it has no support. Finally we have an isomorphism $\mathcal{O}_{C'} \cong \mathcal{O}_C$. We conclude $C = C'$. $\square$

2.1 Linkage Theory

While the definitions and results we will be using in this section can be found in greater generality, we will only state them as they will be needed. For a general reference about liaison theory, the reader can see Migliore's book (17). For more detailed results for space curves, the reader can see the paper by Martin-Deschamps and Perrin (14), and the paper by Peskine and Szpiro (22).

We will start this section with the ideal quotient. This ideal is at the heart of the definition of linkage theory. Furthermore, it will also be very important for many of our proofs.

Definition 2.1.1. Let I and J be ideals of S. The *ideal quotient* is defined as follows:

$$[I : J] = \{s \in S \,|\, Js \subset I\}.$$

Next, we will define complete intersections which will be our linking scheme.

Definition 2.1.2. If X is a curve in $\mathbb{P}^3$, then X is said to be a *complete intersection* if the number of generators of I_X is equal to 2.

Definition 2.1.3. If R is a ring, then a pair F, G of elements in R is called a *regular sequence* in R if F is not a zero divisor in R, multiplication by G induces an injective map on $R/(F)$, and $R/(F, G) \neq 0$.

The following proposition contains a characterization of complete intersections which will be useful for our arguments.

Proposition 2.1.4. *Let* $0 \neq F, G$ *be two homogeneous polynomials in* $S = k[x, y, z, w]$. *Then the following are equivalent:*

1. $(F, G) = I_X$ *form some unique complete intersection* $X \subset \mathbb{P}^3$.

2. (F, G) *is a regular sequence.*

3. F *and* G *have no non-constant common factor.*

Proof. Statements 1 and 2 are equivalent by (17, Example 1.5.1). We will prove statements 2 and 3 are equivalent by proving their negations are equivalent.

($\Rightarrow$) Assume polynomials F and G have a common factor $H \notin k$. So

$$F = HF' \text{ and } G = HG'$$

for some polynomials F' and G'. Notice that $F' \neq 0$ in $S/(F)$ since $H \notin k$, but

$$GF' + (F) = HF'G' + (F) = FG' + (F) = 0 + (F) \in S/(F).$$

Thus G is a zero divisor for $S/(F)$ and thus (F, G) is not a regular sequence.

($\Leftarrow$) Assume that (F, G) is not a regular sequence. Since F is not a zero divisor in S and $S/(F, G) \neq 0$, we must have that $G + (F)$ is a zero divisor in $S/(F)$. There exists a non-zero element $H + (F)$ in $S/(F)$, in particular $F \not| H$, such that

$$GH + (F) = (F) \Rightarrow GH \in (F) \Rightarrow F|GH.$$

F and G have a non-constant common factor since S is a unique factorization domain. $\square$

Definition 2.1.5. Let C and C' be curves in $\mathbb{P}^3$ and let $I_X = (F, G) \subseteq I_C \cap I_{C'}$ be a complete intersection ideal. Then C is *directly linked* to C' by X if and only if we have the following equalities for the ideal quotients $[I_X : I_C] = I_{C'}$ and $[I_X : I_{C'}] = I_C$. We will use the notation $C \overset{X}{\sim} C'$. The curve C is said to be *residual* to C' in X and vice-versa. If $C = C'$, the we say C is *self-linked*.

Example 2.1.6. *Consider the curves $C, C' \subset \mathbb{P}^3$ with ideals $I_C = (x, y)$ and $I_{C'} = (x, z)$. Then C is directly linked to C' by the complete intersection X with ideal (x, yz).*

An equivalent definition of algebraic linkage used by Schenzel (25), Peskine and Szpiro (22), and Rao (23) is presented below. We will not use this definition, but it is presented for context.

Definition 2.1.7. Curves C and C' in $\mathbb{P}^3$ are directly linked by complete intersection X if and only if

1. $I_X \subset I_C \cap I_{C'}$;

2. C and C' do not have embedded components.

3. $\mathcal{I}_{C'}/\mathcal{I}_X = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_C, \mathcal{O}_X)$.

4. $\mathcal{I}_C/\mathcal{I}_X = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{C'}, \mathcal{O}_X)$.

We will not completely prove that both definitions are equivalent. However, we will prove a key step in the proof, namely the isomorphism $[I_X : I_C]/I_X \cong \mathrm{Hom}_S(S/I_C, S/I_X)$. The lemma helps prove that the statement $I_{C'}/I_X = Hom_S(S/I_C, S/I_X)$ is equivalent to the statement $\mathcal{I}_{C'}/\mathcal{I}_X = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_C, \mathcal{O}_X)$. Similar arguments follow by reversing the roles of C' and C.

Lemma 2.1.8. *If $C \subset X$ is a curve in $\mathbb{P}^3$ lying on a complete intersection, then*

$$[I_X : I_C]/I_X \cong Hom_S(S/I_C, S/I_X).$$

Proof. Consider the map

$$\Psi : [I_X : I_C] \to \mathrm{Hom}_S(S/I_C, S/I_X)$$

where

$$s \longmapsto (t + I_C \overset{\Phi_s}{\longmapsto} st + I_X).$$

Notice that $s \in [I_X : I_C]$ implies $I_C \subset I_X$ thus the map Φ_s is well defined. We also have the following:

$$\begin{aligned} s \in I_X \quad &\Leftrightarrow \quad st \in I_X \, \forall\, t \in S \\ &\Leftrightarrow \quad st + I_X = I_X \, \forall\, t \in S \\ &\Leftrightarrow \quad \Phi_s = 0 \in \mathrm{Hom}_S(S/I_C, S/I_X) \\ &\Leftrightarrow \quad s \in \mathrm{Ker}\, \Psi. \end{aligned}$$

Note that the first $\Leftarrow$ can be justified by letting $t = 1$. Thus we have

$$I_X = \mathrm{Ker}\, \Psi.$$

Let $\Phi \in \mathrm{Hom}_S(S/I_C, S/I_X)$. Let $\Phi(1 + I_C) = s + I_X$. Since Φ is an S-homomorphism we have for all $t \in S$ that $\Phi(t + I_C) = t\Phi(1 + I_C) = st + I_X$. Thus $s \in [I_X : I_C]$ and $\Psi(s) = \Phi$. Therefore we have the following short exact sequence

$$0 \to I_X \to [I_X : I_C] \to \mathrm{Hom}_S(S/I_C, S/I_X) \to 0.$$

By the first isomorphism theorem we get

$$[I_X : I_C]/I_X \cong \mathrm{Hom}_S(S/I_C, S/I_X).$$

$\square$

Now that we have the definition, we will state some basic results that will be useful later.

Proposition 2.1.9. *If two curves C and C' in $\mathbb{P}^3$ are directly linked by complete intersection X with ideal $I_X = (F, G)$, then* $\deg C + \deg C' = \deg F \cdot \deg G$.

Proof. A special case of (12, Proposition 4.7) where $n = 3$ and $r = 1$. $\square$

Proposition 2.1.10. *Let $C \subset X$ be a curve in $\mathbb{P}^3$ lying on a complete intersection curve. Consider $J = [I_X : I_C]$. Let $I_{C'}$ be the scheme defined by J, so $J = I_{C'}$. Then*

1. *$C \cup C' = X$ as sets.*

2. *C' is a curve.*

3. *$C \overset{X}{\sim} C'$.*

Proof. A special case of (17, Proposition 5.2.3) using $n = 3$. $\square$

The previous proposition is important because it lets you start with one curve and guarantees that a complete intersection, which contains the original curve, links it to another curve by only having to check one of the ideal quotients in the definition instead of two.

Notice that direct linkage is not necessarily transitive or reflexive. As a matter of fact, we determine which multiple lines are reflexive for this relation, thus determining which multiple lines are self-linked. In general, linkage theory uses direct linkage to generate an equivalence class. We will not focus on the equivalence class. However, the definition will still be provided below as a reference.

Definition 2.1.11. The equivalence relation generated by direct algebraic linkage is called *linkage*. More precisely, we say that C is *linked* to C', denoted $C \sim C'$ if there is a sequence of curves $C_1, \ldots, C_k$ and a sequence of complete intersections $X_1, \ldots X_{k+1}$ such that

$$C \overset{X_1}{\sim} C_1 \overset{X_2}{\sim} \cdots \overset{X_k}{\sim} C_k \overset{X_{k+1}}{\sim} C'.$$

If $k+1$ is even, we say that C and C' are *evenly linked*. Notice that even linkage also generates an equivalence relation, and the equivalence classes are called *even linkage classes*. The notion of even linkage is sometimes called *bilinkage* and evenly linked curves are said to be *bilinked*.

The linkage class of complete intersections is one of the best understood linkage classes. We have the following special case of a proposition from (17, Page 108). While the inclusion of the proof is not necessary as it can be found in (17), we are including it as it is similar to proofs we will find later in this thesis.

Proposition 2.1.12. *Any two complete intersection curves in $\mathbb{P}^3$ are in the same linkage class. Furthermore, they are linked by at most two steps and are directly linked if their ideals share one generator.*

Proof. We prove the last statement first. Assume C and C' are complete intersection curves in $\mathbb{P}^3$ with ideals $I_C = (F, G)$ and $I_{C'} = (F, G')$. Furthermore consider the complete intersection X with ideal $I_X = (F, GG')$. Clearly we have the following

$$I_X \subseteq I_C \cap I_{C'}.$$

For the other necessary equalities we need to do a little more work.

$$\begin{aligned}
[I_X : I_C] &= \{s | sI_C \subset I_X\} \\
&= \{s | s(F, G) \subset (F, GG')\} \\
&= \{s | sG \subset (F, GG')\} \\
&\supset (F, G') \\
&= I_{C'}.
\end{aligned}$$

For the other containment, we have

$$\begin{aligned}
s \in [I_X : I_C] &\Rightarrow s(F, G) \subset (F, GG') \\
&\Rightarrow sG \in (F, GG') \\
&\Rightarrow sG = fF + gGG' \text{ for some } f, g \in k[x, y, z, w] \\
&\Rightarrow sG - gGG' = fF \\
&\Rightarrow G(s - gG') = fF \\
&\Rightarrow G | fF.
\end{aligned}$$

But (F, G) is the ideal of a complete intersection which means G and F do not have any factors in common by Proposition 2.1.4. Thus we can conclude that $f = f'G$ to get

$$\begin{aligned} G(s - gG') = f'GF \quad &\Rightarrow \quad s - gG' = f'F \\ &\Rightarrow \quad s = f'F + gG' \\ &\Rightarrow \quad s \in (F, G') \\ &\Rightarrow \quad s \in I_{C'}. \end{aligned}$$

Thus we have $C \overset{X}{\sim} C'$. Running the argument again, we can produce a sequence of links, changing one generator at a time, from one arbitrary complete intersection curve to the other in two steps. Thus, if we have a third complete intersection curve C'' with $I_{C''} = (F', G')$. We can directly link C' to C'' using complete intersection X' with ideal $I_{X'} = (G, FF')$ to get the following chain:

$$C \overset{X}{\sim} C' \overset{X'}{\sim} C''.$$

□

The above proof also shows that complete intersections of codimension 2 are in the same bilinkage class.

Corollary 2.1.13. *Any complete intersection curve in $\mathbb{P}^3$ is self-linked.*

Proof. Let $G' = G$ in Proposition 2.1.12. □

2.2 Rao Modules and ACM Curves

The Rao module[2] will be a powerful tool for analyzing the support of extremal multiple fold lines. Arithmetically Cohen-Macaulay curves will appear throughout this work. We will normally deal with them separately as they have a different structure from other curves. We will start by stating the definition of the Rao Module.

Definition 2.2.1. Let C be a curve in $\mathbb{P}^3$. The *Rao Module* $M(C)$ of C is the graded S-module defined by the first cohomology of the ideal sheaf of C, namely

$$M(C) = H^1_*(\mathcal{I}_C).$$

We will use both notations as appropriate.

Definition 2.2.2. Let $I_C = H^0_*(\mathcal{I}_C)$ be the saturated ideal of a curve C in $\mathbb{P}^3$. Then Y is *arithmetically Cohen-Macaulay (ACM)* if and only if S/I_C is a Cohen-Macaulay ring.

Arithmetically Cohen-Macaulay curves can be characterized as follows.

Lemma 2.2.3. *Let C be a curve in $\mathbb{P}^3$. Then the following are equivalent:*

1. *C is ACM.*
2. *$H^1_*(\mathcal{I}_C) = 0$.*
3. *The length of a minimal free graded S-resolution of I_C is 2.*
4. *C is in the linkage class of a complete intersection.*

[2]The Rao module is sometimes called the Hartshorne-Rao module because Hartshorne defined it. However, Rao's theorem made it into a useful object.

Proof. Statement 1 is equivalent to statement 2 by (17, Lemma 1.2.3) using $r = 1$ and equivalent to statement 3 by (17, Page 10). Finally, statement 3 and 4 are equivalent by (17, Example 1.5.1). □

By (17, Example 1.5.1) we have that the length of a minimal resolution of a complete intersection is equal to its codimension, in our case that means equal to 2. Thus we conclude that complete intersections are ACM curves.

Lemma 2.2.4. *(7, Proposition 1.1) If a curve C in $\mathbb{P}^3$ lies on a quadric surface, then C is ACM if and only if I_C is generated by three (or fewer) elements.*

While ACM curves will occasionally show up during our arguments, most of the work deals with curves that are not ACM. The following theorem is one of the most important theorems in linkage theory. Rao's theorem states exactly when two curves in $\mathbb{P}^3$ are evenly linked. Even though Hartshorne originally defined the Rao module, the module is named after Rao because this theorem made it into an useful object.

Theorem 2.2.5. *Let C and C' be curves in $\mathbb{P}^3$ with Rao modules $H^1_*(\mathcal{I}_C)$ and $H^1_*(\mathcal{I}_{C'})$. Then C is evenly linked to C' if and only if $H^1_*(\mathcal{I}_C)$ is isomorphic to some shift of $H^1_*(\mathcal{I}_{C'})$.*

Proof. (23, Theorem 2.8) and (17, Theorem 6.2.7). □

Before we present the Hartshorne-Schenzel theorem, we need the following definition.

Definition 2.2.6. Let C be a curve in $\mathbb{P}^3$, then its *dual Rao module* is defined as follows

$$M^*(C) = \left[H^1_*(\mathcal{I}_C)\right]^* = Hom_k(M(C), k).$$

Theorem 2.2.7. *(17, Theorem 5.3.1) Let C, $C' \subset \mathbb{P}^3$ be curves and assume that $C \overset{X}{\sim} C'$ where X is a complete intersection with ideal $I_X = (F, G)$ and sheafified minimal free resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-a-b) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-b) \longrightarrow \mathcal{I}_X \longrightarrow 0$$

where $\deg F = a$ *and* $\deg G = b$*. Then*

$$M(C') \cong M^*(C)(4-a-b).$$

We will end this section with the fact that will be useful for direct linkage but not for self-linkage.

Proposition 2.2.8. *(17, Corollary 5.2.14) Let C and C' be curves in $\mathbb{P}^3$. If C is directly linked to C' by X and* $\deg C = \deg C'$ *then $p_a(C) = p_a(C')$, where $p_a(C)$ is the arithmetic genus of the curve C.*

2.3 Extremal Curves

Before we discuss multiple lines, we will discuss extremal lines. We will need the following two definitions and a result by Martin-Deschamps and Perrin (15) in order to define what an extremal line is.

Definition 2.3.1. Let C be a curve in $\mathbb{P}^3$. Then the *Rao function* $\rho_C(j)$ of C is defined as follows

$$\rho_C(j) = h^1(\mathcal{I}_C(j)) = \dim_k H^1(\mathcal{I}_C(j)) \text{ for } j \in \mathbb{Z}.$$

The function $\rho_C(j)$ is named the Rao function because it gives the dimensions of the graded pieces of the Rao module of C.

Definition 2.3.2. A curve C in $\mathbb{P}^3$ is *degenerate* if C lies on a plane $H \subset \mathbb{P}^3$. Otherwise C is *non-degenerate*.

Theorem 2.3.3. *Let C be a non-degenerate curve in $\mathbb{P}^3$ of degree $d \geq 2$. Then the following inequalities hold for the Rao function*

$$\rho_C(j) \leq \begin{cases} 0 & \textit{if } j \leq g - \frac{(d-2)(d-3)}{2}, \\ \frac{(d-2)(d-3)}{2} - g + j & \textit{if } g - \frac{(d-2)(d-3)}{2} \leq j \leq 0, \\ \frac{(d-2)(d-3)}{2} - g & \textit{if } 0 \leq j \leq d-2, \\ \frac{(d-2)(d-3)}{2} - g - j & \textit{if } d-2 \leq j \leq \frac{(d-2)(d-3)}{2} - g, \\ 0 & \textit{if } \frac{(d-2)(d-3)}{2} - g \leq j. \end{cases}$$

Now we can define what an extremal curve is.

Definition 2.3.4. Let C be non-ACM curve in $\mathbb{P}^3$ of degree d greater than or equal to 2. Then C is *extremal* if the equality holds in Theorem 2.3.3 for each j.

Finally, we will state geometric conditions equivalent to the condition of extremal which will help us to categorize prime-fold lines.

Theorem 2.3.5. *Let $C \subset \mathbb{P}^3$ be a curve of degree $d \geq 5$. The following are equivalent:*

1. *C is extremal.*

2. *C contains a planar subcurve of degree $d-1$.*

3. *C is contained in two independent quadrics.*

Furthermore the first two are equivalent for $d \geq 2$.

Proof. By (8, Theorem 2.2) and (20, Proposition 2.2). □

2.4 Multiplicity Structures on Space Curves

In this section we will quickly review the results of Banica and Forster (2) on multiplicity structures on smooth curves in smooth surfaces.

Definition 2.4.1. If Y is a scheme, then a *multiplicity structure* Z on Y is a locally Cohen-Macaulay scheme Z which contains Y and has the same support as Y.

Consider a non-singular curve Y in $\mathbb{P}^3$ and multiplicity structure $Y \subset Z \subset \mathbb{P}^3$. Banica and Forster defined a filtration Z as follows: let $Y^{(i)}$ denote the scheme defined by I_Y^i. Remove any embedded points from $Z \cap Y^{(i)}$ and call it Z_i. Then Z_i is the unique largest Cohen-Macaulay subscheme contained in $Z \cap Y^{(i)}$. Let k be the smallest integer such that $Z \subset Y^{(k)}$, then we have

$$Y = Z_1 \subset Z_2 \subset \cdots \subset Z_k = Z.$$

Let $\mathcal{I}_i = \mathcal{I}_{Z_i}$. Define $L_i = \mathcal{I}_i/\mathcal{I}_{i+1}$ to be the sheaves associated to this filtration. The L_i's are locally free $\mathcal{O}_Y$-modules. Also, we get generically surjective maps $L_i \otimes L_j \to L_{i+j}$ and $L_1^{\otimes j} \to L_j$. Then we can get short exact sequences

$$0 \longrightarrow \mathcal{I}_{j+1} \longrightarrow \mathcal{I}_j \longrightarrow L_j \longrightarrow 0.$$

Because $Y \subset \mathbb{P}^3$ is a smooth subvariety of codimension two, the conormal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a rank two bundle on Y (10, Theorem II.8.17). Since we have the following diagram

$$\begin{tikzcd} \mathcal{I}_Y \arrow[rr] \arrow[dr] & & L_1 \\ & \mathcal{I}_Y/\mathcal{I}_Y^2 \arrow[ur] & \end{tikzcd}$$

we can conclude that L_1 has rank 0, 1 or 2. The rank 0 case yields that all $L_j = 0$ and thus $Z = Y$. The rank 2 case yields that $\mathcal{I}_Y/\mathcal{I}_Y^2 \to L_1$ is an isomorphism. Thus $Y^{(2)} \subset Z$ and Z has generic embedding dimension 3. Therefore the main case of interest is rank equal to 1. In this case we say that the extension Z is quasi-primitive. The generically surjective maps $L_1^{\otimes j} \to L_j$ imply that there are effective divisors D_j on Y such that $L_j = L^j(D_j)$ for $j \leq k$. We say that $(\deg L, \deg D_2, \ldots, \deg D_{d-1})$ is the type of the extension Z.

Proposition 2.4.2. *Let $Z \subset \mathbb{P}^3$ be a planar multiplicity structure of degree d on a line. Then up to coordinate change $I_Z = (x, y^d)$. In particular, Z is a complete intersection.*

Proof. Let Y be the line with ideal $I_Y = (x, y)$ and Z be contained in the the plane $x = 0$. By the Banica and Forster construction, we consider

$$Z \cap Y^{(j)} \text{ i.e. } I_Z + I_Y^j.$$

Globally we have the ideal $I_Z = (x, f_1, f_2, \ldots, f_n)$

$$\begin{aligned} I_Z + I_Y^j &= (x, f_1, f_2, \ldots, f_n) + (x^j, x^{j-1}y, x^{j-2}y^2, \ldots, xy^{j-1}, y^j) \\ &= (x, f_1, f_2, \ldots, f_n, x^j, x^{j-1}y, x^{j-2}y^2, \ldots, xy^{j-1}, y^j) \\ &= (x, f_1, f_2, \ldots, f_n, y^j). \end{aligned}$$

Furthermore to obtain Z_j, we have to remove any embedded points from $Z \cap Y^j$. Thus we have

$$Z_j \subset Z \cap Y^{(j)} \Rightarrow I_{Z_j} \supset I_Z + I_Y^j.$$

Consider the complete intersection ideal $I_X = (x, y^j)$. We have

$$I_X \subset I_Z + I_Y^j \subset I_{Z_j} \Rightarrow Z_j \subset X.$$

By construction Z_j doesn't have any embedded points, thus it is Cohen-Macaulay, like the complete intersection X. If $\deg Z_j = j$ then by Lemma 2.0.3 we have $X = Z_j$. We can take $j = d$ to obtain the result. Therefore the proof is finished by the proving the following claim.

Claim: $\deg Z_j = j$.

Recall that we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_{j+1} \longrightarrow \mathcal{I}_j \longrightarrow L_j \longrightarrow 0.$$

We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{I}_{j+1} \arrow[r] \arrow[d] & \mathcal{I}_j \arrow[r] \arrow[d] & L_j \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, "="] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & 0 & \\
 & \mathcal{O}_{Z_{j+1}} \arrow[r, "\phi"] \arrow[d] & \mathcal{O}_{Z_j} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{1}$$

By the snake lemma we have the following exact sequence

$$0 \longrightarrow \operatorname{Ker} \phi \longrightarrow L_j \longrightarrow 0 \longrightarrow \operatorname{Coker} \phi \longrightarrow 0$$

which yields the following short exact sequence

$$0 \longrightarrow L_j \longrightarrow \mathcal{O}_{Z_{j+1}} \xrightarrow{\phi} \mathcal{O}_{Z_j} \longrightarrow 0.$$

Recall that L_j's are locally free $\mathcal{O}_Y$-modules. Since Y is a line we have $Y \cong \mathbb{P}^1$. Therefore $L_j \cong \mathcal{O}_{\mathbb{P}^1}(a_j)$ for some integer a_j. A Hilbert polynomial calculation using the short exact sequence yields

$$\begin{aligned} P_{Z_{j+1}}(t) &= P_{Z_j}(t) + P_{\mathbb{P}^1(a_1)}(t) \\ &= (d_j t + C) + (t + 1 + a_1) \\ &= (d_j + 1)t + C + 1 + a_1 \end{aligned}$$

where d_j is the degree of Z_j. Therefore $\deg Z_{j+1} = \deg Z_j + 1$. We complete the proof of the theorem by recalling the fact that $Z_1 = Y$ and thus $\deg Z_1 = 1$. $\square$

Lemma 2.4.3. *(19, Lemma 1.3.c) A multiple curve in $\mathbb{P}^3$ is planar if and only if its of type* $(-1, 0, \ldots, 0)$.

Finally, we will state geometric qualities of extremal lines which will help us to categorize prime-fold lines.

Theorem 2.4.4. *A quasi-primitive d-fold line in $\mathbb{P}^3$ with $d \geq 3$ is extremal if and only if its type is $(-1, 0, \ldots, 0, b)$.*

Proof. Let W be a d-fold line with support on line Y with $d \geq 3$. Then it is of type

$$(\deg L, \deg D_2, \ldots, \deg D_{d-1}).$$

By Theorem 2.3.5, W is extremal if and only if it contains a planar subcurve Z of degree $d-1$. Proposition 2.4.2 implies that up to a coordinate change we have $I_Z = (x, y^{d-1})$. We have

$$I_Z \;=\; (x, y^{d-1}) \supset I_Y^{d-1} \supset I_W + I_Y^{d-1} \Rightarrow Z \subset W \cap Y^{(d-1)}.$$

The Banica and Forster construction for multiplicity structure implies that W_{d-1} is the largest Cohen-Macaulay subscheme contained in $W \cap Y^{(d-1)}$. Complete intersections are Cohen-Macaulay schemes, thus we get $Z \subset W_{d-1}$. By the claim in the proof of Proposition 2.4.2 $\deg W_{d-1} = d-1$. Now by Lemma 2.0.3 we that $Z = W_{d-1}$. Thus W is extremal if and only if W_{d-1} is planar. By Lemma 2.4.3 we have that W_{d-1} is planar if and only if it is of type $(-1, 0, \ldots, 0)$. Finally, that is equivalent to W being of type $(-1, 0, \ldots, 0, b)$. $\square$

Notation 2.4.5. *If W is an extremal line of type $(-1, 0, \ldots, 0, b)$, we will say that W has type b.*

Lemma 2.4.6. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of type b, then its arithmetic genus is $p_a(W) = (d^2 - 3d - 2b + 2)/2$.*

Proof. We get the result from combining (19, Lemma 1.3) and Theorem 2.4.4. $\square$

The previous result allows us to say the following:

Corollary 2.4.7. *Let W and W' in $\mathbb{P}^3$ be extremal curves of degree d and of type $(-1, 0, \ldots, 0, b)$ and $(-1, 0, \ldots, 0, b')$ respectively. If W and W' are are linked, then $b = b'$.*

Proof. By Proposition 2.2.8 we have that $p_a(W) = p_a(W')$. From Lemma 2.4.6 we get $p_a(W) = \frac{d(d-1)}{2} - d + 1 - b$ and $p_a(W') = \frac{d(d-1)}{2} - d + 1 - b'$. Combining both results we get

$$\begin{aligned} p_a(W) &= p_a(W') \\ \iff \frac{d^2 - 3d - 2b + 2}{2} &= \frac{d^2 - 3d - 2b' + 2}{2} \\ \iff b &= b'. \end{aligned}$$

$\square$

3 Quadric Surfaces in $\mathbb{P}^3$

Quadrics surfaces in $\mathbb{P}^3$ over the complex numbers have been studied and used for more than 100 years. Extending a well-known result over $\mathbb{C}$, we will geometrically classify quadric surfaces into four categories: the smooth quadric, the quadric cone, the reducible quadric and the double plane. Extending classical results over $\mathbb{C}$, we prove that the quadrics surfaces in any characteristic classify into these four categories. These surfaces will be necessary for creating complete intersections that can link two prime-fold lines. We start this chapter with a natural lemma.

Lemma 3.1. *Let $F \in k[x_0, x_1, \ldots, x_n]$ be a a non-zero quadratic form and assume $\partial F/\partial x_i$ are linearly independent. Let $\overline{z} = M^{-1}\overline{x}$ be any linear coordinate change with $M \in GL(n+1, k)$. Then $\partial F/\partial z_i$ are also linearly independent.*

Proof. Consider a linear coordinate change given by the matrix

$$M = \begin{bmatrix} b_{0,0} & b_{0,1} & \ldots & b_{0,n} \\ b_{1,0} & b_{1,1} & \ldots & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & \ldots & b_{n,n} \end{bmatrix} \in \mathrm{GL}(n+1, k).$$

Define the z_i's such that

$$\begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n} \\ b_{1,0} & b_{1,1} & \dots & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

In other words we have

$$x_i = \sum_{j=0}^{n} b_{i,j} z_j.$$

Thus the partials of x_i with respect to z_j are

$$\frac{\partial x_i}{\partial z_j} = b_{i,j}.$$

Now suppose that there are constants a_i for which

$$\begin{aligned} 0 &= \sum_{i=0}^{n} a_i \frac{\partial F}{\partial z_i} \\ &= \sum_{i=0}^{n} a_i \left(\sum_{j=0}^{n} \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial z_i} \right) \\ &= \sum_{i=0}^{n} a_i \left(\sum_{j=0}^{n} b_{i,j} \frac{\partial F}{\partial x_j} \right) \\ &= \sum_{j=0}^{n} \left(\sum_{i=0}^{n} a_i b_{i,j} \right) \frac{\partial F}{\partial x_j}. \end{aligned}$$

We have $\partial F/\partial x_j$ are linearly independent by hypothesis and thus

$$\sum_{i=0}^{n} a_i b_{i,j} = 0 \text{ for all } j \Rightarrow \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n} \\ b_{1,1} & b_{1,1} & \dots & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,1} & \dots & b_{n,n} \end{bmatrix}^T \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The first implication is by the definition of matrix multiplication and the second implication follows from the fact that if M is invertible then M^T is also invertible. Therefore, we have $\partial F/\partial z_i$ are also linearly independent. □

I would like to thank again Dr. Mohan Kumar again for providing the statement and a sketch of the proof for the case where the characteristic of the field is equal to 2 which we will present next.

Theorem 3.2. *Let k be an algebraically closed filed of characteristic 2. Up to a linear coordinate change, any quadric $0 \neq F \in k[x_0, \dots, x_n]$ is of the form $\sum x_i y_i$, $z^2 + \sum x_i y_i$ or z^2 where x_i, y_i, z are distinct variables in $\{x_0, \dots, x_n\}$.*

Proof. Let $F(x_0, \dots, x_n)$ be any quadric and let $F_i = \partial F/\partial x_i$. Then the F_is are linear forms.

Claim 1: If the F_i's are linearly independent, then $F = \sum x_i y_i$.

We will prove this by induction on n.

- $n = 0$: We have $F = a_0 x_0^2$. Thus $F_0 = 0$ which is not linearly independent. Thus the hypothesis is not true for $n = 0$. We will start the induction with $n = 1$.

- $n = 1$: We have $F = a_0x_0^2 + bx_0x_1 + a_1x_1^2$. By assumption char $k = 2$ we have that $F_0 = bx_0$ and $F_1 = bx_1$ which are linearly independent and thus $b \neq 0$. Notice that if either a_0 or a_1 equals zero, then $F = (bx_0 + a_1x_1)x_1$ or $F = x_0(a_0x_0 + bx_1)$ respectively. After an easy change of coordinates we get $F = z_0z_1$, therefore we can assume that a_0 and a_1 are non-zero. Consider the following:

$$\begin{aligned} F &= a_0x_0^2 + bx_0x_1 + a_1x_1^2 \\ &= \left(\frac{a_0^2}{b^2}\frac{x_0^2}{x_1^2} + \frac{a_0}{b}\frac{x_0}{x_1} + \frac{a_0a_1}{b^2}\right)\frac{b^2}{a_0}x_1^2 \\ &= \left(z^2 + z + a\right)\frac{b^2}{a_0}x_1^2 \text{ with } z = \frac{a_0}{b}\frac{x_0}{x_1} \text{ and } a = \frac{a_0a_1}{b^2}. \end{aligned}$$

Since the field k is algebraically closed, we have that $z^2 + z + a$ has at least one root, label it r. Now, notice $r^2 + r + a = 0$ implies the following:

$$(r+1)^2 + (r+1) + a = r^2 + 1 + r + 1 + a = r^2 + r + a = 0.$$

Therefore we have two distinct roots: r and $r+1$. Thus $z^2 + z + a$ factors into two distinct linear equations (up to scalar multiple) $z - r$ and $z - r - 1$. Putting it back together we get

$$\begin{aligned} F &= \left(z^2 + z + a\right)\frac{b^2}{a_0}x_1^2 \\ &= (z - r)(z - (r+1))\frac{b^2}{a_0}x_1^2 \\ &= \left(\frac{a_0}{b}\frac{x_0}{x_1} - r\right)\left(\frac{a_0}{b}\frac{x_0}{x_1} - (r+1)\right)\frac{b^2}{a_0}x_1^2 \\ &= \left(x_0 - \frac{br}{a_0}x_1\right)\Big(a_0x_0 - b(r+1)x_1\Big). \end{aligned}$$

Since $z-r$ and $z-r-1$ are distinct (up to scalar multiple), then so are $\left(x_0 - \frac{br}{a_0}x_1\right)$ and $\left(a_0x_0 - b(r+1)x_1\right)$. Consider the coordinate change

$$\begin{aligned} z_0 &= x_0 - \frac{br}{a_0}x_1, \\ z_1 &= a_0x_0 - b(r+1)x_1, \end{aligned}$$

then $F = z_0z_1$.

- $n = k > 2$: We have F and F_i as in the set up. So $F_n \neq 0$ and has no x_n term (remember that char $k = 2$ so the partial derivative of x_n^2 vanishes in F_n). Relabel indices if necessary to make sure that F_n has a nonzero x_{n-1} term. For some quadric polynomial $H_1(x_0, \ldots, x_{n-1})$ and some linear polynomial $H_2(x_0, \ldots, x_{n-1})$ we can separate F as follows:

$$F = H_1(x_0, \ldots, x_{n-1}) + H_2(x_0, \ldots, x_{n-1})x_n + cx_n^2.$$

Thus taking partial derivatives, we see that we have $F_n = H_2(x_0, \ldots, x_{n-1})$. Now consider the following coordinate change:

$$z_{n-1} = H_2(x_0, \ldots, x_{n-1}) \text{ and } z_i = x_i \text{ for } i \neq n-1.$$

Note that H_2 has no x_n term and recall that the x_{n-1} coefficient is nonzero: if

$$z_{n-1} = \sum_{i \le n-1} a_i x_i, \text{ then } x_{n-1} = 1/a_{n-1} \left(z_{n-1} - \sum_{i<n-1} a_i x_i \right),$$

which we can substitute to derive the following:

$$F = H(z_0, \ldots, z_{n-1}) + z_{n-1} z_n + c z_n^2.$$

We can also express H as follows:

$$H(z_0, \ldots, z_{n-1}) = z_{n-1} l(z_0, \ldots, z_{n-1}) + K(z_0, \ldots, z_{n-2})$$

where $l(z_0, \ldots, z_{n-1})$ is a linear polynomial and $K(z_0, \ldots, z_{n-2})$ is a quadratic polynomial. Putting everything together we can say

$$F = K(z_0, \ldots, z_{n-2}) + z_{n-1}(z_n + l(z_0, \ldots, z_{n-1})) + c z_n^2.$$

Use the coordinate change

$$u_n = z_n + l(z_0, \ldots, z_{n-1}) \text{ and } u_i = z_i \text{ for } i \ne n$$

to get

$$F \;=\; K(u_0, \ldots, u_{n-2}) + u_{n-1}(u_n) + c(u_n - l(u_0, \ldots, u_{n-1}))^2.$$

Note that $c(u_n - l(u_0, \ldots, u_{n-1}))^2$ does not contribute any terms to $\partial F/\partial u_i$ because we are in characteristic two. Thus for $0 \leq i \leq n-2$

$$\frac{\partial F}{\partial u_i} = \frac{\partial K}{\partial u_i}.$$

By Lemma 3.1 we have that $\partial F/\partial u_i$ are linearly independent and thus so are $\partial K/\partial u_i$. Therefore the partials of $K \in k[u_0, \ldots, u_{n-2}]$ are linearly independent and K has fewer variables. By the induction hypothesis, after a coordinate change we can assume that $K = \sum v_i w_i$ where v_i, w_i are distinct variables that are linear combinations $u_0, \ldots, u_{n-2}$. For an appropriate integer r we have

$$\begin{aligned} F &= K(u_0, \ldots, u_{n-2}) + u_{n-1}(u_n) + c(u_n - l(u_0, \ldots, u_{n-1}))^2 \\ &= \sum_{i=0}^{r} v_i w_i + u_{n-1}(u_n) + c(u_n - l(u_0, \ldots, u_{n-1}))^2 \\ &= \sum_{i=0}^{r+1} v_i w_i + p^2, \end{aligned}$$

where $p = \sqrt{c}(u_n - l(u_0, \ldots, u_{n-1}))$, $v_{r+1} = u_{n-1}$ and $w_{r+1} = u_n$. Since p is linear we can use the coordinate change above to write p as follows

$$p = \sum_{i=0}^{r+1} (c_i v_i + d_i w_i) \text{ for } c_i, d_i \in k.$$

We can rewrite $p^2 = \sum(c_i v_i + d_i w_i)^2$ since char $k = 2$. Finally we get

$$\begin{aligned} F &= \sum v_i w_i + p^2 \\ &= \sum v_i w_i + \sum(c_i v_i + d_i w_i)^2 \\ &= \sum \left(v_i w_i + (c_i v_i + d_i w_i)^2\right). \end{aligned}$$

Consider polynomials defined as follows

$$p_i = v_i w_i + (c_i v_i + d_i w_i)^2.$$

Notice that

$$\frac{\partial p_i}{\partial v_i} = w_i \text{ and } \frac{\partial p_i}{\partial w_i} = v_i.$$

Thus we conclude that the quadric polynomials in two variables p_i have linearly independent partial derivatives. The base step implies that there is a coordinate change such that p_i is of the form $x_i y_i$ where x_i and y_i are distinct variables that are linear combinations v_i and w_i. Thus we can combine all the coordinate changes for each i to conclude

$$F = \sum x_i y_i$$

where x_i and y_i are distinct variables.

Claim 2: If the F_is are linearly dependent, then $F = \sum x_i y_i + z^2$,$F = \sum x_i y_i$ or z^2 after a coordinate change.

In this case, we have a relation $\sum a_i F_i = 0$ where not all $a_i = 0$. Relabel to make sure that $a_n \neq 0$. Consider the following equations

$$\begin{aligned} x_i &= z_i + a_i z_n \text{ for } i < n, \\ x_n &= a_n x_n. \end{aligned}$$

We can express the equations in matrix form as follows

$$\begin{bmatrix} & I & & a_1 \\ & & & \vdots \\ 0 & \dots & 0 & a_n \end{bmatrix} \begin{bmatrix} z_0 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$$

where I is the identity matrix. Notice that

$$M = \begin{bmatrix} & I & & a_1 \\ & & & \vdots \\ 0 & \dots & 0 & a_n \end{bmatrix}$$

is an invertible matrix and thus M^{-1} is a valid a coordinate change from $x_i's$ to $z_i's$. Finally by the chain rule we have

$$\frac{\partial F}{\partial z_n} = \sum F_i \frac{\partial x_i}{\partial z_n} = \sum a_i F_i = 0.$$

For constants $c_{i,j}$ we can write F as follows

$$\begin{aligned}
F &= \sum_{i=0}^{n} z_i \left(\sum_{j<i} c_{i,j} z_j \right) + \sum_{i=0}^{n} c_{i,i} z_i^2 \\
\Rightarrow \frac{\partial F}{\partial z_n} &= \sum_{j<i} c_{n,j} z_j = 0 \\
\Rightarrow c_{n,j} &= 0 \text{ for all } j \neq n, \\
F &= \sum_{i=0}^{n-1} z_i \left(\sum_{j<i} c_{i,j} z_j \right) + \sum_{i}^{n} c_{i,i} z_i^2.
\end{aligned}$$

So we can write F as follows

$$F = G(z_0, \ldots, z_{n-1}) + c_n z_n^2.$$

If the partials G_i are linearly dependent, we repeat the process until we get

$$F = G(z_0, \ldots z_m) + \sum_{m<k<n} c_k z_k^2$$

and the partial derivatives of G are linearly independent or G is possibly 0. Now we use Claim 1 to get that after a coordinate change, $G = \sum v_i w_i$ which are distinct variables that are linear combinations $z_0, \ldots, z_m$. Putting F back together we get

$$F = \sum v_i w_i + \sum_{m<k<n} c_k z_k^2 \text{ or } F = \sum_{m<k<n} c_k z_k^2.$$

A final coordinate change $u_n = \sum_{m<k<n} c_k z_k^2$ yields

$$F = \sum v_i w_i + u_n^2 \text{ or } F = u_n^2. \qquad \square$$

Now that we have taken care of the special case where char $k = 2$, we will take care of the general case.

Theorem 3.3. *Let k be an algebraically closed field and $0 \neq F \in k[x_0, \dots, x_n]$ a quadratic form. Then there is a linear change of coordinates after which $F = \sum x_i y_i$, $\sum x_i y_i + z^2$ or z^2 with x_i, y_i, z distinct variables in $\{x_0, \dots, x_n\}$.*

Proof. We proved case char $k = 2$ in Corollary 3.2. Now assume char $k \neq 2$. Note that F can be brought into the form $\sum_{j=0}^{r} a_j x_j^2$ for some $r \leq n$ and some nonzero $a_j \in k$ by (21, Corollary 2.4). Since k is algebraically closed, we can assume $a_j = 1$. Now factor pairs of variables as follows.

$$x_j^2 + x_{j+1}^2 = (x_i + ix_{j+1})(x_i - ix_{j+1})$$

where $i = \sqrt{-1}$. Notice that $(x_i + ix_{j+1}) \neq (x_i - ix_{j+1})$ since char $k \neq 2$. Now do coordinate change

$$X_i = x_i + ix_{j+1} \text{ and } Y_i = x_i - ix_{j+1}.$$

If the number of variables is even, then we are done. If the number of variables is odd, then we can replace x_n by z. $\square$

Theorem 3.4. *Let Q be a quadric surface in $\mathbb{P}^3$. Then up to coordinate change F is in one of four categories:*

1. *Smooth quadric with ideal $(zx - yw)$: The image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$.*

2. *Quadric cone with ideal $(xz - y^2)$: The cone over a smooth conic in the plane.*

3. *Reducible quadric with ideal (xy): The union of two intersecting planes.*

4. *Double plane with ideal (x^2): The doubling of a plane.*

Proof. The ideals follow from using $n = 3$ in Theorem 3.3. The geometric description for statement 2 is given by (10, Example II.6.5.2). $\square$

4 Linkage of Multiple Lines

We will now analyze multiple lines. We will primarily focus on self-linkage by quadric surfaces, but mention some direct linkage results. We will see that if we assume the degree to be prime, then self-linkage has to occur (if at all) by using a quadric or a plane. We start this chapter by looking at the general form of complete intersections that can link two prime-fold lines of the same degree. We will analyze extremal multiple lines first. We will finish the chapter by analyzing non-extremal multiple lines.

Definition 4.0.8. A *d-fold* line is a locally Cohen-Macaulay scheme of degree d supported on a line in $\mathbb{P}^3$.

Lemma 4.0.9. *Let $W \subset \mathbb{P}^3$ be a d-fold line and let $X \subset \mathbb{P}^3$ be a complete intersection that links W to another d-fold lined W'. If d is prime, then $I_X = (F, G)$ where* $\deg F = 2$ *and* $\deg G = d$ *or* $\deg F = 1$ *and* $\deg G = 2d$.

Proof. Let X be a complete intersection with ideal (F, G) that directly links W to W'. Thus by Proposition 2.1.9

$$2d = \deg W + \deg W' = \deg F \cdot \deg G.$$

Therefore $\deg F = 2$ and $\deg G = d$ or $\deg F = 1$ and $\deg G = 2d$ since d is prime. $\square$

We first analyze the case where $\deg F = 1$ and $\deg G = 2d$. Later we will analyze the case $\deg F = 2$ and $\deg G = d$, which is more interesting.

Lemma 4.0.10. *Let $W \subset \mathbb{P}^3$ be a d-fold line that is linked to another d-fold line by a complete intersection X with ideal $I_X = (F, G)$ where $\deg F = 1$. Then W is a complete intersection and up to coordinate change it is of the form (x, y^d), and thus ACM.*

Proof. Let W be a d-fold line and let X be a complete intersection with ideal of the form (F, G) where $deg\, F = 1$ and $deg\, G = 2d$. By a change of coordinates, we assume that $F = x$. Thus, we have that X is planar. We have

$$W \subset X \Rightarrow I_X \subset I_W \Rightarrow x \in I_W.$$

Therefore W is also planar. We use Proposition 2.4.2 to conclude $\mathcal{I}_W = (x, y^d)$. Finally we have that W is a complete intersection. □

Corollary 4.0.11. *Let $W \subset \mathbb{P}^3$ be a d-fold line that is linked to another d-fold line W' by a complete intersection X with ideal $I_X = (F, G)$ where $\deg F = 1$. Then*

1. *W is self-linked.*

2. *W and W' are directly linked.*

Proof. By Lemma 4.0.10, we have that W and W' are complete intersections. Lemma 2.1.12 proves statement 1. Corollary 2.1.13 proves statement 2. □

We conclude that if a non-ACM prime-fold line is self-linked, then it is self-linked by using a quadric, and thus lies on a quadric. For this case, we look at $\deg F = 2$ and $\deg G = d$. However, if we limit our scope to links involving a quadric, then the following results are valid in arbitrary degree greater than or equal to 3. We will explicitly mention which results need an odd prime degree. Unless stated otherwise, we will assume that all complete intersections involve a quadric.

4.1 Extremal Multiple Lines

First, we will analyze d-fold line $W \subset \mathbb{P}^3$ contained in two independent quadrics. From Theorem 2.3.5 we have that if $\deg W \geq 5$, then W is extremal. By (19, Proposition 2.1) and (19, Proposition 2.3) we have that the previous statement holds for triple lines as well, unless W is a thick triple line or W lies on the quadric cone. Theorem 2.4.4 states that the type of W is the $(d-1)$-tuple

$$(-1, 0, \ldots, 0, b)$$

and by Theorem 2.3.5 W_{d-1} is a planar curve. We have the following short exact sequence

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_{W_{d-1}} \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow 0$$

Theorem 4.1.1. *Let Y in $\mathbb{P}^3$ be the line $x = y = 0$ and let Z be the multiplicity $d-1$ structure $x = y^{d-1} = 0$ on Y. Let p, q be two homogeneous polynomials of degrees $b-d+2, b$ which have no common zeroes along Y. Then the maps $x \mapsto p, y^{d-1} \mapsto q$ define a surjection $u : \mathcal{I}_Z \to \mathcal{O}_Y(b-d+1)$. The kernel of u is the ideal sheaf of a d-multiplicity structure W on Y. Further, we have*

1. $p_a(W) = (d^2 - 3d - 2b + 2)/2$.

2. $H^1_*(\mathcal{I}_W) \cong (S/(x, y, p, q))(b-d+1)$.

3. $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$.

4. *If p', q' define another d-structure W', then $W = W'$ if and if there exist $c \in k^*$ such that $p' = cp \bmod I_Y$ and $q' = cq \bmod I_Y$.*

5. *W is quasi-primitive unless $q = 0$ and $d \geq 3$.*

Proof. By hypothesis we have Z is a complete intersection with ideal $I_Z = (x, y^{d-1})$. We have the following diagram:

$$0 \longrightarrow S(-d) \xrightarrow{\begin{bmatrix} y^{d-1} \\ -x \end{bmatrix}} S(-1) \oplus S(1-d) \xrightarrow{\begin{bmatrix} x & y^{d-1} \end{bmatrix}} I_Z \longrightarrow 0.$$

Tensor with $S_Y = S/(x, y)$ (which is right exact) to get

$$S_Y(-d) \xrightarrow{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} S_Y(-1) \oplus S_Y(1-d) \xrightarrow{\begin{bmatrix} \bar{x} & \bar{y}^{d-1} \end{bmatrix}} I_Z \otimes S_Y \longrightarrow 0.$$

Therefore

$$\operatorname{Ker}\left(\begin{bmatrix} \bar{x} & \bar{y}^{d-1} \end{bmatrix}\right) = \operatorname{Im}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0$$

which yields the following isomorphism

$$I_Z \otimes S_Y = I_Z/I_Z I_Y \cong S_Y(-1) \oplus S_Y(1-d)$$

where $I_Z/I_Z I_Y$ is freely generated by the images of x and y^{d-1}. Therefore the map defined by $x \mapsto \bar{p}$ and $y^{d-1} \mapsto \bar{q}$ is well defined. This maps defines the following graded homomorphism

$$\begin{tikzcd} \phi : I_Z \arrow[rr] \arrow[dr] & & S_Y(b-d+1) \\ & \Phi : I_Z/I_Z I_Y \arrow[ur] & \end{tikzcd}$$

Since $(\bar{p}, \bar{q})$ is a regular sequence in S_Y, we have the following sequence

$$0 \longrightarrow S_Y(-b-1) \xrightarrow{\begin{bmatrix} \bar{q} \\ -\bar{p} \end{bmatrix}} \begin{matrix} S_Y(-1) \\ \oplus \\ S_Y(1-d) \end{matrix} \longrightarrow S_Y(b-d+1) \xrightarrow{\Phi} S_Y(b-d+1)/(\bar{p}, \bar{q}) \longrightarrow 0$$

Let $\operatorname{Ker}\phi = I_W$. Notice that we have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_Z/I_ZI_Y \arrow[r, "="] & I_Z/I_ZI_Y \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_W \arrow[r] & I_Z \arrow[r, "\phi"] \arrow[d] & (\bar{p},\bar{q})(b-d+1) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & S_Y(-b-1) \arrow[r] & \begin{matrix} S_Y(-1) \\ \oplus \\ S_Y(1-d) \end{matrix} \arrow[r, "\Phi"] \arrow[d] & (\bar{p},\bar{q})(b-d+1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Using the snake lemma we have the following short exact sequence

$$0 \longrightarrow I_Z/I_ZI_Y \longrightarrow I_W \longrightarrow S_Y(-b-1) \longrightarrow 0$$

which we can use to lift a generator from $S_Y(-b-1)$ to I_W. Following the maps we obtain that 1 lifts to $xq - y^{d-1}p$. Therefore the short exact sequence yields

$$I_W = (I_ZI_Y, xq - y^{d-1}p).$$

Sheafify

$$0 \longrightarrow S_Y(-b-1) \xrightarrow{\begin{bmatrix} \bar{q} \\ -\bar{p} \end{bmatrix}} \begin{matrix} S_Y(-1) \\ \oplus \\ S_Y(1-d) \end{matrix} \longrightarrow S_Y(b-d+1) \xrightarrow{\Phi} S_Y(b-d+1)/(\bar{p},\bar{q}) \longrightarrow 0$$

to obtain

$$0 \longrightarrow \mathcal{O}_Y(-b-1) \longrightarrow \begin{matrix} \mathcal{O}_Y(-1) \\ \oplus \\ \mathcal{O}_Y(1-d) \end{matrix} \xrightarrow{\widetilde{\Phi}} \mathcal{O}_Y(b-d+1) \longrightarrow \widetilde{M} \longrightarrow 0$$

where $M = S_Y(b-d+1)/(\bar{p},\bar{q})$. Furthermore, we have that the stalk maps $\widetilde{\Phi}_{p_0}$ are surjective for all points p_0 since $(\bar{p},\bar{q})$ do not have any zeros in common. Therefore $\widetilde{\Phi}$ is surjective and $\widetilde{M} = 0$. We can also set up the exact sequence

$$0 \longrightarrow I_W \longrightarrow I_Z \xrightarrow{\phi} S_Y(b-d+1) \longrightarrow S_Y(b-d+1)/(\bar{p},\bar{q}) \longrightarrow 0.$$

Sheafify the sequence to obtain

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow \widetilde{M} = 0 \longrightarrow 0.$$

Apply functor H^i_*

$$\begin{tikzcd} 0 \arrow[r] & H^0_*(\mathcal{I}_W) \arrow[r] & H^0_*(\mathcal{I}_Z) \arrow[r] & H^0_*(\mathcal{O}_Y(b-d+1)) \arrow[dll] & \\ & H^1_*(\mathcal{I}_W) \arrow[r] & H^1_*(\mathcal{I}_Z) \arrow[r] & H^1_*(\mathcal{O}_Y(b-d+1)) \arrow[r] & \dots \end{tikzcd}$$

By Lemma 2.2.3 we have $H^1_*(\mathcal{I}_Z)$ since Z is a complete intersection, and thus ACM. We have the following exact sequence

$$0 \longrightarrow I_W \longrightarrow I_Z \longrightarrow S_Y(b-d+1)) \longrightarrow H^1_*(\mathcal{I}_W) \longrightarrow 0$$

which yields

$$H^1_*(\mathcal{I}_W) = S_Y(b-d+1))/\operatorname{Im} I_Z = S_Y(b-d+1)/(\bar{p},\bar{q}) = (S/(x,y,p,q))(b-d+1)$$

We use the same snake lemma as in diagram (1) in Proposition 2.4.2 to get

$$0 \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

which yields the fact that $\operatorname{Supp} W = Y$ and $\operatorname{depth} \mathcal{O}_W$ is greater than or equal to 1. Thus we have that W is a multiplicity d structure on Y. Since $p_a(Z) = \frac{1}{2}(d-2)(d-3)$ we get that $p_a(W) = (-b+d+1) + \frac{1}{2}(d-2)(d-3) - 1$ which simplifies to $(d^2-3d-2b+2)/2$.

If p', q' define another d-fold line W' such that $W = W'$. By the construction above we must have $(\bar{p}, \bar{q}) = (\bar{p}', \bar{q}')$ in S_Y. This only occurs if and only if there exist a $c \in k^*$ such that $p' = cp \bmod I_Y$ and $q' = cq \bmod I_Y$. Finally if $q = 0$ and $d \geq 3$, we have

$$I_Y^2 = (x^2, xy, y^2) \supset (x^2, xy, y^d, y^{d-1}p) = I_W.$$

So $Y^{(2)} \subset W$ and thus W is not quasi-primitive. □

Corollary 4.1.2. *Consider multiple line W of type $(-1, 0, \ldots, 0, b)$. Then after a change of coordinates, W is constructed as in Proposition 4.1.1.*

Proof. Any multiple line W of type $(-1, 0, \ldots, 0, b)$ with planar $d-1$ multiplicity structure W_{d-1} has the following short exact sequence.

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_{W_{d-1}} \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow 0$$

Then the construction above gives

$$\mathcal{I}_Z \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow 0$$

□

We will separate the cases by values of b. We will consider d-fold lines where d is greater than or equal to 3.

Corollary 4.1.3. *Let $W \subset \mathbb{P}^3$ be a d-fold line of type $(-1, 0, \ldots, 0, b)$ with $b \geq 0$. Then $b = 0$ or $b > d - 3$. Moreover, up to coordinate change:*

(a) If $b = 0$, then W is planar with ideal (x, y^d).

(b) If $b = d - 2$, then $\deg W = 2$ or W is ACM with ideal $(x^2, xy, xq - y^{d-1})$ where $q \in k[z, w]_{d-2}$.

(c) If $b = d - 1$, then $(\operatorname{Ann} M(W))_1 = (I_{p_0})_1$, where $p_0 \in Z(x, y)$ is a point and the ideal of W is $(x^2, xy, y^d, xq - y^{d-1}z)$ where $q \in k[z, w]_{d-1}$ and p and z have no common zeroes along $Z(x, y)$.

(d) If $b \geq d$, then $(\operatorname{Ann} M(W))_1 = (x, y)_1$ with ideal $(x^2, xy, y^d, xq - y^{d-1}p)$ where $p \in k[z, w]_{b-d+2}$ and $q \in k[z, w]_b$ which have no common zeroes along $Z(x, y)$.

Additionally, W is extremal when $b \geq d - 1$ and ACM when $b = d - 2$ or $b = 0$.

Proof. Consider the Rao Module $M(W) \cong (S/(x, y, p, q))(b - d + 1)$ where W is an extremal d-fold line of type b. First notice that for type $1 \leq b \leq d - 3$, we have $\deg p \leq -1$ and $1 \leq \deg q \leq d - 3$. So we have $p = 0$. For d-fold line to exist then q must be a unit. Thus this case cannot occur because the $1 \leq \deg q \leq d - 3$. We will separate the remaining cases as follows:

(a) If $b = 0$, then $\deg p \leq -1$ and $\deg q = 0$. So we have $p = 0$ and q a unit. So we have

$$I_W = (x^2, xy, y^d, xq - y^{d-1}p) = (x^2, xy, y^d, xq) = (x, y^d).$$

Therefore W is planar (and thus a complete intersection). This case was analyzed in Corollary 4.0.11.

(b) If $b = d - 2$, then $\deg p = 0$ and $\deg q = d - 2$. We have to options

- $p = 0$. For d-fold line to exist then q must be a a unit which means we have $d = 2$. Thus we have a double line with ideal $(x^2, xy, y{}^{,}xq - y^{d-1}z)$. This case was analyzed by Migliore in (18).

- p is a unit. First we have

$$M(W) \cong (S/(x, y, p, q))(b - d + 1) = 0.$$

  Thus we have that W is ACM. Furthermore, we can assume

$$I_W = (x^2, xy, y^d, xq - y^{d-1}p) = (x^2, xy, y^d, xq - y^{d-1}).$$

  But we have

$$y^d = y(xq - y^{d-1} - xq) = y(xq - y^{d-1}) - xyq \in (x^2, xy, xq - y^{d-1}).$$

  Thus we can conclude

$$I_W = (x^2, xy, xq - y^{d-1}).$$

(c) If $b = d-1$, then $\deg p = 1$ and $\deg q = d-1$. After a coordinate change we can assume $p = z$ which yields

$$I_W = (x^2, xy, y^d, xq - y^{d-1}p) = (x^2, xy, y^d, xq - y^{d-1}z),$$
$$M(W) \cong (S/(x,y,z,q))(b-d+1) = 0,$$

and thus

$$\mathrm{Ann}(M(W))_1 = (x,y,z) = (I_p)_1,$$

where $p \in Y$ is a point.

(d) If $b \geq d$, then $\deg p \geq 2$, $\deg q \geq d$ and $\deg xq - y^{d-1}p \geq d+1$. Thus we have

$$\begin{aligned} I_W &= (x^2, xy, y^d, xq - y^{d-1}p) \\ \mathrm{Ann}(M(W))_1 &= (x,y) = I_Y. \end{aligned}$$

$\mathrm{Ann}(M(W))_1$ is the linear vector space of the annihilator of module $M(W)$. This fact will be useful later for direct linkage.

□

Remark 4.1.4. *Since double lines were analyzed by Migliore in (18). We will only consider d-fold lines W where $d \geq 3$.*

We examine each case in the following subsections.

4.1.1 ACM Case

By Corollary 4.1.3, if a d-fold line W is of type (-1,0, ..., 0) or type $(-1, 0, \ldots, 0, d-2)$ then it is ACM. While ACM curves cannot be extremal by definition, we will include them in this section as their construction is similar to extremal lines. If d-fold line W is of type $(-1, 0, \ldots, 0)$, then W is a planar, which has already been analyzed in Corollary 4.0.11. We will now consider W of type $(-1, 0, \ldots, 0, d-2)$.

Proposition 4.1.5. *Let $W \subset \mathbb{P}^3$ be a d-fold line of type $(-1, 0, \ldots, 0, d-2)$. Then W is self-linked.*

Proof. By Corollary 4.1.3 we have

$$I_W = (x^2, xy, xq - y^{d-1}).$$

Consider the complete intersection ideal $I_X = (x^2, y^d)$. Notice the following

$$\begin{aligned}
I_{W'} &= [I_X : I_W] \\
&= [(x^2, y^d) : (x^2, xy, xq - y^{d-1})] \\
&= \{s \in S | s(x^2, xy, xq - y^{d-1}) \subset (x^2, y^d)\} \\
&= \{s \in S | s(xy, xq - y^{d-1}) \subset (x^2, y^d)\}.
\end{aligned}$$

Clearly $x^2 \in I_{W'}$. Now notice the following products are elements of (x^2, y^d)

$$\begin{array}{rcl} & xy & \\ (xy)(xy) & = & x^2y^2 \\ (xy)(xq - y^{d-1}) & = & x^2yq - y^d \\ xq & - & y^{d-1} \\ (xq - y^{d-1}))(xy) & = & x^2yq - y^d \\ (xq - y^{d-1})(xq - y^{d-1}) & = & x^2q^2 - y^{2d-2} \end{array}$$

Therefore we have $(x^2, xy, xq - y^{d-1}) \subset I_{W'}$. We can conclude that $(x^2, xy, xq - y^{d-1}) = I_{W'}$ by Lemma 2.0.3. Therefore W is self-linked. $\square$

4.1.2 Special Case

We now analyze extremal multiple-fold lines of type $b = d - 1$. Up to a coordinate change we have

$$I_W = (x^2, xy, y^d, xq - y^{d-1}z)$$

where we can assume q is an element of $k[z, w]$ that has no common zeros with z along Y and $\deg q = d - 1$, which is equivalent to $z \not| q$. Thus we have $\deg(xq - y^{d-1}z) = d$. As before, we will analyze what kind of complete intersections X, that involve a quadric, can link two d-fold lines of type $b = d - 1$.

Proposition 4.1.6. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of type $b = d-1$ with ideal*

$$I_W = (x^2, xy, y^d, xq - y^{d-1}z)$$

where q is an element of $k[z,w]$ that has no common zeros with z along Y. Let X be a complete intersection that links W to another d-fold line W', then $\operatorname{Supp} W = \operatorname{Supp} W'$*. Furthermore up to coordinate change we have one of the following:*

1. $I_X = (xy, x^d + y^d)$,

2. $I_X = (x^2, gxy + y^d)$ *where* $g \in k[y,z,w]_{d-2}$.

Proof. Assume that X is a complete intersection that links two d-fold lines. As established in Theorem 2.4.7, W' is also of type $b = d-1$. Let $I_X = (F, G)$ where we can assume $\deg F = 2$ and $\deg G = d$. Now notice the following

$$\begin{aligned}
W \subset X \quad &\Leftrightarrow \quad I_X \subset I_W \\
&\Leftrightarrow \quad (F, G) \subset (x^2, xy, y^d, xq - y^{d-1}z) \\
&\Leftrightarrow \quad F \in \langle x^2, xy \rangle \text{ and } G \in \langle x^2, xy, y^d, xq - y^{d-1}z \rangle_d \\
&\Leftrightarrow \quad F = \alpha x^2 + \beta xy \text{ and } G = g_1 x^2 + g_2 xy + c_3 y^d + c_4(xq - y^{d-1}z)
\end{aligned}$$

for $c_3, c_4 \in k$ and $g_1, g_2 \in S_{d-2}$. We will examine X by looking at different cases.

Case 1: $\beta \neq 0$.

We can assume $\beta = 1$. Consider the following coordinate change

$$Y = \alpha x + y.$$

Thus we have

$$
\begin{aligned}
I_W &= (x^2, xy, y^d, xq - y^{d-1}z) \\
&= (x^2, x(Y-\alpha x), (Y-\alpha x)^d, xq - (Y-\alpha x)^{d-1}z) \\
&= \left(x^2, xY - \alpha x^2, \sum_{i=0}^{d} Y^i(-\alpha x)^{d-i}, xq - \left(\sum_{i=0}^{d-1} Y^i(-\alpha x)^{d-1-i}\right) z\right) \\
&= (x^2, xY, Y^d, xq - Y^{d-1}z), \\
I_X &= (F, G) \\
&= (\alpha x^2 + \beta xy, g_1x^2 + g_2xy + c_3y^d + c_4(xq - y^{d-1}z)) \\
&= (x(\alpha x + y), g_1x^2 + g_2xy + c_3y^d + c_4(xq - y^{d-1}z)) \\
&= (xY, g_1x^2 + g_2x(Y-\alpha x) + c_3(Y-\alpha x)^d + c_4(xq - (Y-\alpha X)^{d-1}z)) \\
&= (xY, g_1x^2 + g_2x(Y-\alpha x) + c_3\left(\sum_{i=0}^{d} Y^i(-\alpha x)^{d-i}\right) \\
&\quad + c_4\left(xq - \sum_{i=0}^{d-1} Y^i(-\alpha x)^{d-1-i}\right) z) \\
&= (xY, g_1x^2 + \alpha g_2x^2 + c_3Y^d - \alpha^d c_3x^d + c_4xq - c_4Y^{d-1}z - \alpha^{d-1}c_4x^{d-1}z) \\
&= (xY, (g_1 - \alpha g_2 + \alpha^d c_3x^{d-2} - \alpha^{d-1}c_4x^{d-3}z)x^2 + c_3Y^d + c_4xq - c_4Y^{d-1}z) \\
&= (xY, gx^2 + c_3Y^d + c_4xq - c_4Y^{d-1}z) \text{ where } g \in S_{d-2}.
\end{aligned}
$$

For the sake of simplicity, we will write y instead of Y for the rest of the proof. Thus we have

$$
\begin{aligned}
I_W &= (x^2, xy, y^d, xq - y^{d-1}z), \\
I_X &= (xy, gx^2 + c_3y^d + c_4(xq - y^{d-1}z)) \text{ where } g \in S_{d-2}.
\end{aligned}
$$

Combining the results from above we have

$$\begin{aligned}
\operatorname{Supp} X &= Z(F, G) \\
&= Z(xy, gx^2 + c_3 y^d + c_4(xq - y^{d-1}z)) \\
&= Z(x, gx^2 + c_3 y^d + c_4(xq - y^{d-1}z)) \\
&\quad \cup Z(y, gx^2 + c_3 y^d + c_4(xq - y^{d-1}z)) \\
&= Z(X, c_3 y^d - c_4 y^{d-1} z) \cup Z(y, gx^2 + c_4 xq) \\
&= Z(x, (c_3 y - c_4 z) y^{d-1}) \cup Z(y, (gx + c_4 q)x) \\
&= Z(x, c_3 y - c_4 z) \cup Z(x, y^{d-1}) \\
&\quad \cup Z(y, gx - c_4 q) \cup Z(y, x) \\
&= Z(x, c_3 y - c_4 z) \cup Z(x, y) \cup Z(y, gx + c_4 q).
\end{aligned}$$

Recall that X is a union of two d-fold lines. We already have a d-fold line on $Z(x, y)$. Thus we have three options

1. $Z(x, c_3 y - c_4 z) = Z(x, y)$.

2. $Z(y, gx + c_4 q) = Z(x, y)$.

3. $Z(x, c_3 y - c_4 z) = Z(y, gx + c_4 q)$.

Claim: $c_4 = 0$. We will only prove statement 3 and leave the statements 1 and 2 for the reader. The arguments for 1 and 2 are similar to an argument later in this proof. For statement 3 we have that $Z(x, c_3y - c_4z)$ lies on the x-plane and $Z(y, lx + c_4q)$ must lie on the y-plane. There is only one curve that lies on the x-plane and the y-plane. Thus we have

$$Z(x, c_3y - c_4z) = Z(x, y) = Z(y, gx + c_4q).$$

Thus we conclude $c_4 = 0$ for all three cases. Furthermore $c_3 \neq 0$ since $F = XY$ and $G = gX^2 + c_3Y^d$ can't have a common factor. We can assume $c_3 = 1$. Thus we have

$$I_X = (xy, gx^2 + y^d).$$

Finally we look at the support one more time

$$\begin{aligned}
Z(x, y) &= Z(I_X) \\
&= Z(xy, gx^2 + y^d) \\
&= Z(x, y^d) \cup Z(y, gx^2) \\
&= Z(x, y) \cup Z(y, g) \\
\Rightarrow Z(y, g) &= Z(x, y) \\
\Rightarrow \sqrt{(y, g)} &\subset \sqrt{(x, y)} \\
\Rightarrow (y, g) &\subset (x, y) \\
\Rightarrow g &\in (x, y) \\
\Rightarrow g &= g_1x + g_2y \text{ for some } g_i \in S_{d-3}.
\end{aligned}$$

We get

$$I_X = (xy, g_1x^3 + y^d).$$

We repeat the process on g_1. Since $(xy, gx^2 + y^d)$ is a regular sequence we have that $g \neq 0$ by Proposition 2.1.4. These two facts yield the following:

$$I_X = (xy, c_1x^d + y^d).$$

By a coordinate change we get

$$(xy, x^d + y^d).$$

Case 2: $\beta = 0$.

We can assume $\alpha = 1$ Thus we have

$$\begin{aligned}
I_X &= (F, G) \\
&= (x^2, g_1x^2 + g_2xy + c_3y^d + c_4(xq - y^{d-1}z)) \\
&= (x^2, g_2xy + c_3y^d + c_4(xq - y^{d-1}z)) \\
\operatorname{Supp} X &= Z(F, G) \\
&= Z(x^2, g_2xy + c_3y^d + c_4(xq - y^{d-1}z)) \\
&= Z(x, c_3y^d - c_4y^{d-1}z)) \\
&= Z(x, (c_3y - c_4z)y^{d-1}) \\
&= Z(x, y) \cup Z(x, c_3y - c_4z).
\end{aligned}$$

Notice that X must be two d-fold lines. Since W is a d-fold line on $Z(x, y)$ then $Z(x, c_3y - c_4z)$ must be the support of another d-fold line.

Case 2a: $c_4 = 0$.

So we have $I_X = (x^2, g_2xy + c_3y^d)$. Since $(x^2, g_2xy + c_3y^d)$ is a complete intersection then $c_3 \neq 0$. We may assume $c_3 = 1$. So we have

$$\begin{aligned}
\operatorname{Supp} X &= Z(x, y) \cup Z(x, c_3y - c_4z) \\
&= Z(x, y) \cup Z(x, y), \\
I_X &= (x^2, g_2xy + y^d).
\end{aligned}$$

Case 2b: $c_4 \neq 0$.

We may assume $c_4 = 1$. Thus we have

$$\begin{aligned}
I_X &= (x^2, g_2xy + c_3y^d + xq - y^{d-1}z), \\
I_W &= (x^2, xy, y^d, xq - y^{d-1}z).
\end{aligned}$$

Consider the following coordinate change

$$Z = c_3y - z.$$

We obtain the following:

$$
\begin{aligned}
I_W &= (x^2, xy, y^d, xq - y^{d-1}z) \\
&= (x^2, xy, y^d, xq - y^{d-1}(c_3y - Z)) \\
&= (x^2, xy, y^d, xq - c_3y^d + y^{d-1}Z), \\
I_X &= (x^2, g_2xy + c_3y^d + xq - y^{d-1}z) \\
&= (x^2, g_2xy + c_3y^d + xq - y^{d-1}(c_3y - Z)) \\
&= (x^2, g_2xy + c_3y^d + xq - c_3y^d + y^{d-1}Z) \\
&= (x^2, g_2xy + xq + y^{d-1}Z), \\
I_{W'} &= [I_X : I_W] \\
&= [(x^2, g_2xy + xq + y^{d-1}Z) : (x^2, xy, y^d, xq - c_3y^d + y^{d-1}Z)] \\
&= \{s \in S | s(x^2, xy, y^d, xq - c_3y^d + y^{d-1}Z) \subset (x^2, g_2xy + xq + y^{d-1}Z)\} \\
&= \{s \in S | s(xy, y^d, xq - c_3y^d + y^{d-1}Z) \subset (x^2, g_2xy + xq + y^{d-1}Z)\}.
\end{aligned}
$$

For the sake of simplicity, we will write z instead of Z for the rest of the proof. So we have

$$
\begin{aligned}
I_W &= (x^2, xy, y^d, xq - c_3y^d + y^{d-1}z) \\
I_X &= (x^2, g_2xy + xq + y^{d-1}z) \\
I_{W'} &= \{s \in S | s(xy, y^d, xq - c_3y^d + y^{d-1}z) \subset (x^2, g_2xy + xq + y^{d-1}z)\} \\
\operatorname{Supp} X &= Z(x, y) \cup Z(x, z).
\end{aligned}
$$

Recall that W' is a d-fold line. Thus we have

$$
\begin{aligned}
& Z(I_{W'}) = Z(x,z) \\
\Rightarrow\ & \sqrt{I_{W'}} = \sqrt{(x,z)} \\
\Rightarrow\ & \sqrt{I_{W'}} = (x,z) \\
\Rightarrow\ & z^n \in I_{W'} \text{ for some } n \\
\Rightarrow\ & z^n(xy, y^d, xq - c_3y^d + y^{d-1}z) \subset (x^2, g_2xy + xq + y^{d-1}z) \\
\Rightarrow\ & y^dz^n \in (x^2, g_2xy + xq + y^{d-1}z) \\
\Rightarrow\ & y^dz^n = Fx^2 + G(g_2xy + xq + y^{d-1}z) \text{ for some } F, G \in S.
\end{aligned}
$$

We can assume that G doesn't have any x^2 terms because if it did, we could just move them to Fx^2. So we can write $G = g + g'x$ where $g, g' \in k[y,z,w]$. So we have

$$
\begin{aligned}
& y^3z^n = Fx^2 + (g + g'x)(g_2xy + xq + y^{d-1}z) \\
\Rightarrow\ & y^3z^n = Fx^2 + g_2xyg + xqg + y^{d-1}zg + l_2x^2yg' + x^2qg' + xy^{d-1}zg',
\end{aligned}
$$

We can move l_2x^2yg', x^2qg' and any x term of g_2 to Fx^2 to get

$$
y^3z^n = Fx^2 + g_2xyg + xqg + y^{d-1}zg + xy^{d-1}zg'
$$

where $g_2 \in k[y,z,w]$. Notice that the only x that are not explicitly stated come from F as $g_2, g, g' \in k[y,z,w]$ and $q \in k[z,w]$.

Thus we can do the following:

Term	LHS	RHS
No x	$y^d z^n$	$y^{d-1}zg.$
One x	0	$l_2xyg + xqg + xy^{d-1}zg'.$
Two ore more $x's$	0	$Fx^2.$

The first row yields $g = yz^{n-1}$ which means the second row is

$$0 = l_2xy^{d-1}z^{n-1} + xyz^{n-1}q + xy^{d-1}zg'.$$

Recall that $q \in k[z, w]$, which means $xyz^{n-1}q$ only has one y while every other term has at least y^{d-1}. Thus $xyz^{n-1}q$ cannot cancel with any other terms. This is a contradiction because the left hand side of the equation is equal to 0. Therefore, for this case the complete intersection X does not link two d-fold lines.

□

We will see in the next section that the only complete intersections that link extremal d-fold lines of type $b = d - 1$ to another extremal d-fold line have the same form as complete intersections that link extremal d-fold lines of type $b \geq d$ to other d-fold lines. Thus we will able to put both cases together.

4.1.3 General Case

Now we will examine extremal d-fold lines of type $b \geq d$. From Corollary 4.1.3 we have that the form of the Rao module and ideal follow a similar pattern.

Lemma 4.1.7. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of degree d of type $b \geq d$, then*

1. $M^*(W) = M(W)(d-2)$.

2. $\operatorname{Supp} W = Z((\operatorname{Ann} M(W))_1)$.

Proof. Let W be an extremal d-fold line of type $b \geq d$. Up to a coordinate change, by Proposition 4.1.1 we have $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$, $I_Z = (x, y^{d-1})$ and $I_Y = (x, y)$ where Z its the underlying $d-1$ fold line and Y is its support.

Proof of 1: The fact that $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$ implies that W is contained in the reducible quadric with equation $xy = 0$. Then by (7, Corollary 2.4) we have that $M^*(W) = M(W)(d-2)$.

Proof of 2: We use the following short exact sequence

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y(b-d+1) \longrightarrow 0.$$

Apply H_*

$$0 \longrightarrow H^0_*(\mathcal{I}_W) \longrightarrow H^0_*(\mathcal{I}_Z) \longrightarrow H^0_*(\mathcal{O}_Y(b-d+1) \longrightarrow H^1_*(\mathcal{I}_W) \longrightarrow H^1_*(\mathcal{I}_Z) \longrightarrow \dots$$

which simplifies to

$$0 \longrightarrow I_W \longrightarrow I_Z \xrightarrow{\Phi} S/(I_Y)(b-d+1) \longrightarrow H^1_*(\mathcal{I}_W) \longrightarrow 0.$$

Thus $I_Y \subset (\operatorname{Ann} M(W))_1$. By Corollary 4.1.3 $(\operatorname{Ann} M(W))_1$ has two minimal (linear) generators. Thus we conclude $I_Y = (\operatorname{Ann} M(W))_1$. $\square$

The next result is useful for direct linkage but unnecessary for self-linkage.

Lemma 4.1.8. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of type $b \geq d$ that is directly linked to an extremal d-fold line W'. Then* $\operatorname{Supp} W = \operatorname{Supp} W'$.

Proof. We have

$$(Ann(S/(x,y,p,q)))_1 = (x,y)$$

By part 2 of Lemma 4.1.7 we have that the support of W is (x,y). By Theorem 2.2.7 we have that if W is linked to W', then

$$M(W') = M^*(W)(4-t).$$

But then by part 1 of Lemma 4.1.7 we have that

$$M^*(W) = M(W)(d-2).$$

Combining those two statements we get

$$M(W') = M^*(W)(4-t) = M(W)(d-2)(4-t) = M(W)(d+2-t).$$

Applying part 2 of Lemma 4.1.7 we get that $\operatorname{Supp} W = \operatorname{Supp} W'$. □

We now analyze the equations of the complete intersection necessary to link two extremal d-fold lines of type $b \geq d$.

Proposition 4.1.9. *Let $W \subset \mathbb{P}^3$ be a d-fold line of type $b \geq d$ and let X be a complete intersection that links W to another d-fold line W'. Then up to coordinate change, we have one of the following:*

1. $I_X = (xy, x^d + y^d)$.

2. $I_X = (x^2, gxy + y^d)$ *where* $g \in k[y, z, w]_{d-2}$.

Proof. By Lemma 4.0.9 we have $I_X = (F, G)$ where $\deg F = 2$ and $\deg G = d$. Now consider the following:

$$\begin{aligned} W \subset X &\Rightarrow I_X \subset I_W \\ &\Rightarrow (F, G) \subset (x^2, xy, y^d, xq - y^{d-1}p) \\ &\Rightarrow F \in \langle x^2, xy \rangle_1 \text{ and } G \in \langle x^2, xy, y^d \rangle_d \\ &\Rightarrow F = \alpha x^2 + \beta xy \text{ and } G = fx^2 + gxy + \gamma y^d, \end{aligned}$$

where $\alpha, \beta, \gamma \in k$ and $f, g \in k[y, z, w]_{d-2}$. Notice that G doesn't need the last generator because its degree is greater than d by construction. Also notice that $\gamma \neq 0$, since otherwise, we would have $G = x(fx + gy)$ and $F = x(\alpha x + \beta y)$. F and G can't share a factor by Proposition 2.1.4 since because (F, G) must be a regular sequence. Thus we can assume that $\gamma = 1$ by a coordinate change. Finally we have

$$I_X = (\alpha x^2 + \beta xy, fx^2 + gxy + y^d).$$

We can assume $g \in k[y, z, w]$, since if g had any x terms, then gxy would have x^2 terms which we could move to fx^2.

Case 1: $\beta = 0$.

We can assume $\alpha = 1$ and this yields

$$I_X = (x^2, fx^2 + gxy + y^d) = (x^2, gxy + y^d).$$

Case 2: $\beta \neq 0$.

We can assume $\beta = 1$. Consider the following coordinate change

$$Y = \alpha x + y.$$

Thus we have

$$\begin{aligned}
I_W &= (x^2, xy, y^d, xq - y^{d-1}p) \\
&= (x^2, x(Y - \alpha x), (Y - \alpha x)^d, xq - (Y - \alpha x)^{d-1}p) \\
&= \left(x^2, xY - \alpha x^2, \sum_{i=0}^{d} Y^i(-\alpha x)^{d-i}, xq - \left(\sum_{i=0}^{d-1} Y^i(-\alpha x)^{d-1-i}\right) p\right) \\
&= (x^2, xY, Y^d, xq - Y^{d-1}p), \\
I_X &= (\alpha x^2 + xy, fx^2 + gxy + y^d) \\
&= (x(\alpha x + y), fx^2 + gxy + y^d) \\
&= (xY, fx^2 + gx(Y - \alpha x) + (Y - \alpha x)^d) \\
&= \left(xY, fx^2 + gxY - \alpha gx^2 + \sum_{i=0}^{d} Y^i(-\alpha x)^{d-i}\right) \\
&= (xY, fx^2 - \alpha gx^2 + Y^d + (-\alpha x)^d) \\
&= (xY, (f - \alpha g + (-\alpha x)^{d-2})x^2 + Y^d) \\
&= (xY, f'x^2 + Y^d) \text{ with } f' = f - \alpha g + (-\alpha x)^{d-2}.
\end{aligned}$$

For the sake of simplicity, we will replace Y with y and f' with f for the rest of the proof.

Thus we have

$$\begin{aligned}
I_W &= (x^2, xy, y^d, xq - y^{d-1}p), \\
I_X &= (xy, fx^2 + y^d).
\end{aligned}$$

From Lemma 4.1.8, we have that the support of X is $Z(x, y)$. Thus we get the following

$$\begin{aligned}
Z(x,y) &= Z(I_X) \\
&= Z(xy, fx^2 + y^d) \\
&= Z(x, y^d) \cup Z(y, fx^2) \\
&= Z(x,y) \cup Z(y,f) \\
\Rightarrow Z(y,f) &= Z(x,y) \\
\Rightarrow \sqrt{(y,f)} &\subset \sqrt{(x,y)} \\
\Rightarrow (y,f) &\subset (x,y) \\
\Rightarrow f &\in (x,y).
\end{aligned}$$

Since $\deg f = d-2$ we have

$$f = f_1 x + g_1 y$$

where $f_1, g_1 \in S_{d-3}$. Thus we have

$$I_X = (xy, (f_1 x + g_1 y)x^2 + y^d) = (xy, f_1 x^3 + g_1 x^2 y + y^d) = (xy, f_1 x^3 + y^d).$$

Again we have that the support of X is $Z(x, y)$ and thus

$$\begin{aligned}
Z(x,y) &= Z(I_X) \\
&= Z(xy, f_1x^3 + y^d) \\
&= Z(x, y^d) \cup Z(Y, f_1X^3) \\
&= Z(x,y) \cup Z(y, f_1) \\
\Rightarrow Z(y, f_1) &= Z(x,y) \\
\Rightarrow \sqrt{(y, f_1)} &\subset \sqrt{(x,y)} \\
\Rightarrow (y, f_1) &\subset (x,y) \\
\Rightarrow f_1 &\in (x,y).
\end{aligned}$$

We can run the same argument again and keep running it until f_i is constant. However $f_1 \neq 0$ so that I_X defines a curve. With a coordinate change we can assume

$$I_X = (xy, x^d + y^d).$$

□

Now we can combine the general case with the special case and analyze extremal d-fold lines of type $b \geq d - 1$.

Proposition 4.1.10. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of type $b \geq d-1$ and ideal $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$. Let X be a complete intersection that involves a quadric that links W to another d-fold line W'. Then up to coordinate change, we have one of the following:*

1. *$I_X = (x^2, gxy + y^d)$ where $g \in k[y, z, w]_{d-2}$.*

2. *$I_X = (xy, x^d + y^d)$.*

Proof. We combine the following cases:

1. $b \geq d$ is a special case of Proposition 4.1.9.

2. $b = d - 1$ is taken care by Proposition 4.1.6.

□

Proposition 4.1.10 allows us to analyze both cases for extremal d-fold lines at the same time. We will now prove a of couple algebraic facts.

Observations 4.1.11. *We have the following algebra facts:*

1. *$(x^2, gxy + y^d)$ contains xy^d and y^{2d-1}.*

2. *$(xy, x^d + y^d)$ contains y^{d+1} and x^{d+1}.*

Proof. Observe

$$\begin{aligned} xy^d &= x(gxy + y^d - gxy) = x(gxy + y^d) - gx^2y. \\ y^{2d-1} &= y^{d-1}(gxy + y^d - gxy) = y^{d-1}(gxy + y^d) - gxy^d. \\ y^{d+1} &= y(x^d + y^d - x^d) = y(x^d + y^d) - x^dy. \\ x^{d+1} &= x(x^d + y^d - y^d) = x(x^d + y^d) - xy^d. \end{aligned}$$

□

Proposition 4.1.12. *Let $p, q, p', q' \in k[z,w]$ and $g \in k[y,z,w]$ be homogeneous polynomials with $\deg p = \deg p' = b - d + 2$, $\deg q = \deg q' = b$ and $\deg g = d - 2$. Furthermore assume (p,q) and (p',q') have no common zeroes on Y. Then the following are equivalent.*

1. $(xq' - y^{d-1}p')(xq - y^{d-1}p) \in (x^2, gxy + y^d)$.

2. *There exists $\alpha \in k^*$ and $g' \in k[w,z]$ with $g' = g \bmod I_Y$ such that*

   (a) $p' = \alpha p$

   (b) $q' = -\alpha(g'p + q)$.

Proof. Notice that the first statement is equivalent to

$$xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' \in (x^2, gxy + y^d).$$

($\Rightarrow$) Assume $(xq' - y^{d-1}p')(xq - y^{d-1}p) \in (x^2, gxy + y^d)$. Thus we have

$$xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' = Fx^2 + G(gxy + y^d)$$

where $F \in S_{2b}$ and $G \in S_{2b-d+2}$. We can assume that G and g have no x^2 terms since if it did, we can just relabel and make them part of F. Thus we have $G = G' + G''x$ where G' and G'' are elements of $k[y, z, w]$. We will also rewrite $g = g' + g''y$ where $g' \in k[z, w]$ and $g'' \in k[y, z, w]$. Thus we have the following equation

$$\begin{aligned}
xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' &= Fx^2 + (G' + G''x)((g' + g''y)xy + y^d) \\
\Rightarrow xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' &= Fx^2 + (G' + G''x)(g'xy + g''xy^2 + y^d) \\
\Rightarrow xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' &= Fx^2 + G'g'xy + G'g''xy^2 + G'y^d \\
&\quad + G''g'x^2y + G''g''x^2y^2 + G''xy^d.
\end{aligned}$$

Now we can move $G''g'x^2y$ and $G''g''x^2y^2$ to Fx^2 to have

$$xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' = Fx^2 + G'g'xy + G'g''xy^2 + G'y^d + G''xy^d.$$

Recall that we can assume p, p', q, q' and g' are elements of $k[z, w]$ and g'', G' and g'' are elements of $k[y, z, w]$. Therefore the only term that can have x's that are not explicitly stated is F. By comparing the x terms of the previous equation, we get the following:

Term	LHS	RHS
No x	$-y^{2d-2}pp'$	$G'y^d$.
One x	$xy^{d-1}pq' + xy^{d-1}p'q$	$G'g'xy + G'g''xy^2 + G''xy^d$.
Two or more $x's$	0	Fx^2.

Thus we can see that the left hand side of the equation has no x^2 terms, and thus neither does the right hand side. So we get that $F = 0$. We also get that $G' = -y^{d-2}pp'$ which makes sense since $d \geq 5$. We get that following

$$\begin{aligned} xy^{d-1}pq' + xy^{d-1}p'q &= G'g'xy + G'g''xy^2 + G''xy^d \\ \Rightarrow xy^{d-1}pq' + xy^{d-1}p'q &= -xy^{d-1}pp'g' - xy^d pp'g'' + G''xy^d. \end{aligned}$$

Notice that the only term that can have y terms that are not explicitly stated is G'' and g'' as p, p', q, q' are elements of $k[z, w]$. Now we can compare y terms.

Term	LHS	RHS
Exactly $(d-1)\, y's$	$xy^{d-1}pq' + xy^{d-1}p'q$	$-xy^{d-1}pp'g'$.
More than $d\, y's$	0	$-xy^d pp'g'' + G''xy^d$.

From the first row we get

$$
\begin{aligned}
qp' + pq' &= -pp'g' \\
&\Rightarrow p'|qp' + pq' \\
&\Rightarrow p'|pq' \\
&\Rightarrow p'|p \text{ since } p' \text{ and } q' \text{ do not have common factors} \\
&\Rightarrow p' = \alpha p \text{ for some } \alpha \in k^* \\
&\Rightarrow \alpha qp + pq' = -\alpha g' p^2 \\
&\Rightarrow \alpha q + q' = -\alpha g' p \\
&\Rightarrow q' = -\alpha g' p - \alpha q \\
&\Rightarrow q' = -\alpha (g' p + q).
\end{aligned}
$$

(⇐) Assume $p' = \alpha p$ for some $\alpha \in k^*$ and $q' = -\alpha(g'p + q)$.

Notice

$$
\begin{aligned}
& xy^{d-1}pq' + xy^{d-1}p'q - y^{2d-2}pp' \\
= \ & -\alpha xy^{d-1}p(g'p+q) + \alpha xy^{d-1}pq - \alpha y^{2d-2}p^2 \\
= \ & -\alpha g'xy^{d-1}p^2 - \alpha xy^{d-1}pq + \alpha xy^{d-1}pq - \alpha y^{2d-2}p^2 \\
= \ & -\alpha(g' + g''y - g''y)xy^{d-1}p^2 - \alpha xy^{d-1}pq + \alpha xy^{d-1}pq - \alpha y^{2d-2}p^2 \\
= \ & -\alpha gxy^{d-1}p^2 + \alpha g''xy^dp^2 - \alpha xy^{d-1}pq + \alpha xy^{d-1}pq - \alpha y^{2d-2}p^2 \\
= \ & -\alpha gxy^{d-1}p^2 - \alpha y^{2d-2}p^2 + \alpha g''x(y^d + gxy - gxy)p^2 \\
= \ & -\alpha y^{d-2}p^2(gxy + y^d) + \alpha g''x(y^d + gxy)p^2 - \alpha gg''x^2yp^2 \\
= \ & (-\alpha y^{d-2}p^2 + \alpha g''x)(gxy + y^d) - \alpha gg''x^2yp^2 \\
\in \ & (x^2, gxy + y^d).
\end{aligned}
$$

□

Proposition 4.1.13. *Let $p, q, p', q' \in k[z, w]$ be homogeneous polynomials with* $\deg p = \deg p' = b - d + 2$ *and* $\deg q = \deg q' = b$*. Furthermore suppose (p, q) and (p', q') have no zeroes on Y. Then the following are equivalent:*

1. $(xq - y^{d-1}p)(yq' - x^{d-1}p') \in (xy, x^d + y^d)$.

2. *There exists $\alpha \in k^*$ such that $p' = \alpha p$ and $q' = \alpha q$.*

Proof. First notice that (1) is equivalent to $x^dqp' + y^dpq' \in (xy, x^d + y^d)$.

($\Rightarrow$) Assume $x^d qp' + y^d pq' \in (xy, x^d + y^d)$. Then we have

$$x^d qp' + y^d pq' = Fxy + G(x^d + y^d)$$

where $\deg x^d qp' = 2b + 2$, and thus $\deg F = 2b$ and $\deg G = 2b - 2$ Recall that we can assume $p, p', q, q' \in k[z, w]$. Notice that we can assume that G does not have any terms that are a multiple of xy, since if it did, then we would just be able to move them to F. Furthermore we can assume that G doesn't have any terms that are multiples of x or y since if it did, then $G(x^d + y^d)$ would have terms that are multiples of x^{d+1} without a y factor or multiples of y^{d+1} without an x factor. However, this cannot be the case because both the left hand side and Fxy do not have any terms of this form. Therefore we may assume $G \in k[z, w]$. Thus we have that $F = 0$ since the rest of the equation doesn't have any terms that are multiples of xy. This yields the following equation:

$$x^d qp' + y^d pq' = G(x^d + y^d).$$

Since $p, q, p', q' \in k[z, w]$ we get that $qp' = G = pq'$. Now we have $p' | pq'$. However, q' and p' do not have any factors in common, so we get that $p' | p$. But they are of the same degree, thus we conclude $p' = \alpha p$ where $\alpha \in k^*$. Similarly we get $q' = \beta q$ where $\beta \in k^*$.

Putting these equations together we get

$$\begin{aligned} qp' &= pq' \\ \Rightarrow \beta qp &= \alpha pq \\ \Rightarrow \beta &= \alpha. \end{aligned}$$

($\Leftarrow$) Assume $p' = \alpha p$ and $q' = \alpha q$. Then we have

$$x^d qp' + y^d pq' = x^d \alpha qp + y^d \alpha pq = \alpha qp(x^d + y^d).$$

$\square$

Theorem 4.1.14. *Let $W \subset \mathbb{P}^3$ be an extremal d-fold line of type $b \geq d$ that is linked to another extremal d-fold line W'. If the ideal of W is $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$, then we have one of the following*

1. $I_{W'} = (x^2, xy, y^d, xq' - y^{d-1}p')$ *with* $p' = \alpha p$ *and* $q' = -\alpha(gp + q)$ *for some* $\alpha \in k^*$.
2. $I_{W'} = (y^2, xy, x^d, yq - x^{d-1}p)$.

Proof. As established in Proposition 4.1.9 we have two cases: $I_X = (x^2, gxy + y^d)$ where $g \in k[y, z, w]_{d-2}$ or $I_X = (xy, x^d + y^d)$.

Case 1: $I_X = (xy, x^d + y^d)$.

$$\begin{aligned} I_{W'} &= [I_X : I_W] \\ &= [(xy, x^d + y^d) : (x^2, xy, y^d, xq - y^{d-1}p)] \\ &= \{s \in S | s(x^2, xy, y^d, xq - y^{d-1}p) \subset (xy, x^d + y^d)\} \\ &= \{s \in S | s(x^2, y^d, xq - y^{d-1}p) \subset (xy, x^d + y^d)\}. \end{aligned}$$

Claim: $I_{W'} = (y^2, xy, x^d, yq' - x^{d-1}p')$.

Recall that by Observation 4.1.11 we have $(xy, x^d + y^d)$ contains y^{d+1} and x^{d+1}. To check the containment we are going to check each of the generators. Clearly $xy \in I_{W'}$. Now notice that the following products are elements of $(xy, x^d + y^d)$:

$$\begin{aligned} & y^2 \\ (y^2)(x^2) &= x^2y^2 \\ (y^2)(y^d) &= y^{d+2} \\ (y^2)(xq - y^{d-1}p) &= xy^2q - y^{d+1}p \\ & x^d \\ (x^d)(x^2) &= x^{d+2} \\ (x^d)(y^d) &= x^dy^d \\ (x^d)(xq - y^{d-1}p) &= x^{d+1}q - x^dy^{d-1}p \end{aligned}$$

$$\begin{aligned}
yq' &- x^{d-1}p' \\
(yq' - x^{d-1}p')(x^2) &= x^2yq' - x^{d+1}p' \\
(yq' - x^{d-1}p')(y^d) &= y^{d+1}q' - x^{d-1}y^dp' \\
(yq' - x^{d-1}p')(xq - y^{d-1}p) &= xyqq' - x^dp'q - y^dpq' - y^{d-1}x^{d-1}p'p
\end{aligned}$$

Using Observation 4.1.11 we have that $(y^2, xy, x^d, yq' - x^{d-1}p') \subset I_{W'}$ if and only if $(yq' - x^{d-1}p')(xq - y^{d-1}p) \in (xy, x^d + y^d)$. But Proposition 4.1.13 tells us this is exactly when $p' = \alpha p$ and $q' = \alpha q$. However, by Theorem 4.1.1 we can assume $p' = p$ and $q' = q$.

Case 2: $I_X = (x^2, gxy + y^d)$ where $g \in k[y, z, w]_{d-2}$.

$$\begin{aligned}
I_{W'} &= [I_X; I_W] \\
&= [(x^2, gxy + y^d); (x^2, xy, y^d, xq - y^{d-1}p)] \\
&= s \in S | s(x^2, xy, y^d, xq - y^{d-1}p) \subset (x^2, gxy + y^d) \\
&= s \in S | s(xy, y^d, xq - y^{d-1}p) \subset (x^2, gxy + y^d).
\end{aligned}$$

Claim: $I_{W'} = (x^2, xy, y^d, xq' - y^{d-1}p')$.

Recall that by Observation 4.1.11 we have $(x^2, gxy + y^d)$ contains xy^d and y^{2d-1}. To check the containment we are going to check each of the generators. Clearly x^2 is in $I_{W'}$.

Now notice that the following products are elements of $(x^2, gxy + y^d)$:

$$
\begin{aligned}
& xy \\
(xy)(xy) &= x^2y^2 \\
(xy)(y^d) &= xy^{d+1} \\
(xy)(xq - y^{d-1}p) &= x^2yq - xy^dp \\
& y^d \\
(y^d)(xy) &= xy^{(}d+1) \\
(y^d)(y^d) &= y^{2d} \\
(y^d)(xq - y^{d-1}p) &= xy^dq - y^{2d-1}p \\
xq' &- y^{d-1}p' \\
(xq' - y^{d-1}p')(xy) &= x^2yq' - xy^dp' \\
(xq' - y^{d-1}p')(y^d) &= xy^dq' - y^{2d-1}p' \\
(xq' - y^{d-1}p')(xq - y^{d-1}p) &= x^2qq' - xy^{d-1}pq' - xy^{d-1}p'q + y^{2d-2}pp'
\end{aligned}
$$

Using Observation 4.1.11 we have that $(x^2, xy, y^d, xq' - y^{d-1}p') \subset I_{W'}$ if and only if $(xq' - y^{d-1}p')(xq - y^{d-1}p) \in (x^2, gxy + y^d)$. But Proposition 4.1.12 states that this is the case exactly when $p' = \alpha p$ for some $\alpha \in k^*$ and $q' = -\alpha(gp + q)$. Finally we have that X links a d-fold line W to another d-fold line and by Lemma 2.0.3 we have equality and it completes this case and thus the theorem. $\square$

Observations 4.1.15. *The first observation we need to make is that self-linkage is impossible in case 1 of Theorem 4.1.14. The construction in Theorem 4.1.1 implies that* $W'_{d-1} \neq W_{d-1}$ *(the underlying planar curves of degree* $d-1$ *don't agree). Thus, to answer the question of self linkage, we exclusively focus on case 2.*

Next we have the proof of Theorem 1.

Corollary 4.1.16. *(Theorem 1) Let* $W \subset \mathbb{P}^3$ *be an extremal prime-fold line in* $\mathbb{P}^3$ *is self-linked if and only if the in characteristic of the ground field is two.*

Proof. Up to a coordinate change, by Theorem 4.1.1 we have

$$I_W = (x^2, xy, y^d, xq - y^{d-1}p).$$

($\Rightarrow$) Assume extremal d-fold line W is self-linked.

By case 2 of Theorem 4.1.14 we have

$$I_W = (x^2, xy, y^d, xq' - y^{d-1}p') = (x^2, xy, y^d, xq - y^{d-1}p)$$

where $p' = \alpha p$ and $q' = -\alpha(gp + q)$ for some $\alpha \in k^*$. However, we also have $q' = \alpha q$ by Theorem 4.1.1

$$\begin{aligned} & \alpha q = -\alpha(gp + q) \\ \Rightarrow\quad & q + \alpha q = -gp \\ \Rightarrow\quad & q(1 + \alpha) = -gp \\ \Rightarrow\quad & q = -(1 + \alpha)^{-1} gp. \end{aligned}$$

Which is a contradiction because p and q don't have common factors. However, $(1+\alpha)^{-1}$ only exists if $\alpha \neq -1$. Thus we have $\alpha = -1$. Then we have $q = gp + q$ or equivalently $gp = 0$. But $p \neq 0$ thus we have $g = 0$. Now notice that we only have self linkage if $p = -p$. This is only the case if $char\, k = 2$.

($\Leftarrow$) Assume that $char\, k = 2$.

Consider complete intersection ideal (x^2, y^d). By case 2 of Theorem 4.1.14 we have

$$I_{W'} = (x^2, xy, y^d, xq' - y^{d-1}p')$$

where $p' = \alpha p$ and $q' = -\alpha q$ for some $\alpha \in k^*$. But since $char\, k = 2$ we have $q' = \alpha q$. By Theorem 4.1.1 we have that $W = W'$.

$\square$

Remark 4.1.17. When $d = 2$, Corollary 4.1.16 recovers Migliore's theorem on self-linkage of double lines (18, Theorem 4.4).

4.2 Non-Extremal Prime-Fold Lines

Now that we have analyzed extremal d-fold lines W, we will analyze other kinds of d-fold lines with $d \geq 5$. By Theorem 2.3.5, W is extremal if it lies on two quadrics. Therefore, we will focus on multiple-fold lines that are only contained in one quadric.

4.2.1 Multiple Lines on the Smooth Quadric

First we will analyze d-fold lines on the smooth quadric Q.

Proposition 4.2.1. *Let $W \subset \mathbb{P}^3$ be a d-fold line only contained in the smooth quadric. Then W is not self-linked. Furthermore, W can only be directly linked to another d-fold line in the opposite ruling.*

Proof. Let $Q = Z(F)$ the smooth quadric. We have from (10, Example II.6.6.1) that the divisor class group of Q is $\operatorname{Cl} Q \cong \mathbb{Z} \oplus \mathbb{Z}$. From (10, Example II.6.6.2), we have a homomorphism form $\operatorname{Cl} \mathbb{P}^3 \to \operatorname{Cl} Q$ whose image is generated by the element $(1,1)$. Let X be a complete intersection with ideal $I_X = (F, G)$ for some polynomial G and thus can be consider a divisor of Q. Therefore X is represented by a multiple of $(1,1)$ and therefore it has two components, each coming from the different family of lines in Q. So we have that the support of X has to be more than one line, and therefore W is not self-linked. $\square$

We will see in the last chapter of this thesis that in the case of triple lines, we have an algebraic description of the ideals of the triple lines that are directly linked.

4.2.2 Multiple Lines on the Quadric Cone

We will analyze multiple lines on the quadric cone Q.

Proposition 4.2.2. *Let W and W' be d-fold lines in $\mathbb{P}^3$ contained in the same quadric cone. Then W is directly linked to W'. In particular, W is self-linked.*

Proof. Let $Q = Z(F)$ be the quadric cone and let W be a d-fold line on Q. From (10, Exercise V.2.9), curves with even degree that lie on the quadric cone are complete intersections. However, the degree of any two d-fold lines is $2d$, and thus the union is a complete intersection. Finally we conclude that any two d-fold lines that lie on the same quadric cone are directly linked. Furthermore, if $W = W'$ then W is self-linked. □

4.2.3 Multiple Lines on the Reducible Quadric

The third quadric surface Q that we will analyze is the reducible quadric. We will first investigate the form of the complete intersection needed to self-link a multiple line W which is contained by the unique quadric $Z(xy)$.

Lemma 4.2.3. *Let (F, G) be a complete intersection ideal such that $F = xy$. Furthermore assume that (F, G) self-links a d-fold line in $\mathbb{P}^3$. Then G can be assumed to be $x^d + y^d$.*

Proof. Since (F, G) is a complete intersection that links W to itself we have the following

$$
\begin{aligned}
Z(F, G) &= Z(xy, G) \\
&= Z(x, G) \cup Z(y, G) \\
&= Z(x, y).
\end{aligned}
$$

Thus we have

$$\begin{aligned}
Z(x,G) &= Z(x,y) \\
\Rightarrow \sqrt{(x,G)} &\subset \sqrt{(x,y)} \\
\Rightarrow (x,G) &\subset (x,y) \\
\Rightarrow G &= G_1x + G_2y,
\end{aligned}$$

where $G_i \in S_{d-1}$ for $i = 1, 2$. So we have

$$\begin{aligned}
Z(F,G) &= Z(xy, G_1x + G_2y) \\
&= Z(x, G_2y) \cup Z(y, G_1) \\
&= Z(x,y),
\end{aligned}$$

which yields

$$\begin{array}{rcl}
Z(x,G_1) = Z(x,y) & \text{and} & Z(y,G_2) = Z(x,y) \\
\Rightarrow \sqrt{(x,G_1)} \subset \sqrt{(x,y)} & \text{and} & \sqrt{(y,G_2)} \subset \sqrt{(x,y)} \\
\Rightarrow (x,G_1) \subset (x,y) & \text{and} & (y,G_2) \subset (x,y) \\
\Rightarrow G_1 = G_3x + G_4y & \text{and} & G_2 = G_5x + G_6y,
\end{array}$$

where $G_i \in S_{d-2}$ for $i = 3, 4, 5, 6$.

Again we have

$$\begin{aligned}(F,G) &= (xy,(G_3x+G_4y)x+(G_5x+G_6y)y)\\ &= (xy,G_3x^2+G_4xy+G_5xy+G_6y^2)\\ &= (xy,G_3x^2+G_6y^2).\end{aligned}$$

We repeat the process and combine that result with the fact that the degree of G has to be d. After a coordinate change if necessary we get

$$(F,G) = (xy, x^d + y^d).$$

Therefore we can assume that the complete intersection has the form (xy, x^d+y^d). $\square$

We will disregard the case where W is ACM and focus on the non-ACM case. We need the non-ACM condition for the next lemma.

Lemma 4.2.4. *Let $W \subset \mathbb{P}^3$ be a non-ACM d-fold line with $\sqrt{I_W} = (x,y)$ and contained in the reducible quadric with equation $xy = 0$. Then up to a coordinate change,*

$$I_W = (xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f)$$

where $f, g \in k[z,w]$ and $a + b + 2 = d$.

Proof. From (7, Lemma 2.1) we get the homogenous ideal

$$I_W = (xy, x^2A + xhf, y^2B + yhg, xAg + yBf + hfg)$$

where $h, g, f \in k[z,w], f, g \notin K, A \in k[x,z,w], B \in k[y,z,w]$ and $AB \neq 0$ in case of $h = 0$. From (7, Theorem 1.2) we get that (f,g) has to be a regular sequence. From the fact that I_W is a homogenous ideal we get the following formulas for the degrees

$$\begin{aligned}
\deg A &= \deg h + \deg f - 1 \\
\deg B &= \deg h + \deg g - 1.
\end{aligned}$$

Furthermore, (7, Lemma 2.2) we have

$$\begin{aligned}
d &= 2\deg h + \deg f + \deg g \\
&= (\deg h + \deg f - 1) + (\deg h + \deg g - 1) + 2 \\
&= \deg A + \deg B + 2.
\end{aligned}$$

From (7, Remark 2.3) we get that $h = 0$ since the support of W is $Z(x,y)$. This yields

$$I_W = (xy, x^2A, y^2B, xAg + yBf).$$

Recall that the support of W is $Z(x, y)$. We have the following

$$\begin{aligned}
Z(x,y) &= Z(xy, x^2A, y^2B, xAg + yBf) \\
&= Z(x, y^2B, yBf) \cup (y, x^2A, xAg) \\
&= Z(x, y^2, yBf) \cup Z(x, B) \cup (y, x^2, xAg) \cup (y, x^2A, xAg) \cup (y, A).
\end{aligned}$$

Which implies the following

$$Z(x, B) = Z(x, y) \quad \text{and} \quad Z(y, A) = Z(x, y).$$

We will only analyze the case for B as the case for A follows similarly.

$$\begin{aligned}
Z(x,y) &= Z(x, B) \\
&\Rightarrow \sqrt{(x, B)} \subset \sqrt{(x, y)} \\
&\Rightarrow (x, B) \subset (x, y) \\
&\Rightarrow B = B_1x + B_2y,
\end{aligned}$$

where $B_i \in S_{b-1}$ for $i = 1, 2$. So we have

$$(x, B) = (x, B_1x + B_2y) = (x, B_2y).$$

This equation yields:

$$
\begin{aligned}
Z(x,y) &= Z(x, B_2 y) \\
&= Z(x, B_2) \cup Z(x,y) \\
&\Rightarrow \sqrt{(x, B_2)} \subset \sqrt{(x,y)} \\
&\Rightarrow (x, B_2) \subset (x,y) \\
&\Rightarrow B_2 = B_3 x + B_4 y,
\end{aligned}
$$

where $B_i \in S_{b-2}$ for $i = 2, 3$. We can repeat the process and combine that with the fact that degree of B has to be b. After a coordinate change we have $B = y^b$. A similar argument shows that $A = x^a$. Putting everything together we have

$$I_W = (xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f)$$

where $a = \deg A$ and $b = \deg B$ and $a + b + 2 = d$.

□

Consider d-fold line W in $\mathbb{P}^3$ with ideal that lies on the reducible quadric with ideal $I_W = (xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f)$. Note that W is almost never quasi-primitive except for when $a = 0$ or $b = 0$ as otherwise $I_W \subset (x,y)^2$. Interestingly, these cases are exactly the cases where W is contained in two quadrics. Therefore they are exactly the cases where W is extremal. Now we can use Lemma 4.2.4 to prove the following proposition.

Proposition 4.2.5. *If $W \subset \mathbb{P}^3$ is a d-fold line contained in a unique quadric which is reducible, then W is not self-linked.*

Proof. Assume that W is a d-fold line contained by the unique quadric $Z(xy)$ that is self-linked by complete intersection X. By Lemmas 4.2.3 and 4.2.4 we have that

$$\begin{aligned} I_W &= (xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f), \\ I_X &= (xy, x^d + y^d). \end{aligned}$$

where $f, g \in k[z, w]$. Now we have

$$\begin{aligned} I_W &= [I_X : I_W] \\ &= [(xy, x^d + y^d) : (xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f)] \\ &= \{s \in S | s(xy, x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f) \subset (xy, x^d + y^d)\} \\ &= \{s \in S | s(x^{a+2}, y^{b+2}, x^{a+1}g + y^{b+1}f) \subset (xy, x^d + y^d)\}. \end{aligned}$$

In particular we have

$$\begin{aligned} (x^{a+1}g + y^{b+1}f)(x^{a+1}g + y^{b+1}f) &\in (xy, x^d + y^d) \\ \Rightarrow x^{2a+2}g^2 + 2x^{a+1}y^{b+1}fg + y^{2b+2}f^2 &\in (xy, x^d + y^d) \\ \Rightarrow x^{2a+2}g^2 + y^{2b+2}f^2 &\in (xy, x^d + y^d) \\ \Rightarrow x^{2a+2}g^2 + y^{2b+2}f^2 &= Pxy + Q(x^d + y^d) \end{aligned}$$

for some P and Q in $k[x, y, z, w]$.

We can move any xy terms from the left hand side to Pxy. Similarly, we can assume that Q doesn't have any xy terms. Thus we can write

$$x^{2a+2}g^2 + y^{2b+2}f^2 = Pxy + (Q' + Q''x + Q'''y)(x^d + y^d)$$

where $Q' \in k[z, w], Q'' \in k[x, z, w]$ and $Q''' \in k[y, z, w]$. When we expand the expression we get

$$x^{2a+2}g^2 + y^{2b+2}f^2 = Pxy + Q'x^d + Q'y^d + Q''x^{d+1} + Q''xy^d + Q'''x^dy + Q'''y^{d+1}.$$

Again, any xy terms we can move to Pxy.

$$x^{2a+2}g^2 + y^{2b+2}f^2 = Pxy + Q'x^d + Q'y^d + Q''x^{d+1} + Q'''y^{d+1}.$$

Notice that the left hand side does not have any xy terms, and thus we can conclude that $P = 0$. Furthermore we can separate the equations as follows:

$$\begin{aligned} LHS &= RHS \\ \text{no } y \text{ terms } x^{2a+2}g^2 &= Q'x^d + Q''x^{d+1}. \\ \text{no } x \text{ terms } y^{2b+2}f^2 &= Q'y^d + Q'''y^{d+1}. \end{aligned}$$

Since $f, g \in k[z, w]$, then one of the terms on the right hand side has to vanish on both equations which means that

$$2a + 2 \geq d \text{ and } 2b + 2 \geq d \Rightarrow a + b + 2 \geq d.$$

But from the construction we have $a + b + 2 = d$, and thus $2a + 2 = d = 2b + 2$. Putting this back together we get

$$x^d g^2 + y^d f^2 = Q(x^d + y^d).$$

Recall that $f, g \in k[z, w]$ so we can separate as follows

$$\begin{aligned} LHS &= RHS \\ \text{No } y \text{ terms: } x^d g^2 &= Qx^d. \\ \text{No } x \text{ terms: } y^d f^2 &= Qx^d. \end{aligned}$$

Both imply that $g^2 = Q = f^2$ which is impossible since (f, g) is a regular sequence and Proposition 2.1.4. Therefore W is not self-linked. □

4.2.4 Multiple Lines on the Double Plane

The last quadric surface $Q = 2H$ that we will analyze is the double plane. Self-linkage of multiple lines using the double plane is still an open problem. We will present partial work towards a conjecture. Again we will look at the complete intersections that can self-link a d-fold line W to itself and then we will look at the ideal of W.

Lemma 4.2.6. *Let (F, G) be the ideal of a complete intersection that links a d-fold line $W \subset \mathbb{P}^3$ to itself. Furthermore assume that $F = x^2$. Then G can be assumed to be $gx + y^d$.*

Proof. Since (F, G) is a complete intersection that links W to itself, then we have the following

$$\begin{aligned} Z(F, G) &= Z(x^2, G) \\ &= Z(x, G) \\ &= Z(x, y). \end{aligned}$$

So we have the following

$$\begin{aligned} Z(x, G) &= Z(x, y) \\ \Rightarrow \sqrt{(x, G)} &\subset \sqrt{(x, y)} \\ \Rightarrow (x, G) &\subset (x, y) \\ \Rightarrow G &= G_1 x + G_2 y, \end{aligned}$$

where $G_i \in S_{d-2}$ for $i = 1, 2$.

Again we have

$$\begin{aligned} Z(F,G) &= Z(x^2, G_1x + G_2y) \\ &= Z(x, G_2y) \\ &= Z(x,y) \cup Z(x, G_2). \end{aligned}$$

Thus repeat the logic to get

$$\begin{aligned} Z(x,G_2) &= Z(x,y) \\ \Rightarrow \sqrt{(x,G_2)} &\subset \sqrt{(x,y)} \\ \Rightarrow (x,G_2) &\subset (x,y) \\ \Rightarrow G_2 &= G_3x + G_4y. \end{aligned}$$

Combining the results we get

$$\begin{aligned} (F,G) &= (x^2, G_1x + (G_3x + G_4y)y) \\ &= (x^2, G_1x + G_3xy + G_4y^2) \\ &= (x^2, (G_1 + G_3y)x + G_4y^2). \end{aligned}$$

We repeat the process and combine that result with the fact that the degree of G has to be d. After a coordinate change if necessary we get

$$(F,G) = (x^2, gx + y^d).$$

Therefore we can assume that the complete intersection has the form $(x^2, gx+y^d)$. $\square$

Now we will look at the ideal of W.

Proposition 4.2.7. *(11, Proposition 2.1) To each curve $W \subset \mathbb{P}^3$ in the double plane $2H$, we can associate a triple $T(W) = \{Z, Y, P\}$ where $Z \subset Y \subset P \subset H$ and Z is a locally complete intersection zero-dimensional subscheme in the reduced plane Z. P is the intersection of W with H without embedded points (which if any end up being Z, and Y is the residual intersection. Let g be the arithmetic genus of W and d, y, p the degrees of W, Y, P respectively then*

$$d = y + p.$$

We will also need the following definition before we can talk about the ideal of curves on the double plane.

Definition 4.2.8. For any $u \times (u+1)$ matrix N, we denote by N_i the determinant of the matrix obtained from N by deleting its i-th column.

As before, we will disregard the case where W is ACM and focus on the non-ACM case, as the condition is necessary for the following lemma.

Lemma 4.2.9. *Let $W \subset \mathbb{P}^3$ be a non-ACM d-fold line which is contained in the double plane $x^2 = 0$ with support $Z(x, y)$. Let $T(W) = \{Z, Y, P\}$ be the associated triple as in Proposition 4.2.7. After a coordinate change, its ideal is*

$$I_W = (x^2, xy^n, y^{n+m}A_1 + xB_1, \dots, y^{n+m}A_{s+1} + xB_{s+1})$$

where

1. $$B = \begin{bmatrix} & & & f_1 \\ & A & & \\ & & & \vdots \\ & & & \\ p_1 & \dots & p_{s+1} & f_{s+1} \end{bmatrix} \in k[y, z, w]^{s+1, s+2}$$

   with $s \geq 1$.

2. *$A_i \neq 0$ for any i.*

3. *$I_Y = (x, y^n)$.*

4. *$I_P = (x, y^{n+m})$.*

5. *$d = 2n + m$.*

Proof. Number 1 is a direct result from (4, Theorem 1.1). Furthermore we get that there exists a matrix M such that

$$M = \left[\begin{array}{cc} & f_1 \\ A & \vdots \\ & f_s \end{array} \right] \in k[y,z,w]^{s,s+2}$$

where the maximal minors of M do not have common zeros on $H = Z(x)$. In particular we can say that $A_i \neq 0$ which proves number 2. Combining (4, Section 2) and the fact that the support of W is $Z(x,y)$ yields number 3 and 4. Combining these results with (4, Theorem 1.1) yields that

$$I_W = (x^2, xy^n, y^{n+m}A_1 + xB_1, \ldots, y^{n+m}A_{s+1} + xB_{s+1}). \qquad \square$$

Conjecture 4.2.10. *A quasi-primitive d-fold line only contained by the double plane is only self-linked in characteristic two.*

4.3 Conclusion

We will summarize the results of this chapter by stating the proof of Theorem 2.

Theorem 4.3.1. *(Theorem 2) Consider a multiple line $W \subset \mathbb{P}^3$ of degree greater than or equal to 3.*

1. *W is self-linked if one of the following occurs. The multiple line:*

   (a) *is a complete intersection.*

   (b) *up to coordinate change has ideal of the form $(x^2, xy, xq - y^{d-1})$.*

   (c) *is extremal and the characteristic of the ground field is two.*

   (d) *lies on a quadric cone.*

2. *W is not self-linked by a quadric if one of the following occurs. The multiple line:*

   (a) *is extremal and the characteristic of the ground field is not two.*

   (b) *lies on the smooth quadric and no other quadric.*

   (c) *lies on the reducible quadric and no other quadric.*

Proof. The proof is given by combining the following results:

1. Self-linkage:

   (a) Corollary 2.1.13.

   (b) Proposition 4.1.5.

   (c) Corollary 4.1.16.

   (d) Proposition 4.2.2

2. Not self-linked by a quadric:

   (a) Corollary 4.1.16.

   (b) Proposition 4.2.1.

   (c) Proposition 4.2.5. □

If we assume the multiple line is extremal of prime degree, we obtain Theorem 1.

Corollary 4.3.2. *(Theorem 1) An extremal prime-fold line in $\mathbb{P}^3$ is self-linked if and only if the characteristic of the ground field is two.*

Proof. By Corollary 4.1.16. □

Theorem 1 extends a result by Migliore, who proved it for double lines.

Theorem 4.3.3. *Consider d-fold lines $W, W' \subset \mathbb{P}^3$ with $d \geq 3$. They are directly linked if one of the following occurs:*

1. *They are complete intersections whose ideals share a generator.*

2. *They are supported on opposite rulings of a smooth quadric.*

3. *They lie on a quadric cone.*

4. *Up to coordinate change, the ideal of W is $I_W = (x^2, xy, y^d, xq - y^{d-1}p)$ and one of the following:*

   (a) the ideal of W' is $I_{W'} = (x^2, xy, y^d, xq' - y^{d-1}p')$ with $p' = \alpha p$ and $q' = -\alpha(gp + q)$ for some $\alpha \in k^*$.

   (b) The ideal of W' is $I_{W'} = (y^2, xy, x^d, yq' - x^{d-1}p')$.

Proof. The proof is given by combining the following results:

1. Proposition 2.1.12.

2. Proposition 4.2.1.

3. Proposition 4.2.2.

4. Theorem 4.1.14. □

5 Linkage of Triple Lines

5.1 Triple Lines

Now we will provide a more careful analysis of a special case $d = 3$. While many of the results in this chapter are special cases of results of the previous chapter, we will still state them in this chapter for reference. The main difference from the previous chapter is that we characterize self-linkage of all triple lines and can determine direct linkages on the double plane. This difference is due to having an explicit formula for the ideal of any such triple line from (19).

Definition 5.1.1. A *triple line* is a non-reduced locally Cohen-Macaulay curve of degree 3 in $\mathbb{P}^3$ whose support consists of a line. There is only one case when the triple line is not quasi-primitive which we will call a *thick triple line*.

From (19) we have the following classification of triple lines on a fixed line $Y \subset \mathbb{P}^3$. If W is a triple line of type $(L, \mathcal{D}_2)$, then there are two exact sequences

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_Y(a) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y(2a+b) \longrightarrow 0$$

where Z is a double line supported on Y described in (19) and $L = \mathcal{O}_Y(a)$. In particular $a \geq -1$ and $\deg D_2 = b \geq 0$. We say that W is of type (a, b). Also from (19) we have the following propositions and their corresponding corollaries.

Proposition 5.1.2. *Let $Z \subset \mathbb{P}^3$ be the double line with ideal $I_Z = (I_Y^2, xg - yf)$, where Y is the line $x = y = 0$ and f, g are homogenous polynomials of degree $a+1$ having no common zeroes along Y. Let p and q be homogenous polynomials of degrees $b, 3a+b+2$ having no common zeroes along Y. Then p and q define a surjection $u : I_Z \to \mathcal{O}_Y(2a+b)$ by $x^2 \mapsto pf^2, xy \mapsto pfg, y^2 \mapsto pg^2$, and $xg - yf \mapsto q$. The kernel of u is the ideal sheaf of a quasi-primitive multiplicity three structure on Y with second Cohen-Macaulay filtrant Z. Further we have:*

1. $p_a(W) = -2 - 3a - b$.

2. $I_W = (I_Y^3, x(xg - yf), y(xg - yf), p(xg - yf) - rx^2 - sxy - ty^2)$ *where* r, s, t *are chosen so that* $q = rf^2 + sfg + tg^2 \bmod I_Y$.

3. *If* p', q' *define another three structure* W', *then* $W = W'$ *if and if there exist* $c \in k^*$ *such that* $p' = cp \bmod I_Y$ *and* $q' = cq \bmod I_Y$.

4. W *is quasi-primitive with second CM filtrant* Z, *unless* $b = 1$ *and* $q = 0$, *in which case* $W = Y^{(2)}$.

Proof. We can find the full proof in (19, Proposition 2.3) but we will present the main ideas below.

The ideal $I_Z = (x^2, xy, y^2, xg - yf)$ has S-presentation

$$0 \to S(-a-4) \to S(-3)^2 \oplus S(-a-3)^2 \xrightarrow{\Phi} S(-2) \oplus S(-a-2) \to I_Z \to 0$$

where Φ is the matrix given by

$$\Phi = \begin{bmatrix} y & 0 & -g & 0 \\ -x & y & f & -g \\ y & -x & 0 & f \\ y & 0 & x & y \end{bmatrix}.$$

Tensoring the exact sequence with S_Y we get $I_Z/I_Z I_Y = \operatorname{Coker}\Phi \otimes S_Y$ is isomorphic to $S_Y(-a-2) \oplus (f^2, fg, g^2)(2a)$ where $\overline{x}^2, \overline{xy}, \overline{y}^2$ are identified with f^2, fg, g^2 respectively. We have

$$0 \longrightarrow I_Z/I_Z I_Y \longrightarrow S_Y(-a-2) \oplus S_Y(2a) \longrightarrow \mathrm{Coker} \longrightarrow 0$$

where Coker is the cokernel of the the inclusion map and has finite length. If we sheafify the short exact sequence we have

$$0 \longrightarrow \widetilde{I_Z/I_Z I_Y} \longrightarrow \mathcal{O}_Y(-a-2) \oplus \mathcal{O}_Y(2a) \longrightarrow 0 \longrightarrow 0.$$

So $\widetilde{I_Z/I_ZI_Y} \cong \mathcal{O}_Y(-a-2) \oplus \mathcal{O}_Y(2a)$ freely generated by $xg-yf$ and an element e such that $ef^2 = \bar{x^2}, efg = \bar{xy}, eg^2 = \bar{x^2}$. The polynomial p and q give a graded homomorphism

$$\phi : I_Z \longrightarrow I_Z/I_ZI_Y \longrightarrow S_Y(-a-2) \oplus S_Y(2a) \xrightarrow{(\bar{q},\bar{p})} S_Y(2a+b).$$

The kernel of the map $(\bar{q}, \bar{p})$ is generated by the Koszul relation $qe - p(xg - yf)$. Since f and g are relatively prime of degree $a+1$ then the map $S_Y(-2a-2)^3 \to S_Y$ defined by sending the generators to (f^2m, fg, g^2) is surjective for degrees greater than $3(a+1)-1$. Therefore there exists (r, s, t) such that $q = rf^2 + sfg + tg^2 \mod I_Y$. We conclude

$$(rx^2 + sxy + ty^2 - p(xg - yf)) = I_Z/I_ZI_Y \cap \mathrm{Ker}(\bar{q}, \bar{p})$$

and then get

$$\mathrm{Ker}\,\phi = (x^3, x^2y, xy^2, y^3, x(xg - yg), y(xg - yf), rx^2 + sxy + ty^2 - p(xg - yf)).$$

$\mathrm{Coker}\,\phi$ has finite length, so ϕ sheafifies to a surjection $u : I_Z \to \mathcal{O}_Y(2a+b)$. Let W be the subscheme such that $I_W = \mathrm{Ker}\,u$. The snake lemma yields the following short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(2a+b) \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

which yields the fact that $\mathrm{Supp}\,W = Y$ and $\mathrm{depth}\,\mathcal{O}_W$ is greater than or equal to 1. Thus we have that W is a multiplicity 3 structure on Y.

Since $p_a(Z) = -1 - a$, the short exact sequence yields $p_a(W) = -2 - 3a - b$. We also have the short exact sequence

$$0 \longrightarrow I_W \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y(2a+b) \longrightarrow 0$$

which shows $I_W = \operatorname{Ker} \phi$. Finally, if p' and q' define W' by the construction above and $W = W'$, then $eq' - (xg - yf)p'$ and $eq - (xg - yf)p$ generates the same S_Y-module of $I_Z/I_Z I_Y \subset S_Y(-a-2) \oplus S_Y(2a)$. Finally, since e and $(xg - yf)$ are free generators, then there exist a $d \in k^*$ such that $p' = pd \bmod I_Y$ and $q' = qd \bmod I_Y$. $\square$

Corollary 5.1.3. *(19, Corollary 2.6) Let $W \subset \mathbb{P}^3$ be a triple line with support Y of type (a, b) with $a, b \geq 0$. Then, after a suitable change of coordinates, W is constructed by Proposition 5.1.2.*

Observations 5.1.4. *Notice that part* (3) *of Proposition 5.1.2 implies that we have the following:*

- *We can assume $p, q \in S/I_Y = S/(x, y) = k[z, w]$.*
- $\deg x(xg - yf) = \deg y(xg - yf) = a + 3$.
- $\deg p(xg - yf) - rx^2 - sxy - ty^2 = a + b + 2$.
- $\deg r = \deg s = \deg t = a + b$.

A special case of triple lines is of type $(-1, b)$ as their corresponding double line Z is planar (hence a complete intersection) while this is not the case for $a \geq 0$. By Theorem 2.3.5 we get that the following lines are the only triple lines that are extremal. We will provide the following statement for reference as it is a special case of Theorem 4.1.1.

Proposition 5.1.5. *Let $Y \subset \mathbb{P}^3$ be the line $x = y = 0$ and let Z be the multiplicity two structure $x = y^2 = 0$ on Y. Let p, q be two homogeneous polynomials of degrees $b-1, b$ which have no common zeroes along Y. Then the maps $x \mapsto p, y^2 \mapsto q$ define a surjection $u : \mathcal{I}_Z \to \mathcal{O}_Y(b-2)$. The kernel of u is the ideal sheaf of a multiplicity three structure W on Y. Further, we have*

1. *$p_a(W) = 1 - b$.*
2. *$H^1_*(\mathcal{I}_W) \cong (S/(x, y, p, q))(b-2)$.*
3. *$I_W = (x^2, xy, y^3, xq - y^2 p)$.*
4. *If p', q' define another three structure W', then $W = W'$ if and if there exist $c \in k^*$ such that $p' = cp \bmod I_Y$ and $q' = cq \bmod I_Y$.*
5. *W is quasi-primitive, unless $b = 1$ and $q = 0$ in which case $W = Y^{(2)}$.*

Moreover, a triple line W with support Y of type $(-1, b)$ with $b \geq 0$ or the thick triple line. Then, after a suitable change of coordinates, W is constructed as above.

Proof. By Proposition 4.1.1 and Corollary 4.1.2. ∎

Observations 5.1.6. *Notice that part* (4) *implies that we have the following*

- *We can assume that* p *and* q *are in* $S/I_Y = S/(x, y) = k[z, w]$.

- $\deg(xq - y^2p) = b + 1$.

The following theorem separates cases by value of b.

Corollary 5.1.7. *Let* $W \subset \mathbb{P}^3$ *be an triple line of type* $(-1, b)$ *with* $b \geq 0$. *Then up to coordinate change:*

(a) If $b = 0$, *then* W *is planar with ideal* (x, y^3).

(b) If $b = 1$, *then* W *is ACM with ideal* $(x^2, xy, xz - y^2)$ *and lies on the quadric cone.*

(c) If $b = 2$, *then* W *has ideal* $(x^2, xy, y^3, xq - y^2z)$ *where* $q \in k[z, w]_2$ *and* p *and* z *have no common zeroes along* $Z(x, y)$ *and* $(\operatorname{Ann} M(W))_1 = (I_{p_0})_1$, *where* $p_0 \in Y$ *is a point.*

(d) If $b \geq 3$, *then* w *has ideal* $(x^2, xy, y^3, xq - y^2p)$ *where* $p \in k[z, w]_{b-1}$ *and* $q \in k[z, w]_b$ *which have no common zeroes along* $Z(x, y)$ *and* $(\operatorname{Ann} M(W))_1 = (I_Y)_1$.

Additionally W *is extremal exactly when* $b \geq 2$ *and ACM when* $b = 0, 1$.

Proof. The result is a special case of Corollary 4.1.3 with $d = 3$. □

We have analyzed the complete intersections that can link prime-fold lines to other prime-fold lines. We will recall that result for triple lines as it will help us determine which triple lines can link to other triple lines.

Observations 5.1.8. *If $X \subset \mathbb{P}^3$ is a complete intersection that links two triple lines, then $I_X = (F, G)$ where we can assume we have either:*

1. *deg $F = 1$ and deg $G = 6$.*

2. *deg $F = 2$ and deg $G = 3$.*

Proof. Use Lemma 4.0.9 with $d = 3$. □

While Observation 5.1.8 is a simple result, it allows us to say that many triple lines do not directly link to other triple lines, and thus are not self-linked either. First we will recall a special case of an earlier statement for reference.

Corollary 5.1.9. *If $W \subset \mathbb{P}^3$ is a triple line, then the following are equivalent:*

1. *W is planar.*

2. *W is a complete intersection.*

3. *W is of type $(-1, 0)$.*

4. *Up to coordinate change, the ideal of W is (x, y^3).*

Proof. Lemma 2.4.3. □

Finally, the next lemma will get us closer to answering the question of when triple lines are directly linked.

Lemma 5.1.10. *If triple lines W and W' in $\mathbb{P}^3$ are linked by a complete intersection X, then the types of W or W' is either $(0, 0)$ or $(-1, b)$ for $b \geq 0$.*

Proof. From Observation 5.1.8 we have two cases.

1. deg $F = 1$ and deg $G = 6$.

2. deg $F = 2$ and deg $G = 3$.

Case 1: deg $F = 1$ and deg $G = 6$.

Using Corollary 5.1.9 we have that W is planar, and hence a complete intersection.

Case 2: deg $F = 2$ and deg $G = 3$.

The containment $W \subset C$ implies $I_X \subset I_W$. Therefore $F \in I_W$ which means I_W must contain a quadratic. By Proposition 5.1.2 we have

$$I_W = (x^3, x^2y, xy^2, y^3, x(xg - yf), y(xg - yf), p(xg - yf) - rx^2 - sxy - ty^2).$$

Case a: We have $\deg x(xg - yf) = 2$, thus $\deg f = \deg g = 0 = a + 1$. Which yields the type of W is $(-1, b)$.

Case b: $\deg p(xg - yf) - rx^2 - sxy - ty^2 = 2$ which can happen in two ways. The first way is $a + 1 = \deg g = \deg f = \deg r = \deg s = \deg t = 0$ and $\deg p = 1 = b$. which yields the type of W is $(-1, 1)$ which is included in case a. The second way is $a + 1 = \deg g = \deg f = 2$ and $b = \deg p = \deg r = \deg s = \deg t = 0$. Which yields the type of W is $(0, 0)$. $\square$

We will now make an observations about the type of two triple lines that are linked to each other.

Observations 5.1.11. *Let $W \subset \mathbb{P}^3$ be a triple line of type (a, b) and genus g. By part (a) of Proposition 5.1.2 we have $g = -2 - 3a - b$ Using the results of Lemma 5.1.10*

$$a = -1 \;\; \Rightarrow \;\; g = 1 - b,$$

$$a = 0 \;\; \Rightarrow \;\; g = -2 - b.$$

If triple lines W and W' are directly linked and they are of type $(-1, b)$ and $(0, b')$ and genus g and g' respectively. Then

$$\begin{aligned} g &= g' \\ \Rightarrow 1 - b &= 2 - b' \\ \Rightarrow b' &= b - 3. \end{aligned}$$

By Observation 5.1.11 we have that for triple lines if the first component of the type (a, b) and the genus g of a curve is fixed, then so is the second component of the type. We also have that if triple lines W and W' of type $(-1, b)$ and $(0, b')$ respectively, are directly linked by a complete intersection C then $b' = b - 3$. Now we have that the only triple lines that link to other triple lines have type $(0, 0)$ and $(-1, b)$.

Remark 5.1.12. Combining Lemma 5.1.10 and Observation 5.1.11 we get that the if triple lines of different type are directly linked then they are of type $(0, 0)$ and $(-1, 3)$. We will see later that this is not the case. This is not a direct corollary from statements from chapter 4, since we are examining all triple lines, and not just extremal triple lines.

5.2 Smooth Quadric

Now we consider triple lines of type (0,0). We start this section by recalling the results for general multiple-fold lines. We then present an argument that produces an explicit description of the ideal of the residual scheme.

Proposition 5.2.1. *Up to a coordinate change, triple line $W \subset \mathbb{P}^3$ of type (0,0) has ideal of the form $I_W = (x^3, x^2y, xy^2, y^3, xz - yw)$. Specifically, the only quadric that W lies on is the smooth quadric $xz - yw = 0$.*

Proof. Let the support of W be $Y(x = y = 0)$. From Proposition 5.1.2 we have

$$I_W = (x^3, x^2y, xy^2, y^3, x(xg - yf), y(xg - yf), p(xg - yf) - rx^2 - sxy - ty^2)$$

such that

$$\begin{aligned} \deg p &= b = 0 = a + b = \deg r = \deg s = \deg t, \\ \deg g &= \deg f = a + 1 = 1, \\ \deg q &= 3a + b + 2 = 2. \end{aligned}$$

By a coordinate change we may assume $f = w$ and $g = z$ since (f, g) makes a regular sequence in $k[z, w]$. Furthermore they do not have common zeros along Y.

If $p = 0$ implies q is a unit (p and q don't share zeros on Y). This leads to a contradiction because $\deg q = 2$. Thus p is a unit. Thus

$$I_W = (x^3, x^2y, xy^2, y^3, x(xz - yw), y(xz - yw), (xz - yw) - rx^2 - sxy - ty^2).$$

Notice the following products are elements of

$$(x^3, x^2y, xy^2, y^3, (xz-yw) - rx^2 - sxy - ty^2):$$

$$\begin{aligned}
x(xg - yf) &= x((xz-yw) - rx^2 - sxy - ty^2 + rx^2 + sxy + ty^2) \\
&= x((xz-yw) - rx^2 - sxy - ty^2) + x(rx^2 + sxy + ty^2) \\
&= x((xz-yw) - rx^2 - sxy - ty^2) + rx^3 + sx^2y + txy^2 \\
y(xg - yf) &= y((xz-yw) - rx^2 - sxy - ty^2 + rx^2 + sxy + ty^2) \\
&= y((xz-yw) - rx^2 - sxy - ty^2) + y(rx^2 + sxy + ty^2) \\
&= y((xz-yw) - rx^2 - sxy - ty^2) + rx^2y + sxy^2 + txy^3.
\end{aligned}$$

Thus

$$I_W = (x^3, x^2y, xy^2, y^3, (xz-yw) - rx^2 - sxy - ty^2).$$

Consider the following coordinate change

$$\begin{aligned}
z' &= z - rx - sy, \\
w' &= w + ty.
\end{aligned}$$

This coordinate change is well-defined since M is invertible where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -r & -s & 1 & 0 \\ 0 & t & 0 & 1 \end{bmatrix}.$$

The following finishes the proof:

$$\begin{aligned} I_W &= (x^3, x^2y, xy^2, y^3, (xz - yw) - rx^2 - sxy - ty^2) \\ &= (x^3, x^2y, xy^2, y^3, xz - rx^2 - sxy - yw - ty^2)) \\ &= (x^3, x^2y, xy^2, y^3, x(z - rx - sy) - y(w - ty)) \\ &= (x^3, x^2y, xy^2, y^3, xz' - yw'). \end{aligned}$$

□

The fact that the smooth quadric is the only quadric that triple lines of type $(0, 0)$ lies on is very important to the rest of this subsection. We will be able to use results from the general multiple-fold case.

Proposition 5.2.2. *Any triple line in* $\mathbb{P}^3$ *of type* $(0, 0)$ *is not self-linked. Furthermore, it is only directly linked to a triple line with different support.*

Proof. Let W be a triple line of type $(0, 0)$, them by Proposition 5.2.1 we have that W is only contained by the smooth quadric. Finally by Proposition 4.2.1 we have that W can only directly link to another triple line with different support. □

Now we change our focus from triple lines of type $(0,0)$ to consider a result that will be important for the rest of this chapter.

Theorem 5.2.3. *If two triple lines in $\mathbb{P}^3$ are directly linked, then they are of the same type.*

Proof. From Remark 5.1.12 we have already establish that the only case left to examine is the case where triple lines W and W' have type $(0,0)$ and $(-1,3)$ respectively. As established in Theorem 4.1.8 a triple line of the type $(-1,3)$ can only be linked to a triple line with the same support. However we also have from Proposition 5.2.2 that a triple line of type $(0,0)$ can only be directly linked to another triple line with different support. Therefore triple lines of type $(0,0)$ cannot be linked to triple lines of type $(-1,3)$, and the desired result follows. $\square$

Question 5.2.4. *Can Theorem 5.2.3 be generalized to all quasi-primitive multiple lines?*

While the following proofs will still use geometric arguments, the approach will be more algebraic focused in order to obtain an explicit description of the ideals.

Corollary 5.2.5. *Up to a change of coordinates, complete intersections $X \subset \mathbb{P}^3$ that contain a triple line W of type $(0,0)$ and directly link to another triple line W' have ideal of the form $I_X = (xz - yw, l^3)$ where l is an element of the vector space $\langle x, y\rangle$.*

Proof. By Proposition 5.2.1 we have

$$I_W = (x^3, x^2y, xy^2, y^3, xz - yw).$$

We also have

$$I_X = (F, G)$$

where by Observation 5.1.8 we have $\deg F = 2$ and $deg\, G = 3$. As before we have $I_X \subset I_W$. Thus G is an element of the vector space $\langle x^3, x^2y, xy^2, y^3\rangle$ and we may assume $F = xz - yw$.

Claim: We can factor G into three linear elements.

Notice the following:

$$\begin{aligned}
G &= c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 \text{ where } c_i \in k \\
&= \left(c_1\frac{x^3}{y^3} + c_2\frac{x^2}{y^2} + c_3\frac{x}{y} + c_4\right)y^3 \\
&= (c_1Z^3 + c_2Z^2 + c_3Z + c_4)y^3 \text{ where } Z = \frac{x}{y} \\
&= \prod_{i=1}^{3}(\alpha_iZ + \beta_i)y^3 \text{ since our ground field is algebraically closed} \\
&= \prod_{i=1}^{3}(\alpha_i\frac{x}{y} + \beta_i)y^3 \text{ where } \alpha_i, \beta_i \in k \\
&= \prod_{i=1}^{3}(\alpha_ix + \beta_iy).
\end{aligned}$$

Now we have $I_X = (xz - yw, l_1l_2l_3)$ where l_i is an element of the vector space $\langle x, y\rangle$.

Claim: If all l_i are the same (up to scalar multiple), then the support of C is two lines. However, if $l_i \neq l_j$ (up to scalar multiple) for some i and some j, then the support of C is more than two lines.

Notice that $(l_i, zx - yw)$ is a complete intersection of degree 2. The components of this complete intersection have to have degrees that add up to 2. Thus the complete intersection is two single lines:

$$Z(xz - yw, l_1 l_2 l_3) = Z(xz - yw, l_1) \cup Z(xz - yw, l_2) \cup Z(xz - yw, l_3).$$

Thus the supports of $Z(xz - yw, l_1)$, $Z(xz - yw, l_2)$ and $Z(xz - yw, l_3)$ must be the same two lines. However since two lines define a unique plane, we have that all $l_i's$ define the same plane, and thus they are equal (up to unit). $\square$

Notice that as the plane $l = 0$ where $l \in (x, y)$ varies, it describes the family of planes P_l containing the line $Y = \operatorname{Supp} W$. Geometrically $P_l \cap Q$ is a plane quadric containing the line Y, hence is the reducible union of two lines, the second line being the support of the linked curve seen in Proposition 5.2.2. Before we compute the ideal of the residual scheme, we will show an easy example.

Example 5.2.6. Let $l = y$ in the Lemma 5.2.5.

Consider the support of C:

$$\begin{aligned} Z(C) &= Z(zx - yw, y^3) \\ &= Z(zx, y) \\ &= Z(x, y) \cup Z(y, z). \end{aligned}$$

Thus by Proposition 5.2.2 we have that $I_W = (x^3, x^2y, xy^2, y^3, xz - yw)$ is linked to $I_W = (z^3, z^2y, zy^2, y^3, xz - yw)$.

We finally have all the tools necessary to describe the ideal of the residual scheme when linking a triple line W of type $(0, 0)$. Now, up to coordinate change by Proposition 5.2.1, we have that a triple line W of type $(0, 0)$ has ideal $I_W = (x^3, x^2y, xy^2, y^3, xz - yw)$, and a complete intersection C linking it to another triple line has ideal $I_X = (xz - yw, l^3)$ where l is in the vector space $\langle x, y \rangle$.

We first take care of an algebra fact that is used several times in the main theorem.

Observations 5.2.7. *We will need the following monomial memberships:*

1. $(xz - yw, x^3)$ *contains* x^2yw, xy^2w^2 *and* y^3w^3.

2. $(xz - yw, y^3)$ *contains* xy^2z, x^2yz^2 *and* x^3z^3.

Proof. Notice

$$
\begin{aligned}
x^2yw &= -x^2(xz - yw - xz) \\
&= -x^2(xz - yw) + x^3z \in (xz - yw, x^3), \\
xy^2w^2 &= -xyw(xz - yw - xz) \\
&= -xyw(xz - yw) + x^2ywz \in (xz - yw, x^3), \\
y^3w^3 &= -y^2w^2(xz - yw - xz) \\
&= -y^2w^2(xz - yw) + xy^2w^2z \in (xz - yw, x^3),
\end{aligned}
$$

$$
\begin{aligned}
xy^2z &= y^2(xz - yw + yw) \\
&= y^2(xz - yw) + y^3w \in (xz - yw, y^3), \\
x^2yz^2 &= xyz(xz - yw + yw) \\
&= xyw(xz - yw) + (xy^2z)w \in (xz - yw, y^3), \\
x^3z^3 &= x^2z^2(xz - yw + yw) \\
&= x^2z^2(xz - yw) + (x^2yz^2)w \in (xz - yw, y^3).
\end{aligned}
$$

$\square$

Theorem 5.2.8. *Let $W \subset \mathbb{P}^3$ be a triple line of type $(0,0)$ with ideal*

$$I_W = (x^3, x^2y, xy^2, y^3, xz - yw).$$

Then it can only be directly linked to another triple line W' if

$$\begin{aligned}
\operatorname{Supp} W' &= Z(\alpha x + \beta y, \alpha w + \beta z) \\
I_{W'} &= ((\alpha x + \beta y)^3, (\alpha x + \beta y)^2(\alpha w + \beta z), (\alpha x + \beta y)(\alpha w + \beta z)^2, \\
&\quad (\alpha w + \beta z)^3, xz - yw)
\end{aligned}$$

where $(\alpha, \beta) \in \mathbb{P}^1$.

Proof. By corollary 5.2.5 we have $I_X = (xz - yw, l^3)$ where l is in the vector space $\langle x, y \rangle$ say $l = \alpha x + \beta y$. We will first take care of the case $\beta = 0$ and then $\beta \neq 0$.

Case 1: $\beta = 0$.

We can assume $\alpha = 1$ so we have $l = x$ and we have $I_X = (xz - yw, x^3)$. Thus we have

$$\begin{aligned}
I_{W'} &= [I_X : I_W] \\
&= [(xz - yw, x^3) : (x^3, x^2y, xy^2, y^3, xz - yw)] \\
&= \{s \in S | s(x^3, x^2y, xy^2, y^3, xz - yw) \subset (xz - yw, x^3)\} \\
&= \{s \in S | s(x^2y, xy^2, y^3) \subset (xz - yw, x^3)\}.
\end{aligned}$$

Claim: $I_{W'} = (x^3, x^2w, xw^2, w^3, xz - yw)$.

It is clear that $xz - yw$ and x^3 are in $I_{W'}$. Recall that Observation 5.2.7 proves that $(xz - yw, x^3)$ contains x^2yw, xy^2w^2 and y^3w^3. Therefore the following products are elements of $(xz - yw, x^3)$:

$$x^2w$$

$$\begin{aligned}
(x^2w)(x^2y) &= x^4yw = x^2(x^2yw) \\
(x^2w)(xy^2) &= x^3y^2w = xy(x^2yw) \\
(x^2w)(y^3) &= x^2y^3w = y^2(x^2yw)
\end{aligned}$$

$$xw^2$$

$$\begin{aligned}
(xw^2)(x^2y) &= x^3yw^2 = xw(x^2yw) \\
(xw^2)(xy^2) &= x^2y^2w^2 = yw(x^2yw) \\
(xw^2)(y^3) &= xy^3w^2 = y(xy^2w^2)
\end{aligned}$$

$$w^3$$

$$\begin{aligned}
(w^3)(x^2y) &= x^2yw^3 = w^2(x^2yw) \\
(w^3)(xy^2) &= xy^2w^3 = w(xy^2w^2) \\
(w^3)(y^3) &= y^3w^3
\end{aligned}$$

Thus we have $I_{W'} \supset (x^3, x^2w, xw^2, w^3, xz - yw)$. By Lemma 2.0.3 the ideal of W' is $I_{W'} = (x^3, x^2w, xw^2, w^3, xz - yw)$.

Case 2: $\beta \neq 0$.

We may assume $l = \alpha x + y$. Consider the following coordinate change:

$$\begin{aligned} X &= x, \\ Y &= \alpha x + y, \\ Z &= z + \alpha w, \\ W &= w. \end{aligned}$$

Thus we have

$$\begin{aligned} I_W &= (x^3, x^2y, xy^2, y^3, xz - yw) \\ &= (X^3, X^2(Y - \alpha X), X(Y - \alpha X)^2, (Y - \alpha X)^3, X(Z - \alpha W) - (Y - \alpha X)W) \\ &= (X^3, X^2Y - \alpha X^3, X(Y^2 - 2\alpha XY + \alpha^2 X^2), \\ &\quad Y^3 - 3\alpha XY^2 + 3\alpha^2 X^2 - \alpha^3 Y^3, XZ - \alpha XW - YW + \alpha XW) \\ &= (X^3, X^2Y - \alpha X^3, XY^2 - 2\alpha X^2Y + \alpha^2 X^3, \\ &\quad Y^3 - 3\alpha XY^2 + 3\alpha^2 X^2 - \alpha^3 X^3, XZ - YW) \\ &= (X^3, X^2Y, XY^2, Y^3, XZ - YW). \end{aligned}$$

We also have

$$\begin{aligned} I_X &= (xz - yw, l^3) \\ &= (X(Z - \alpha W) - (Y - \alpha X)W, (\alpha x + y)^3) \\ &= (XZ - \alpha XW - YW + \alpha XW, (\alpha x + y)^3) \\ &= (XZ - YW, Y^3). \end{aligned}$$

Thus we have

$$\begin{aligned} I_{W'} &= [I_X : I_W] \\ &= [(XZ - YW, Y^3) : (X^3, X^2Y, XY^2, Y^3, XZ - YW)] \\ &= \{s \in S | s(X^3, X^2Y, XY^2, Y^3, XZ - YW)] \subset (XZ - YW, Y^3)\} \\ &= \{s \in S | s(X^3, X^2Y, XY^2)] \subset (XZ - YW, Y^3)\}. \end{aligned}$$

Claim: $I_{W'} = (Y^3, Y^2Z, YZ^2, Z^3, ZX - YW)$.

It is clear that $ZX - YW$ and Y^3 are in $I_{W'}$. Recall that Observation 5.2.7 proves that $(xz - yw, y^3)$ contains xy^2z, x^2yz^2 and x^3z^3. Therefore the following products are elements of $(XZ - YW, Y^3)$:

$$
\begin{aligned}
& Y^2Z \\
(Y^2Z)(X^3) &= X^3Y^2Z = X^2(XY^2Z) \\
(Y^2Z)(X^2Y) &= X^2Y^3Z = XY(XY^2Z) \\
(Y^2Z)(XY^2) &= XY^4Z = Y^2(XY^2Z) \\
& YZ^2 \\
(YZ^2)(X^3) &= X^3YZ^2 = X(X^2YZ^2) \\
(YZ^2)(X^2Y) &= X^2Y^2Z^2 = Y(X^2YZ^2) \\
(YZ^2)(XY^2) &= XY^3Z^2 = YZ(XY^2Z) \\
& Z^3 \\
(Z^3)(X^3) &= X^3Z^3 \\
(Z^3)(X^2Y) &= X^2YZ^3 = Z(X^2YZ^2) \\
(Z^3)(XY^2) &= XY^2Z^3 = Z^2(XY^2Z)
\end{aligned}
$$

Thus we have $I_{W'} \supset (Y^3, Y^2Z, YZ^2, Z^3, ZX - YW)$. By Lemma 2.0.3 we have the ideal of W' is $I_{W'} = (Y^3, Y^2Z, YZ^2, Z^3, ZX - YW)$. Finally, notice that the support of W' is $Z(Y, Z) = Z(\alpha x + y, \alpha w + z)$.

Furthermore

$$
\begin{aligned}
I_{W'} &= ((\alpha x+y)^3, (\alpha x+y)^2(\alpha w+z), (\alpha x+y)z^2, (\alpha w+z)^3, (\alpha w+z)x-(\alpha x+y)w) \\
&= ((\alpha x+y)^3, (\alpha x+y)^2(\alpha w+z), (\alpha x+y)z^2, (\alpha w+z)^3, zx-yw).
\end{aligned}
$$

Combining both cases we get the desired result. $\square$

Example 5.2.9. In case 2, if we have $\alpha = 0$ then we get $I_{W'} = (y^3, y^2z, yz^2, z^3, zx-yw)$ with support of W being (y, z).

5.3 ACM Triple Lines

We will now look at triple lines that are ACM. Recall that Observation 5.1.7 that triple lines of type (-1,0) and (-1,1) are ACM. We will present both direct linkage results and self-linkage results for both cases.

5.3.1 Planar Triple Lines

By Lemma 2.4.3 triple lines of type $(-1, 0)$ are planar triple lines (which are also complete intersections). We will just summarize the results in this section.

Corollary 5.3.1. *Let W and W' be triple lines in $\mathbb{P}^3$ of type $(-1, 0)$, then we have the following:*

1. *Each of W and W' is self-linked.*
2. *W is directly linked to W' if their ideals share a generator.*
3. *If W and W' are disjoint, then they can be linked in two steps.*

Proof. By Corollary 5.1.9, we know that a triple line W of type (-1,0) is a complete intersection. Using Proposition 2.1.12 with Corollary 2.1.13 we have our result. $\square$

5.3.2 Quadric Cone and Thick Triple Lines

Let W be a triple line of type $(-1, 1)$, then we have

$$I_W = (x^2, xy, y^3, xq - y^2p),$$

$$H^1_*(\mathcal{I}_W) \cong (S/(x, y, p, q))(b-2),$$

where we can assume q, p are elements of $k[z, w]$ that have no common zeros along Y where $\deg p = b - 1 = 0$ and $\deg q = b = 1$. Therefore p must be a unit. So we may assume $I_W = (x^2, xy, y^3, xq - y^2)$. Now notice that

$$y^3 = -y(xq - y^2 - xq) = -y(xq - y^2) - q(xy).$$

So we have

$$I_W = (x^2, xy, xq - y^2).$$

If $q = 0$ then W is the thick triple line with ideal

$$I_W = (x^2, xy, y^2).$$

Recall from Proposition 5.1.2 that the triple line is not quasi-primitive. We will include its analysis in this chapter with triple lines of type $(-1,1)$ even though technically, the phrase type of Banica and Forster only refers to quasi-primitive schemes. If $q \neq 0$, then q is linear. By a coordinate change we can assume $q = z$, then we have

$$I_W = (x^2, xy, xz - y^2).$$

Then W is a triple line of type $(-1,1)$.

Proposition 5.3.2. *The thick triple line in $\mathbb{P}^3$ is self-linked.*

Proof. Let W be the thick triple line. We have

$$I_W = (x^2, xy, y^2).$$

Consider the complete intersection ideal $I_X = (x^2, y^3)$. Notice

$$\begin{aligned}
I_{W'} &= [I_X : I_W] \\
&= \{s \in S | sI_W \subset I_X\} \\
&= \{s \in S | s(x^2, xy, y^2) \subset (x^2, y^3)\} \\
&= \{s \in S | s(xy, y^2) \subset (x^2, y^3)\}.
\end{aligned}$$

Clearly $x^2 \in I_{W'}$. Notice the following products are elements of (x^2, y^3) :

$$\begin{array}{rcl} & xy & \\ (xy)(xy) & = & x^2y^2 \\ (xy)(y^2) & = & xy^3 \\ & y^2 & \\ (y^2)(xy) & = & y^3 \\ (y^2)(y^2) & = & y^4 \end{array}$$

So we have $I_{W'} \supset (x^2, xy, y^2)$. By Lemma 2.0.3 we have the equality. $\square$

Triple line of type $(-1, 1)$ are a special case that we have already taken care of. We will include the result below for completeness.

Proposition 5.3.3. *Let W and W' be triple lines in $\mathbb{P}^3$ of type $(-1, 1)$ that lie on the same quadric. Then W is directly linked to W'. Furthermore, W is self-linked.*

Proof. As shown earlier, triple lines of type $(-1, 1)$ lie on the quadric cone. By Proposition 4.2.2 we have the result. $\square$

5.4 Extremal Triple Lines

This section is already covered by the general prime-fold case.

5.5 Conclusion

We summarize the results of this chapter by stating the proof of Theorem 3.

Theorem 5.5.1. *Consider triple lines $W, W' \subset \mathbb{P}^3$. They are directly linked if and only if one of the following occurs:*

1. *They are intersecting complete intersections.*

2. *They are supported on intersecting rulings of a smooth quadric.*

3. *They lie on a quadric cone.*

4. *Up to coordinate change, the ideal of W is $I_W = (x^2, xy, y^3, xq - y^2p)$ and one of the following:*

   (a) *the ideal of W' is $I_{W'} = (x^2, xy, y^3, xq' - y^2p')$ with $p' = \alpha p$ and $q' = -\alpha(gp+q)$ for some $\alpha \in k^*$.*

   (b) *The ideal of W' is $I_{W'} = (y^2, xy, x^3, yq' - x^2p')$ with $p' = \alpha p$ and $q' = \alpha q$ for some $\alpha \in k^*$.*

Proof. Combining Lemma 4.0.9 and Theorem 4.3.3 yields the result. □

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ABSTRACT

ON LINKING MULTIPLE LINES

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We study non-reduced locally Cohen-Macaulay quasi-primitive curves supported on a line in three dimensional projective space over an algebraically closed field k. For an odd prime p, we determine exactly when two extremal p-fold lines W and V are directly linked. In particular, W is self-linked if and only if char $k = 2$, analogous to Migliore's result for $p = 2$. The results hold for multiple lines of any degree if we add the extra hypothesis of using quadric surfaces to do the linking. We provide a complete analysis for triple lines.